MATH 1060 Sample Final Exam

The following three formulas may be helpful in solving problems from this exam.

$$\vec{v}\cdot\vec{w} = ||\vec{v}||\cdot||\vec{w}||\ \cos\theta$$
$$\text{proj}_{\vec{w}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{||\vec{w}||^2}\right)\cdot\vec{w}$$
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Sum and Difference Identities

- $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$
- $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$
- $\cos(u+v) = \cos(u)\cos(v) - \sin u \sin(v)$
- $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$
- $\tan(u+v) = \dfrac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}$
- $\tan(u-v) = \dfrac{\tan(u)-\tan(v)}{1+\tan(u)\tan(v)}$

Double Angle Identities

- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\cos(2\theta) = 1 - 2\sin^2\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Half-Angle Identities

- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1+\cos\theta}$

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1. Evaluate each of the following.

a.) $\sin\left(\frac{9\pi}{4}\right)$

b.) $\cos\left(-\frac{7\pi}{6}\right)$

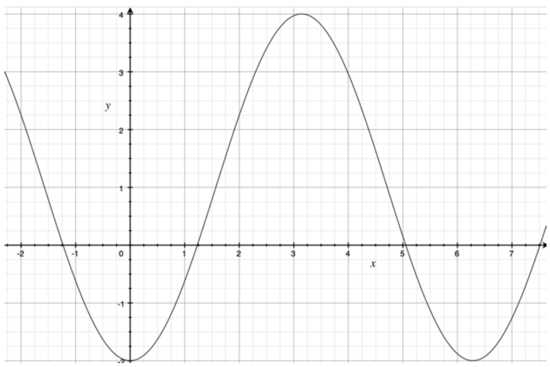
c.) $\tan(-330°)$

d.) $\sec\left(\frac{25\pi}{6}\right)$

2. Let $\sin u = \frac{3}{5}$ with $0 < u < \pi/2$ and $\cos v = 12/13$ with $\frac{3\pi}{2} < v < 2\pi$.
a.) Find $\sin(2u)$.
b.) Find $\cos(u - v)$.

3. Given *f*(x) = -3 cos(2x + π) + 1, find the following:
a.) Amplitude.
b.) Period.
c.) Phase Shift.
d.) Vertical Shift.
e.) Choose the correct graph of *f*(x) from the four options below.

A



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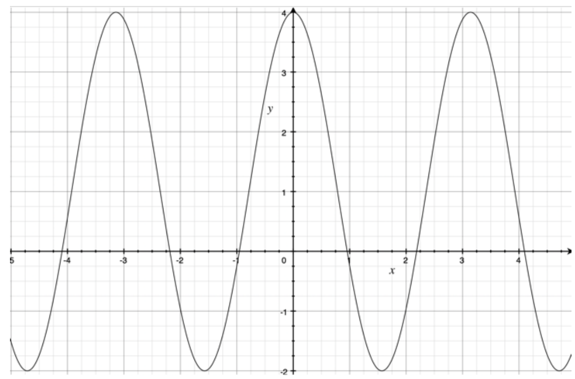

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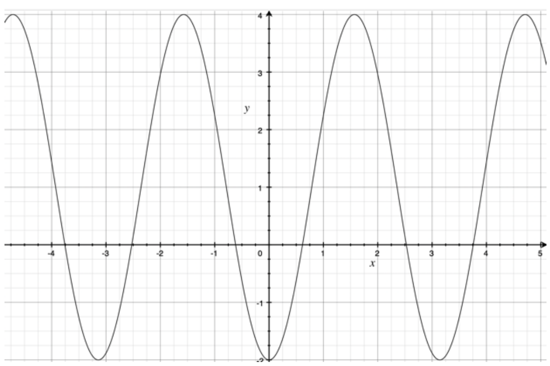

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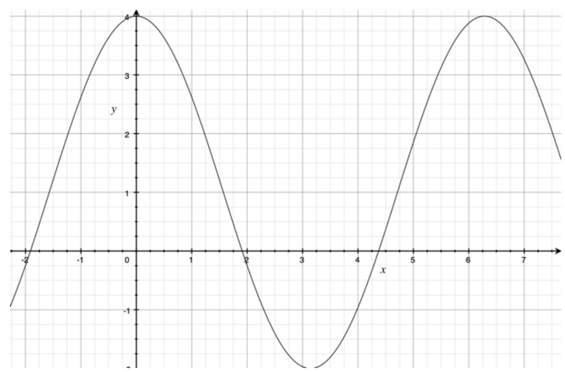


4. A triangle has one of its angles equal to 120°. The sides adjacent to that angle have length 5 and 3.

a.) Find the length of the third side.

b.) Find the angle, γ, that is opposite from the side of length 3. (Your answer should include an unevaluated inverse trig function.)

5. Given $\boldsymbol{u} = \langle 3,6 \rangle$ and $\boldsymbol{v} = \langle 5,2 \rangle$.

a.) Calculate $\boldsymbol{u} \cdot \boldsymbol{v}$.

b.) Find a unit vector in the direction of $\boldsymbol{u}$.

c.) Find the projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$.

6. Find the complex third roots of -8. State the answers in rectangular form of complex numbers.

7. If $\sin\theta = \frac{1}{5}$ and $\frac{\pi}{2} < \theta < \pi$,

a.) Sketch θ in standard position.

b.) Find $\cos\theta$.

c.) Find $\tan\theta$.

d.) Find $\sec\theta$.

e.) Find $\csc\theta$.

f.) Find $\cot\theta$.

8.) Rewrite $\sin(\tan^{-1}(x))$ as an algebraic expression of x.

9. Evaluate the following.

a.) $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$

b.) $\tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

10.) Convert $\left(-\sqrt{2},-\sqrt{2}\right)$ to a polar coordinate (r,θ) with $r < 0$ and $0 < \theta < \pi$.

11.) Given $z = 9i$ and $w = -2\sqrt{3} + 2i$ calculate $\frac{z}{w}$. Your answer must be given in the form $r \cdot cis(\theta)$.

12.) a.) Find all the solutions of the equation $\sin^2(\theta) = \frac{1}{2}$.

b.) Find all the solutions of the equation $\tan\left(x + \frac{\pi}{6}\right) = \sqrt{3}$.

Solutions to MATH 1060 Sample Final Exam

1.) a.) $\sin \frac{9\pi}{4} = \sin\left(\pi/4 + 2\pi\right) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

(or $\frac{\sqrt{2}}{2}$).

b.) $\cos\left(-\frac{7\pi}{6}\right) = \cos\left(-\frac{7\pi}{6} + 2\pi\right) = \cos\left(5\pi/6\right) = -\frac{\sqrt{3}}{2}$.

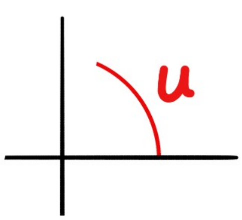
c.) $\tan(-330^\circ) = \tan(30^\circ) = \tan(\pi/6)$

$$= \frac{\sin \pi/6}{\cos \pi/6} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

d.) $\sec \frac{25\pi}{6} = \sec\left(\frac{25\pi}{6} - 4\pi\right) = \sec \frac{\pi}{6}$

$$= \frac{1}{\cos \pi/6} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

2.) a.) $1 = \sin^2 u + \cos^2 u = \frac{9}{25} + \cos^2 u$.



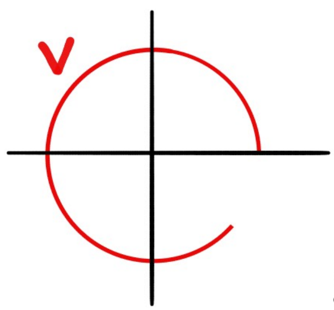
so $\cos u > 0$.

thus, $\cos u = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

By double-angle formula,

$\sin(2u) = 2 \sin u \cos u = 2 \, \frac{3}{5} \, \frac{4}{5} = \frac{24}{25}$.

b.) $1 = \sin^2 v + \cos^2 v = \sin^2 v + \frac{12^2}{13^2}$.



so $\sin v < 0$. Thus,

$$\sin v = -\sqrt{1 - \frac{144}{169}} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}$$

By Difference Identity,

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$= \frac{4}{5}\,\frac{12}{13} + \frac{3}{5}\left(-\frac{5}{13}\right) = \frac{48-15}{65} = \frac{33}{65}.$$

3.) a.) amplitude of $-3\cos(2x+\pi)+1$ is $|-3| = 3$.

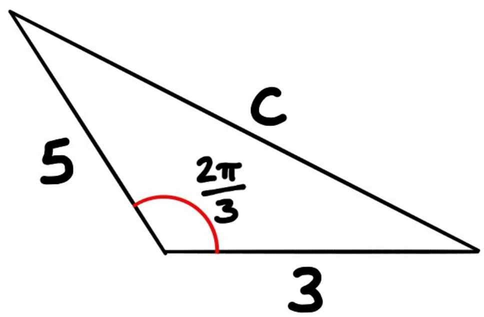
b.) period is $\frac{2\pi}{2} = \pi$.

c.) phase shift is $\frac{\pi}{2}$ (to the left, or $-\pi/2$ to the right).

d.) vertical shift is 1.

e.) B is the correct graph.

4.) a.) $120°$ is $\frac{2\pi}{3}$ radians, so we have



By the law of cosines,

$$\begin{aligned} c &= \sqrt{3^2+5^2-2\cdot 3\cdot 5\cos\left(\tfrac{2\pi}{3}\right)} \\ &= \sqrt{9+25-30\left(-\tfrac{1}{2}\right)} \\ &= \sqrt{49} = 7 \end{aligned}$$

b.) By the law of sines,

$$\frac{\sin\left(\frac{2\pi}{3}\right)}{7} = \frac{\sin\gamma}{3}$$

so $\sin\gamma = \frac{3}{7}\sin\left(\frac{2\pi}{3}\right) = \frac{3}{7}\,\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{14}$.

Hence, $\gamma = \sin^{-1}\left(\frac{3\sqrt{3}}{14}\right)$.

5.) a.) $u \cdot v = \langle 3,6 \rangle \cdot \langle 5,2 \rangle = 3 \cdot 5 + 6 \cdot 2 = 27$

b.) $\|u\| = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$,

$$\text{so } \frac{u}{\|u\|} = \frac{\langle 3,6 \rangle}{3\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

c.) $$\text{proj}_u v = \frac{u \cdot v}{\|u\|^2} u = \frac{27}{(3\sqrt{5})^2} \langle 3,6 \rangle$$

$$= \frac{3}{5} \langle 3,6 \rangle$$

$$= \left\langle \frac{9}{5}, \frac{18}{5} \right\rangle$$

6.) Third roots of $-8 = 8 \operatorname{cis} \pi$ are

$$\sqrt[3]{8} \operatorname{cis}\left(\pi/3\right) = 2\left(\cos \pi/3 + i \sin \pi/3\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

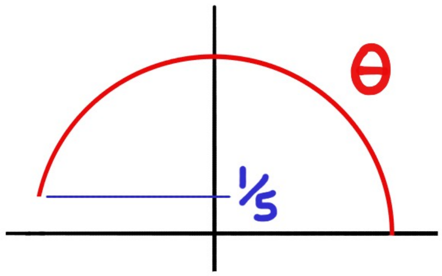
$$= 1 + i\sqrt{3}$$

$$\sqrt[3]{8} \operatorname{cis}\left(2\pi/3 + \pi/3\right) = 2 \operatorname{cis} \pi = 2(-1 + i0) = -2$$

$$\sqrt[3]{8} \operatorname{cis}\left(4\pi/3 + \pi/3\right) = 2 \operatorname{cis} \frac{5\pi}{3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 1 - i\sqrt{3}$$

7.) a.)



b.) $1 = \sin^2\theta + \cos^2\theta = \frac{1}{25} + \cos^2\theta$

By a.), $\cos\theta < 0$, so

$$\cos\theta = -\sqrt{1 - \frac{1}{25}} = -\sqrt{\frac{24}{25}} = -\frac{2\sqrt{6}}{5}$$

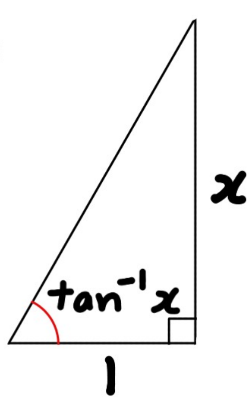
c.) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{5}}{-\frac{2\sqrt{6}}{5}} = -\frac{1}{2\sqrt{6}}$

d.) $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{2\sqrt{6}}{5}} = -\frac{5}{2\sqrt{6}}$

e.) $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{1/5} = 5$

f.) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{-2\sqrt{6}}{5}}{1/5} = -2\sqrt{6}$

8.)

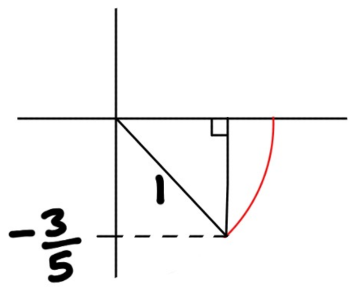


By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{x^2+1}$. So $\sin(\tan^{-1}x) = \frac{\text{opp.}}{\text{hyp.}}$

$$= \frac{x}{\sqrt{x^2+1}}.$$

9.) a.) $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \cos^{-1}(0) = \pi/2$.

b.) $\sin\left(\sin^{-1}\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$



$$1 = \sin^2\left(\sin^{-1}\left(-\frac{3}{5}\right)\right) + \cos^2\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

$$= \left(-\frac{3}{5}\right)^2 + \cos^2\left(\sin^{-1}\frac{3}{5}\right)$$

Notice in the diagram that

$\cos^2\left(\sin^{-1}\left(\frac{3}{5}\right)\right) > 0$, so

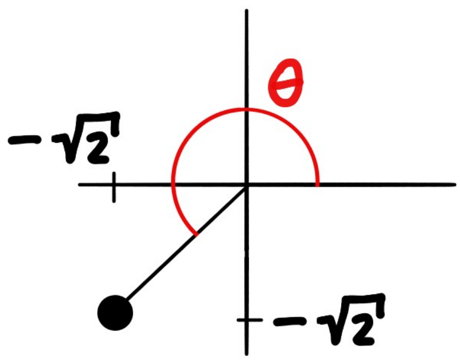
$$\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Thus, $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \dfrac{\sin\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)}{\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)}$

$$= \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}.$$

10.) $r = \|(-\sqrt{2}, -\sqrt{2})\| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2}$

$= \sqrt{4} = 2$



From the diagram,
$\theta = \frac{5\pi}{4}$, so
$(r, \theta) = \left(2, \frac{5\pi}{4}\right)$.

11.) $|w| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{4 \cdot 3 + 4} = 4$.

Thus $\frac{w}{|w|} = \frac{1}{4}\left(-2\sqrt{3} + 2i\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$= \cos(5\pi/6) + i\sin(5\pi/6)$

$= \text{cis}(5\pi/6)$.

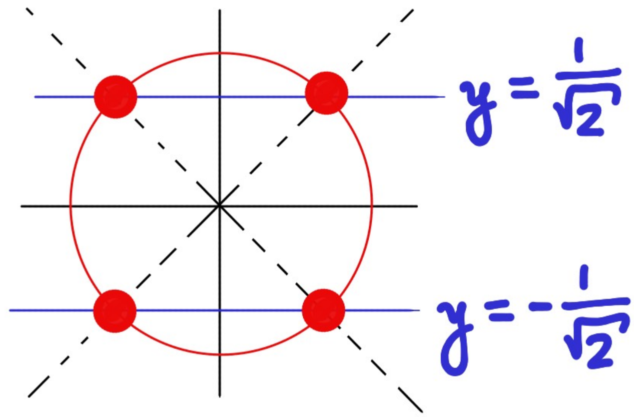
Hence, $w = |w| \frac{w}{|w|} = 4\,\text{cis}(5\pi/6)$.

Note that $z = 9i = 9(0 + i \cdot 1) = 9\,\text{cis}\,\frac{\pi}{2}$.

$$\frac{z}{w} = \frac{9\,\text{cis}\left(\frac{\pi}{2}\right)}{4\,\text{cis}(5\pi/6)} = \frac{9}{4}\,\text{cis}\left(\pi/2 - 5\pi/6\right)$$

$$= \frac{9}{4}\,\text{cis}\left(-\pi/3\right).$$

12.) a.) If $\sin^2\theta = \frac{1}{2}$, then $\sin\theta = \frac{1}{\sqrt{2}}$ or $\sin\theta = -\frac{1}{\sqrt{2}}$.



From the diagram,

$$\theta = \pi/4 + k\frac{\pi}{2} \text{ for some integer } k.$$

b.) If $\tan(x + \pi/6) = \sqrt{3}$, then

$x + \pi/6 = \tan^{-1}(\sqrt{3}) + k\pi$ for some integer k,

since tangent is π-periodic.

Since $\tan(\pi/3) = \frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$,

we have $\tan^{-1}(\sqrt{3}) = \pi/3$, so we have

$x + \pi/6 = \pi/3 + k\pi$ and thus, $x = \pi/6 + k\pi$.