# INTEGRALS OVER COMPONENTS OF THE SPRINGER FIBER FOR $\mathfrak{sl}(n,\mathbb{C})$

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The main result in [BaZ, Section 6] is an algorithm to compute the cohomology of a certain class of irreducible components of the Springer fiber for $\mathfrak{sl}(n,\mathbb{C})$. As explained, for instance, in [J3, Corollary 6.7], this is related to the computation of the integrals over such components of exponentiated Chern classes of homogeneous line bundles on the flag variety. In turn, [Ch, Section 2] implies results about multiplicities in associated cycles of irreducible discrete series representations of $\mathrm{SU}(p,q)$. The algorithm relies crucially on the geometric description of the relevant class of components given in [BaZ, Section 4] (which is of independent interest). The most computationally intensive portion of the algorithm involves a classical branching problem from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{GL}(m,\mathbb{C})$.

The purpose of this note is to describe an algorithm to compute the relevant integrals over *any* component of the Springer fiber for $\mathfrak{sl}(n,\mathbb{C})$. We do this in two steps. First we present an algorithm to compute the multiplicity in the associated cycle of an arbitrary irreducible Harish-Chandra module for $SU(p,q)$ with regular integral infinitesimal character in the block of a finite-dimensional representation[1]. (The argument does applies with superficial changes to $SL(n,\mathbb{C})$.) This algorithm has been known to a handful of experts for some time, and relies on combining results of many people, most notably Barbasch, Joseph, King, and Vogan. The next step is to use an observation about characteristic cycles for $SU(p,q)$ to translate effectively this calculation into a calculation of the relevant integrals. The main subtlety is nailing down certain rational scale factors precisely.

In contrast to the results of [BaZ, Section 6], the algorithm given here depends on the Kazhdan-Lusztig algorithm for $\mathfrak{sl}(n,\mathbb{C})$ and $\mathrm{SU}(p,q)$, and thus is computationally much more intensive. In particular, I know of no way to recover the simpler algorithm of [BaZ, Section 6] (which, recall, works only for special cases) from the general, more complicated one given here.

We begin in the general setting of a connected reductive group $G_{\mathbb{R}}$ and use standard notation, as in [BaZ] (with one exception: the flag variety for $\mathfrak{g}$ will now be denoted $\mathfrak{B}$, not $X$). We need to define the multiplicity polynomial for an arbitrary irreducible Harish-Chandra module $X$. Fix a fundamental Cartan $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$, write $\eta \in \mathfrak{h}^*$ for a representative of the infinitesimal character of $X$. Assume that $\eta$ is regular and integral. (Some parts of the discussion below require nontrivial modification for nonintegral infinitesimal character.) Choose a system of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$ such that $\eta$ is dominant. Let $\Lambda \subset \widehat{H}_{\mathbb{R}}$ denote the set of weights of finite-dimensional representation of $G_{\mathbb{R}}$ (e.g. [V2, Section 0.4]). Since

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[1] If $p \neq q$, there is a unique block of representations with regular infinitesimal character, and so the the hypothesis of being contained in the block of a finite-dimensional representation is empty. If $p = q$, however, there is another such block (as can already be seen for $SU(1,1)$). This block does not exist for $U(p,p)$, and the extra hypothesis about the block of a finite-dimensional is also empty in this setting.

$H$ is fundamental, it is connected, and hence we may naturally view $\Lambda \subset \mathfrak{h}^*$. Let $\Phi$ denote a coherent family for $G_{\mathbb{R}}$ such that $\Phi(\eta) = X$ as in [V2, Lemma 7.2.6] and [V2, Corollary 7.3.23], for instance. Write $X(\lambda) = \Phi(\lambda)$, $\lambda \in \eta + \Lambda$. Thus $X(\lambda)$ is an irreducible Harish-Chandra module if $\lambda$ is dominant and regular.

It follows easily from the definitions that $\mathrm{AV}(X) = \mathrm{AV}(X(\lambda))$ for any dominant regular element $\lambda \in \eta + \Lambda$ (e.g. [BoBr1, Lemma 4.1]). Fix an irreducible component of $\mathrm{AV}(X)$ and consider the function that assigns to each dominant $\lambda$ the multiplicity, say $p_X(\lambda)$, of this component in the associated cycle of $X$. Then $p_X$ extends to a harmonic polynomial on $\mathfrak{h}^*$ (by the general criterion of [V1, Lemma 4.3], for instance). Although $p_X$ depends on a choice of an irreducible component of $\mathrm{AV}(X)$, we suppress this choice from the notation.

Let $q'_{\mathrm{Ann}(X)} \in S(\mathfrak{h}^*)$ denote the Goldie rank polynomial of the annihilator of $X$ [J1]. The arguments in [Ch, Section 1] (for instance) prove that $p_X = c'_X q'_{\mathrm{Ann}(X)}$ for some constant $c'_X$. Meanwhile [J2, Theorem 5.1] defines a polynomial $q_{\mathrm{Ann}(X)}$ (which is explicitly computable using the Kazhdan-Lusztig algorithm for $\mathfrak{g}$ at infinitesimal character $\eta$) so that $q_{\mathrm{Ann}(X)}$ is proportional to $q'_{\mathrm{Ann}(X)}$. Write $p_X = c_X q_{\mathrm{Ann}(X)}$. The scale factor $c_X$ is rational, and there is no known algorithm to compute it, except in favorable instances.

We next recall (e.g. [BV2]) the definition of cells of Harish-Chandra modules. Suppose $X'$ and $X''$ are irreducible Harish-Chandra modules with the same infinitesimal character. Write $X' > X''$ if $X''$ is a subquotient of $X' \otimes F$ where $F$ is a finite-dimensional representation appearing in the tensor algebra of $\mathfrak{g}$. Write $X' \sim X''$ if $X' > X''$ and $X'' > X$. Then $\sim$ is an equivalence relation and its equivalence classes are called cells.

Let $\mathcal{C}$ denote the cell containing our fixed Harish-Chandra module $X$. The elements of $\mathcal{C}$ index a basis of a subquotient of the full coherent continuation representation of the Weyl group $W = W(\mathfrak{h}, \mathfrak{g})$. We write $\mathrm{Coh}(\mathcal{C})$ for this subquotient, and $[Y] \in \mathrm{Coh}(\mathcal{C})$ for the basis element indexed by $Y \in \mathcal{C}$. Meanwhile, we can consider the span, say $\mathrm{GR}(\mathcal{C})$ of the various Goldie rank polynomials $q_{\mathrm{Ann}(Y)}$ for $Y \in \mathcal{C}$. Then $\mathrm{GR}(\mathcal{C})$, with the natural action extending the $W$ action $\mathfrak{h}^*$, is an irreducible (special) representation of $W$ [J2].

If $Y \in \mathcal{C}$, then $\mathrm{AV}(Y) = \mathrm{AV}(X)$ (once again by [BoBr1, Lemma 4.1], for instance). Recall that we have fixed an irreducible component of $\mathrm{AV}(X)$. So we can consider the corresponding multiplicity polynomial $p_Y$ for $Y$.

**Theorem A.1.** *Retain the setting above for a connected reductive real group $G_{\mathbb{R}}$. The map*

$$\mathrm{Coh}(\mathcal{C}) \longrightarrow \mathrm{GR}(\mathcal{C})$$
$$\sum_{Y \in \mathcal{C}} n_Y [Y] \longrightarrow \sum_{Y \in \mathcal{C}} n_Y p_Y$$

*is a $W$-equivariant surjection.*

**Sketch.** The only account that appears in print is roundabout: the statement of the theorem is the main result of [Ki] combined with [BV1] and the Barbach-Vogan conjecture [SV]. (A direct proof can perhaps be deduced from the equivariance results of [KT] and [Ta], together with the interpretation of multiplicities given in [Ch, Section 2].)

□

Since the representation $\mathrm{Coh}(\mathcal{C})$ is explicitly computable using the Kazhdan-Lusztig-Vogan algorithm for $G_{\mathbb{R}}$, and since (as we remarked above) $\mathrm{GR}(\mathcal{C})$ is computable using the Kazhdan-Lusztig algorithm for $\mathfrak{g}$, Theorem A.1 provides explicitly computable restrictions on multiplicity polynomials. To get started, we need to be able to compute some multiplicities independently. Here is a special case where such a computation is easy (and well-known).

**Proposition A.2.** *Let $G_{\mathbb{R}}$ be an arbitrary reductive group, and let $A_{\mathfrak{q}}$ be a derived functor module of the form considered in* [VZu] *for a $\theta$-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Then*

$$\mathrm{AV}(A_{\mathfrak{q}}) = K \cdot (\mathfrak{u} \cap \mathfrak{p}),$$

*and hence is the closure of a single nilpotent $K$ orbit $\mathcal{O}_K$ on $\mathfrak{p}$. (Here we are using the convention of* [BaZ, Section 6] *where associated varieties are subvarieties of the nilpotent cone, rather than its dual.) If we further assume that*

$$G \cdot \mathrm{AV}(A_{\mathfrak{q}}) = G \cdot \mathfrak{u}, \tag{A.1}$$

*then the multiplicity of $\overline{\mathcal{O}_K}$ in the associated cycle of $A_{\mathfrak{q}}(\lambda)$ is exactly one.*

**Sketch.** Temporarily set $X = A_{\mathfrak{q}}$. Let $\mathcal{D}$ denote the sheaf of algebraic differential operators on $\mathfrak{B}$, and $\mathcal{X} = \mathcal{D} \otimes_{\mathrm{U}(\mathfrak{g})} X$. Let $Z = \mathrm{supp}(X)$ denote the dense $K$ orbit in the support of $\mathcal{X}$. Consider the characteristic cycle of $\mathcal{X}$ (e.g. [BoBr3, Section 2] which states results in the setting of complex groups, but whose proofs carry over without change to the real case). The closure of $T^*_Z\mathfrak{B}$ always appears in the characteristic cycle of $X$ with multiplicity one (e.g. [BoBr3, Proposition 2.8(a)]). Since $X$ is a derived functor module, its characteristic variety is irreducible are no other components besides $\overline{T^*_Z\mathfrak{B}}$. Unwinding the definitions shows

$$\mu\left(\overline{T^*_Z\mathfrak{B}}\right) = K \cdot (\mathfrak{u} \cap \mathfrak{p}).$$

Since the moment map image of the characteristic variety of $X$ is the associated variety of $X$ (e.g. [BoBr3, Theorem 1.9(c)]), $\mathrm{AV}(X) = K \cdot (\mathfrak{u} \cap \mathfrak{p})$, as claimed. Meanwhile if $f$ denotes a generic point of the moment map image, as in [BaZ, Section 2], then (A.1) implies that the intersection of the $\mu^{-1}(f)$ with $\overline{T^*_Z\mathfrak{B}}$ identifies with the flag variety for $\mathfrak{l}$. Given the characteristic cycle computation, the results of [Ch, Section 2] (recalled in more detail in (A.2) below) show that the multiplicity in the associated cycle is the dimension of the space of holomorphic functions on the flag variety for $\mathfrak{l}$. Hence it is one. □

Next we recall the relationship between integrals over the Springer fiber and multiplicities in associated cycles. Let $e^\lambda$ denote the exponential of the first Chern class of the homogeneous line bundle on $\mathfrak{B}$ parametrized by $\lambda \in \mathfrak{h}^*$ (and our fixed choice of positive roots). Let $C$ be an irreducible component of the Springer fiber. The discussion around [SV, Equation 5.6], for instance, carefully explains how to define the integral $\int_C e^\lambda$ of $e^\lambda$ over $C$.

Now suppose $X$ is an irreducible Harish-Chandra module with regular integral infinitesimal character. Write the characteristic cycle of its localization, e.g. [BoBr3, Section 2], as

$$\sum_j m_j [\overline{T^*_{Z_j}\mathfrak{B}}].$$

Recall the fixed component $\overline{\mathcal{O}_K}$ of $\mathrm{AV}(X)$, and choose $f \in \mathcal{O}_K$. Let $S = S(X, \mathcal{O}_K)$ denote the subset of indices $j$ such that

$$\mu\left(\overline{T^*_{Z_j}\mathfrak{B}}\right) = \overline{\mathcal{O}_K}.$$

Then [Ch, Proposition 2.5.6] shows that

$$p_X(\lambda) = \sum_{j \in S} \left( m_j \int_{C_j} e^\lambda \right), \tag{A.2}$$

where
$$C_j = \overline{T^*_{Z_j}\mathfrak{B}} \cap \mu^{-1}(f);$$
see also the exposition around [SV, Equation 7.23].

We specialize to the setting of $SU(p,q)$ and trivial infinitesimal character $\eta = \rho$. By [BV2], each cell representation $\mathrm{Coh}(\mathcal{C})$ is irreducible. (Such cells are reducible for general groups.) Hence the map in Theorem A.1 is an isomorphism, and the scale factors $c_Y$, $Y \in \mathcal{C}$, are determined by any one of them. Thus we are reduced to computing the associated cycle of one representation in each cell at trivial infinitesimal character. But [BV2] shows that each cell $\mathcal{C}$ of representations in the block of the trivial representation contains a derived functor module of the form $A_{\mathfrak{q}}$ satisfying the condition (A.1), and thus Proposition A.2 computes its associated cycle. This specifies all scale factors for representations in the block of a finite-dimensional representation, and implies the existence of an effective algorithm to compute associated cycles of such irreducible Harish-Chandra modules for $SU(p,q)$[2]. Note, in particular, that associated varieties of such modules are irreducible.

(If one considers $G_{\mathbb{R}} = \mathrm{SL}(n,\mathbb{C})$ and left cells $\mathcal{C}$, the results of the previous paragraph carry over with only superficial modifications. The relevant cell calculations in this context are due to Joseph.)

To conclude, we also give an effective means to compute $\int_C e^{\lambda}$ for any component of the Springer fiber for $\mathfrak{sl}(n,\mathbb{C})$. This relies on a key geometric fact for $SU(p,q)$. (Again, the results of this paragraph carry over with superficial modifications for $SL(n,\mathbb{C})$.) Let $X$ be an irreducible Harish-Chandra module with infinitesimal character $\lambda$ in the block of a finite-dimensional representation. As we remarked above, $\mathrm{AV}(X)$ is irreducible, so write $\mathrm{AV}(X) = \overline{\mathcal{O}_K}$ and fix $f \in \mathcal{O}_K$. Write the characteristic variety of its appropriate localization $\mathcal{X}$ as
$$\overline{T^*_{Z_1}\mathfrak{B}} \cup \cdots \cup \overline{T^*_{Z_k}\mathfrak{B}}$$
for $K$ orbits $Z_i$ on $\mathfrak{B}$. There may be multiple terms here. But we claim that the set $S = S(X, \mathcal{O}_K)$ entering (A.2) consists of a single element in our setting. (This certainly fails in general.) First we locate one element of $S$, and then indicate that there can be no others. Let $\mathrm{supp}(X)$ denote the dense $K$ orbit in the support of $\mathcal{X}$. As in the proof of Proposition A.2, the closure of $T^*_{\mathrm{supp}(X)}\mathfrak{B}$ always appears as an irreducible component of the characteristic variety of $X$; moreover it appears with multiplicity one in the characteristic cycle (e.g. [BoBr3, Proposition 2.8(a)]). In [Tr1, Theorem 5.6(a)], it is proved that
$$\mu\left(\overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}\right) = \mathrm{AV}(X);$$
so indeed $T^*_{\mathrm{supp}(X)}\mathfrak{B}$ belongs to $S$. Set
$$C(X) = \overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}} \cap \mu^{-1}(f).$$
We remark that the map $X \mapsto (\mathrm{AV}(X), C(X))$ is explicitly computed in [Tr1, Theorem 5.6(a)]; in particular, each $C(X)$ is a single irreducible component of the Springer fiber $\mu^{-1}(f)$, and every such component arises in this way for some $X$. We now argue that $S$ can contain no other elements besides the conormal bundle to $\mathrm{supp}(X)$. This can be deduced from the characteristic cycle computation for derived functor modules recalled in the proof of Proposition A.2, the fact that each cell contains such a derived functor module, and the

[2]I do not know how to compute the scale factors for the other block of $SU(p,p)$

equivariance results of [KT] and [Ta]. (Alternatively, the introduction of [Tr2] explains how the assertion is equivalent to the main result of [Me].) We conclude that (A.2) reduces to

$$p_X(\lambda) = 1 \cdot \int_{C(X)} e^{\lambda}. \tag{A.3}$$

Since $p_X$ is know by the algorithm given above, since $X \mapsto C(X)$ is explicitly computable, and since every component of the Springer fiber for $\mathfrak{sl}(n,\mathbb{C})$ arises as some $C(X)$, (A.3) gives an algorithm to compute the integral over any such component.

## References


[BV1] D. Barbasch and D. A. Vogan, Jr., The local structure of characters, *J. of Func. Anal.*, **34**(1), 1980, 27–55.

[BV2] D. Barbasch and D. A. Vogan, Jr., Weyl Group Representations and Nilpotent Orbits, *Proceeding of the Park City Conference on Harmonic Analysis*, P. Trombi ed., Progress in Math, Birkhäuser (Boston), 1982, pp 21–33

[BaZ] L. Barchini and R. Zierau, Certain components of the Springer fiber and associated cycles for discrete series representations of $\mathrm{SU}(p,q)$, *Representation Theory*, **12** (2008), 403–434.

[BoBr1] W. Borho, J.-L. Brylinski, Differential operators on homogeneous spaces I. Irreducibility of the associated variety for annihilators of induced modules, *Invent. Math,* **69** (1982), no. 3, 437–476.

[BoBr3] W. Borho, J.-L. Brylinski, Differential operators on homogeneous spaces III. Characteristic varieties of Harish-Chandra modules and of primitive ideals, *Invent. Math,* **80** (1985), no. 1, 1–68.

[Ch] J.-T. Chang, Asymptotics and characteristic cycles for representations of complex groups. *Compositio Math.*, **88** (1993), no. 3, 265–283.

[J1] A. Joseph, Goldie rank in the enveloping algebra of a semisimple Lie algebra. I, *J. Algebra*, **65** (1980), no. 2, 269–283.

[J2] A. Joseph, Goldie rank in the enveloping algebra of a semisimple Lie algebra. II, *J. Algebra*, **65** (1980), no. 2, 284–306.

[J3] A. Joseph, On the characteristic polynomials of orbital varieties, Ann. Sci. École Norm. Sup. (4), **22** (1989), no. 4, 569–603.

[KT] M. Kashiwara, T. Tanisaki, The characteristic cycles of holonomic systems on a flag manifold related to the Weyl group algebra, *Invent. Math.*, **77** (1984), no. 1, 185–198.

[Ki] D. R. King, The character polynomial of the annihilator of an irreducible Harish-Chandra module, *Amer. J. Math.*, **103** (1980), 1195–1240.

[Me] A. Melnikov, Irreducibility of the associated varieties of simple highest weight modules in $\mathfrak{sl}(n)$, *C. R. Acad. Sci. Paris Sr. I, Math*, **316** (1993), no. 1, 53–57.

[SV] W. Schmid and K. Vilonen, Characteristic cycles and wave front cycles of representations of reductive groups, *Annals of Math.*, **151** (2000), 1071-1118.

[Ta] T. Tanisaki, Holonomic systems on a flag variety associated to Harish-Chandra modules and representations of a Weyl group, in *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, Adv. Stud. Pure Math., **6** (1985), 139–154.

[Tr1] P. E. Trapa, Generalized Robinson-Schensted algorithms for real Lie groups *Intern. Math. Res. Not.*, **1999**, no. 15, 803–834.

[Tr2] P. E. Trapa, Leading terms cycles of Harish-Chandra modules and partial orders on irreducible components of the Springer fiber, *Compositio Math.*, **143** (2007), 515–540.

[V1] D. A. Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, *Inventiones Math.*, **48** (1978), 75–98.

[V2] D. A. Vogan, Jr., *Representations of Real Reductive Lie Groups*, Progress in Mathematics, **15** (1981), Birkhäuser (Boston).

[VZu] D. A. Vogan, G. Zuckerman, Unitary representations with nonzero cohomology, *Compositio Math.*, **53** (1984), no. 1, 51–90.


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