OER Math 1060 – Trigonometry

1st Edition

Created Fall 2017

Salt Lake Community College
&
University of Utah

Acknowledgements

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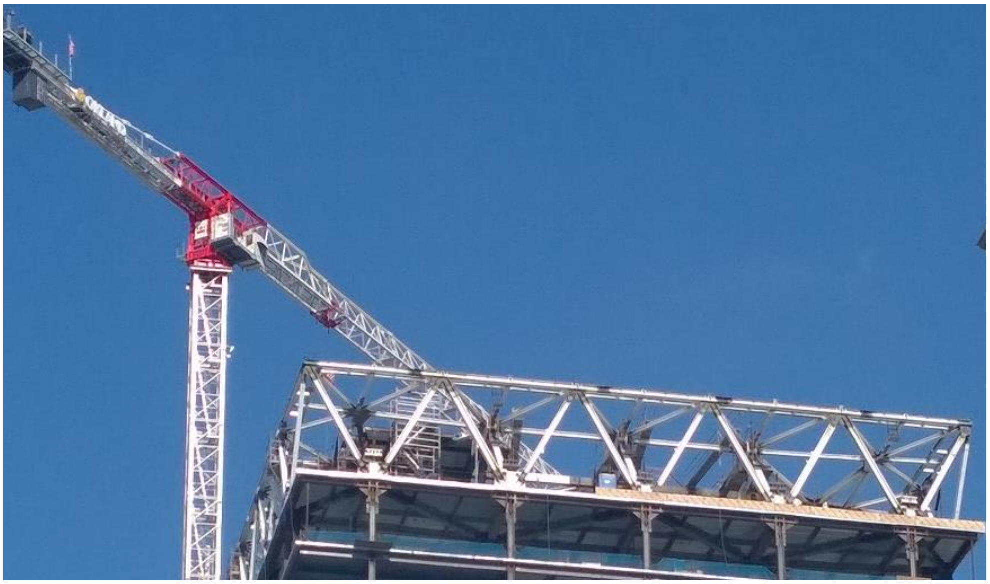
CHAPTER 1
ANGLES AND THEIR MEASURE



Chapter Outline

1.1 Degree Measure of Angles

1.2 Radian Measure of Angles

1.3 Applications of Radian Measure

Introduction

In this chapter we introduce angles in preparation for their critical role in the study of trigonometry. Beginning with the degree measure of angles in Section 1.1, we move on to radian measure in Section 1.2. The inclusion of conversion between degrees and radians leads to Section 1.3, where radian measure proves useful in solving applications of circular arc length and area, as well as linear and angular speed. Throughout this chapter, emphasis remains on angle measure and graphing. In preparation for Chapter 2, angles are graphed in standard position and coterminal angles are identified. The applications at the end of Chapter 1 will be the first of many real-world uses to occur throughout our study of trigonometry.

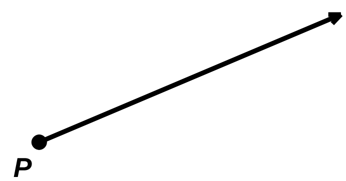
1.1 Degree Measure of Angles

Learning Objectives

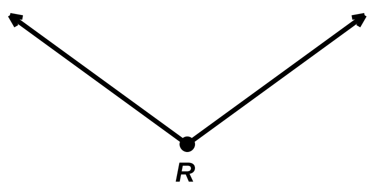
In this section you will:

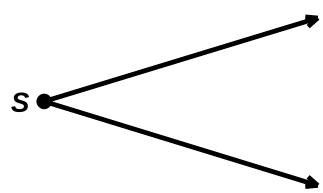
- **Convert between the Degree-Minute-Second system and decimal degrees.**
- **Determine supplementary and complementary angle measures.**
- **Graph angles in standard position.**
- **Determine coterminal angle measures.**

To get started, we recall some basic definitions from geometry. A **ray**, often described as a half-line, is a subset of a line that contains a point *P* along with all points lying to one side of *P*. The point *P* from which the ray originates is called the **initial point** of the ray, as pictured below.

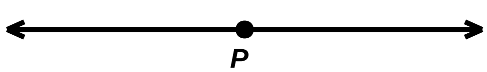


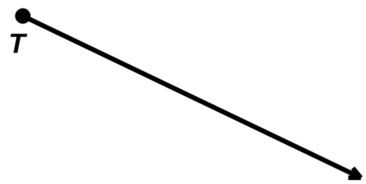
When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Following are two examples of angles, the first with vertex *R* and the second with vertex *S*.



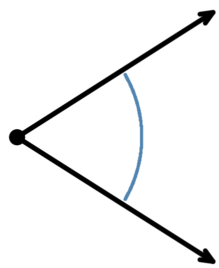


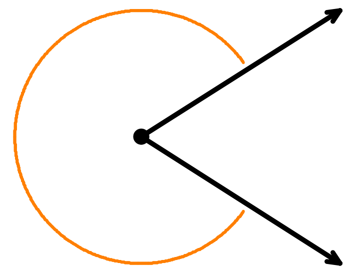
The two figures below also depict angles. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**. In the second, the rays are identical so the angle is indistinguishable from the ray itself.



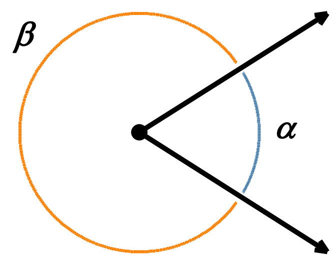


The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as the following pictures indicate.





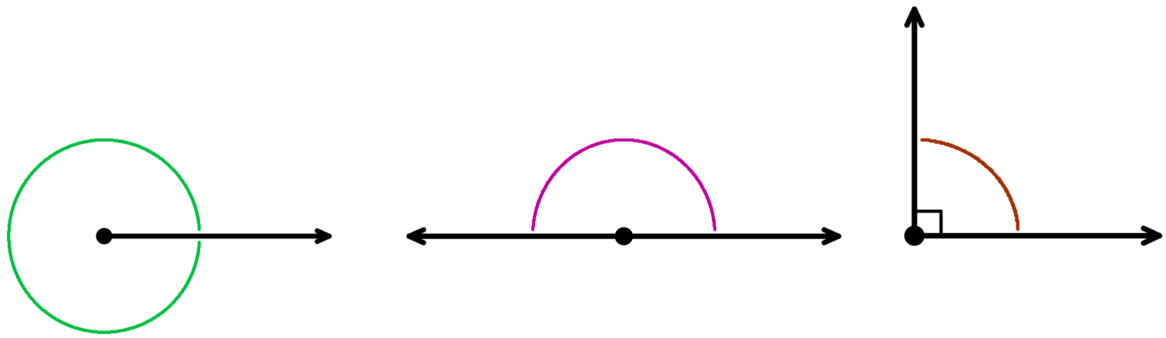
Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.[1] Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have



Degree Measure

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by a small circle displayed as a superscript. One complete revolution is 360°, and parts of a revolution are measured proportionately.[2] Thus half of a revolution (a straight angle) measures $\frac{1}{2}\left(360^\circ\right)=180^\circ$; a quarter of a revolution measures $\frac{1}{4}\left(360^\circ\right)=90^\circ$, and so on.

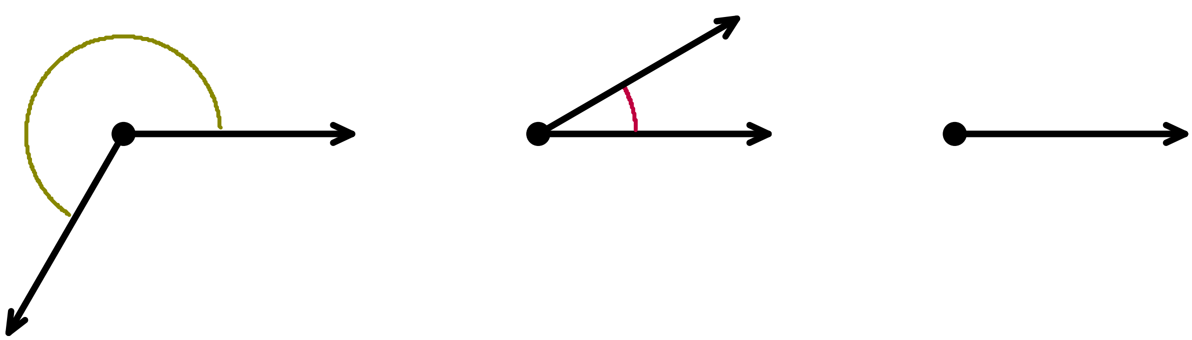
[1] The phrase 'at least' will be justified in short order.
[2] The choice of 360 is most often attributed to the Babylonians.



One revolution ↔ 360° 180° 90°

Note that in the previous figure we have used the small square to denote a right angle, as is commonplace in geometry. If an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measure strictly between 90° and 180° it is called an **obtuse angle**.

We can determine the measure of any angle as long as we know the proportion it represents of an entire revolution.[3] For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}\left(360^\circ\right)=240^\circ$. The measure of an angle which constitutes only $\frac{1}{12}$ of a revolution would measure $\frac{1}{12}\left(360^\circ\right)=30^\circ$. An angle which indicates no rotation at all is measured as 0°.



240° 30° 0°

[3] This is how a protractor is graded.

Using our definition of degree measure, 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than 1°.

There are two ways to subdivide degrees.

1. The first, and most familiar, is **decimal degrees**. For example, an angle with a measure of 30.5° would represent a rotation halfway between 30° and 31° or, equivalently $\frac{30.5}{360}=\frac{61}{720}$ of a full rotation.

2. The second way to divide degrees is the **Degree-Minute-Second** (**DMS**) system, used in surveying, global positioning and other applications requiring measurements of longitude and latitude. In the DMS system, one degree is divided equally into sixty minutes, and in turn each minute is divided equally into sixty seconds. In symbols, we write $1^\circ = 60'$ and $1' = 60''$, from which it follows that $1^\circ = 3600''$.

EXAMPLE 1.1.1. Convert 42.125° to the DMS system.

SOLUTION. To convert 42.125° to the DMS system, we first note that $42.125^\circ = 42^\circ + 0.125^\circ$. Converting the partial amount of degrees to minutes, we find

$$\begin{aligned} 0.125^\circ\left(\frac{60'}{1^\circ}\right) &= 7.5' \\ &= 7' + 0.5'. \end{aligned}$$

Next, converting the partial amount of minutes to seconds gives

$$0.5'\left(\frac{60''}{1'}\right) = 30''.$$

The result is

$$\begin{aligned} 42.125^\circ &= 42^\circ + 0.125^\circ \\ &= 42^\circ + 7.5' \\ &= 42^\circ + 7' + 0.5' \\ &= 42^\circ + 7' + 30'' \\ &= 42^\circ 7' 30''. \end{aligned}$$

□

EXAMPLE 1.1.2. Convert 117°15'45" to decimal degrees.

SOLUTION. To convert 117°15'45" to decimal degrees, we first compute

$$15'\left(\frac{1^\circ}{60'}\right) = \frac{1}{4}^\circ$$
$$= 0.25^\circ$$

and

$$45''\left(\frac{1^\circ}{3600''}\right) = \frac{1}{80}^\circ$$
$$= 0.0125^\circ.$$

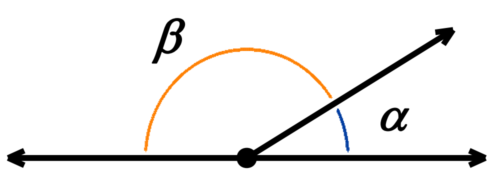
Then we find

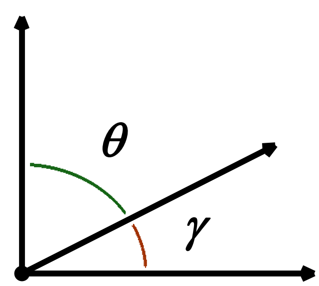
$$\begin{aligned} 117^\circ 15'45'' &= 117^\circ + 15' + 45'' \\ &= 117^\circ + 0.25^\circ + 0.0125^\circ \\ &= 117.2625^\circ \end{aligned}$$

□

Supplementary and Complementary Angles

Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In the diagrams below, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.





In practice, the distinction between the angle itself and its measure is blurred so that the statement 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$'.

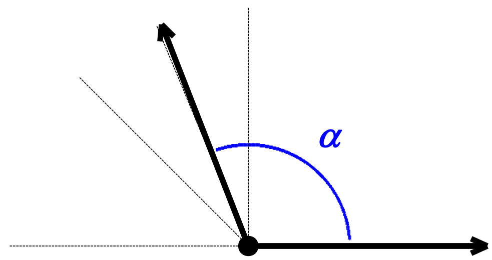
EXAMPLE 1.1.3. Let $\alpha = 111.371^\circ$ and $\beta = 37^\circ 28'17''$.

1. Sketch α and β.

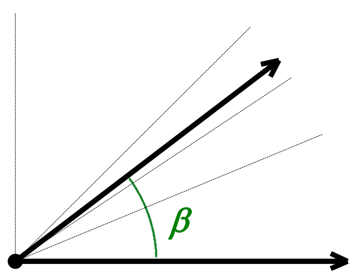
2. Find a supplementary angle for α.
3. Find a complementary angle for β.

SOLUTION.

1. To sketch $\alpha = 111.371^\circ$, we first note that $90^\circ < \alpha < 180^\circ$. If we divide this range in half, we observe that $90^\circ < \alpha < 135^\circ$. After one more division, we get $90^\circ < \alpha < 112.5^\circ$. This gives us a fairly good estimate for α, as shown in the figure.



For $\beta = 37^\circ 28'17''$, converting to decimal degrees results in approximately 37.47°. We find $0^\circ < \beta < 90^\circ$. After dividing this range in half, we get $0^\circ < \beta < 45^\circ$, followed by $22.5^\circ < \beta < 45^\circ$, and lastly $33.75^\circ < \beta < 45^\circ$.



2. To find a supplementary angle for $\alpha = 111.371^\circ$, we seek an angle θ so that $\alpha + \theta = 180^\circ$. We get

$$\begin{aligned}\theta &= 180^\circ - \alpha \\ &= 180^\circ - 111.371^\circ \\ &= 68.629^\circ.\end{aligned}$$

3. To find a complementary angle for $\beta = 37^\circ 28'17''$, we seek an angle γ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ 28'17''$. While we could reach for the calculator to obtain an approximate answer, we choose instead to do a bit of sexagesimal[4] arithmetic. We first rewrite

[4] This is a base-60 system.

$$\begin{aligned} 90^\circ &= 90^\circ 0' 0'' \\ &= 89^\circ 60' 0'' \\ &= 89^\circ 59' 60''. \end{aligned}$$

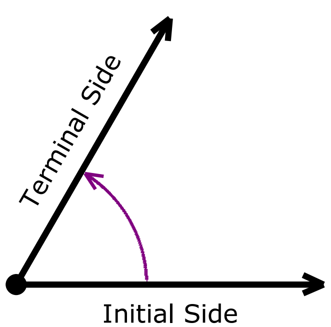
In essence, we are 'borrowing' 1°=60' from the degree place, and then borrowing 1'=60" from the minutes place.[5] This yields

$$\begin{aligned} \gamma &= 90^\circ - 37^\circ 28' 17'' \\ &= 89^\circ 59' 60'' - 37^\circ 28' 17'' \\ &= 52^\circ 31' 43''. \end{aligned}$$

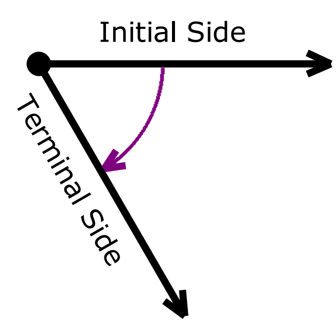
□

Oriented Angles

Up to this point, we have discussed only angles which measure between 0° and 360°, inclusive. Ultimately, we want to extend their applicability to other real-world phenomena. A first step in this direction is to introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say the angle is **negative**.



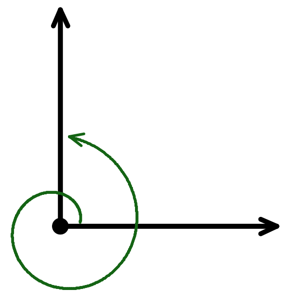
A positive angle, 60°



A negative angle, –60°

We also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the first 360°), then continue with an additional 90° counter-clockwise rotation, as seen below.

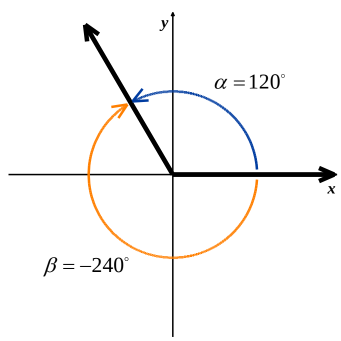
[5] This is the exact kind of borrowing often taught in elementary school when trying to find 300–125. The difference is that a base ten system is used to find 300–125; here, it is base sixty.



Standard Position

To further connect angles with the algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive *x*-axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a Quadrant I angle. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side.[6]

In the following figure, $\alpha = 120^\circ$ and $\beta = -240^\circ$ are two coterminal Quadrant II angles drawn in standard position. Note that $\alpha = \beta + 360^\circ$, or equivalently $\beta = \alpha - 360^\circ$. We leave it as an exercise for the reader to verify that coterminal angles always differ by a multiple of 360°.[7] More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360^\circ \cdot k$ where k is an integer.[8]



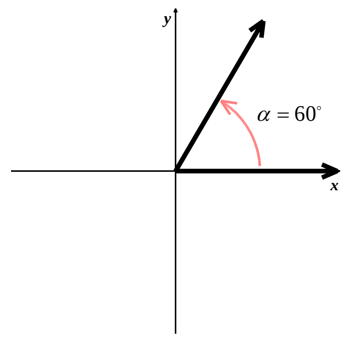
[6] Note that by being in standard position they automatically share the same initial side which is the positive *x*-axis.
[7] It is worth noting that all of the pathologies of Analytic Trigonometry result from this innocuous fact.
[8] Recall that this means $k = 0, \pm 1, \pm 2, \cdots$.

EXAMPLE 1.1.4. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = 60^\circ$ 2. $\beta = -225^\circ$ 3. $\gamma = 540^\circ$ 4. $\varphi = -750^\circ$

SOLUTION.

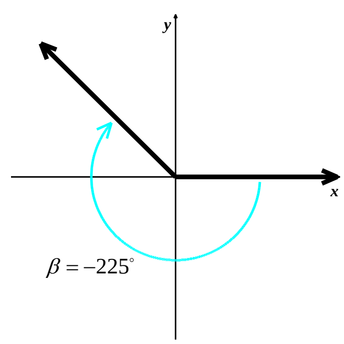
1. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive *x*-axis and rotate counter-clockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle.



To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$ for some integer *k*.

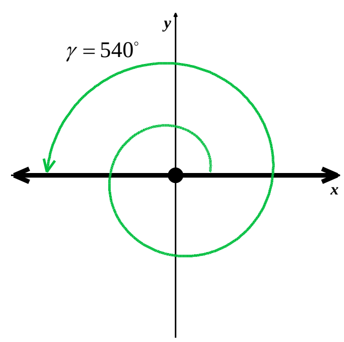
- When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$.
- Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$.
- If we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

2. Since $\beta = -225^\circ$ is negative, we start at the positive *x*-axis and rotate clockwise $\frac{225^\circ}{360^\circ} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle.



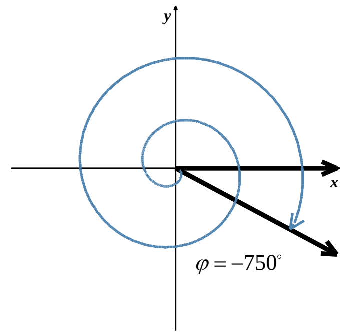
To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of *k*. Letting $k = 1$, $k = -1$ and $k = 2$, we find 135°, –585° and 495° are all coterminal with –225°.

3. Since $\gamma = 540^\circ$ is positive, we rotate counter-clockwise from the positive *x*-axis. One full revolution accounts for 360°, with 180°, or half of a revolution, remaining. Since the terminal side of γ lies on the negative *x*-axis, γ is a quadrantal angle.



All angles coterminal with γ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where *k* is an integer. Working through the arithmetic, we find three such angles: 900°, 180° and –180°.

4. The Greek letter φ is pronounced 'fee' or 'fie'[9] and since $\varphi = -750^{\circ}$ is negative, we begin our rotation clockwise from the positive x-axis. Two full rotations account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go.



We find that φ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^{\circ} + 360^{\circ} \cdot k$ for a few integers k and obtain –390°, –30° and 330°.

□

Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in **Example 1.1.4** to see this.

[9] The symbol φ represents the small Greek letter phi. We will occasionally use the symbol ϕ to represent the uppercase Greek letter phi.

1.1 Exercises

In Exercises 1 – 4, convert the angles into the DMS system. Round each of your answers to the nearest second.

1. 63.75°
2. 200.325°
3. -317.06°
4. 179.999°

In Exercises 5 – 8, convert the angles into decimal degrees. Round each of your answers to three decimal places.

5. $125^\circ 50'$
6. $-32^\circ 10' 12''$
7. $502^\circ 35'$
8. $237^\circ 58' 43''$

In Exercises 9 – 12, sketch the angles using the technique presented in **Example 1.1.3**.

9. 100.491°
10. 39.273°
11. $172^\circ 5' 3''$
12. $82^\circ 9' 27''$

In Exercises 13 – 16, find a supplementary angle for each given angle.

13. 100.491°
14. 39.273°
15. $172^\circ 5' 3''$
16. $82^\circ 9' 27''$

In Exercises 17 – 20, find a complementary angle for each given angle.

17. 10.491°
18. 39.273°
19. $27^\circ 5' 3''$
20. $82^\circ 9' 27''$

In Exercises 21 – 28, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

21. 330°
22. -135°
23. 120°
24. 405°
25. -270°
26. 300°
27. -150°
28. 135°

1.2 Radian Measure of Angles

Learning Objectives

In this section you will:

- **Graph angles in standard position.**
- **Determine coterminal angle measures.**
- **Convert between degree and radian measures.**

The number π is a mathematical constant. That is, it doesn't matter which circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle.

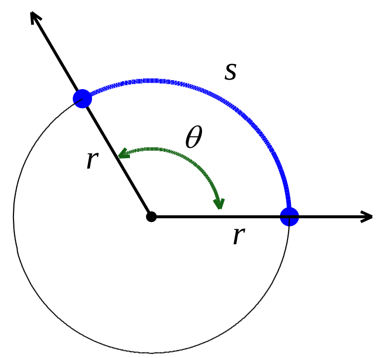
The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

$$\pi = \frac{C}{d}$$

Since the diameter of a circle is twice its radius, we can quickly rearrange the equation $\pi = \frac{C}{d}$ to get a formula more useful for our purposes, namely $2\pi = \frac{C}{r}$. This tells us that for any circle, the ratio of its circumference to its radius is also always constant and that the constant is 2π.

Radian Measure

Suppose now we take a portion of the circle, so instead of comparing the entire circumference C to the radius r, we compare some arc measuring s units in length to the radius r, as depicted below. Let θ be the **central angle** subtended by this arc; that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality arguments, it stands to reason that the ratio $\frac{s}{r}$ should also be a constant among all circles with the same central angle θ. This ratio, $\frac{s}{r}$, defines the **radian measure** of an angle.

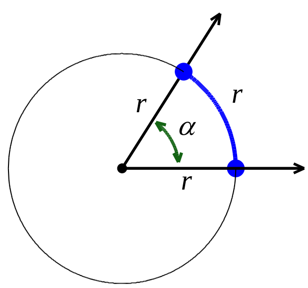


The radian measure of an angle is $\theta = \frac{s}{r}$.

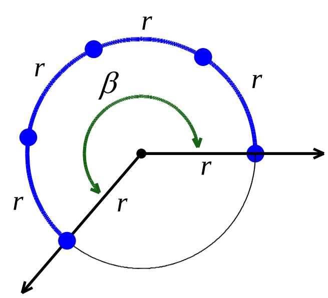
To get a better feel for radian measure, we note that

- An angle with radian measure 1 means the corresponding arc length *s* equals the radius *r* of the circle. Hence $s = r$.
- When the radian measure is 2, we have $s = 2r$.
- When the radian measure is 3, $s = 3r$, and so forth.

Thus, the radian measure of an angle *θ* tells us how many *radius lengths* we need to sweep out along the circle to subtend the angle *θ*.



α has radian measure 1



β has radian measure 4

Since one revolution sweeps out the entire 2*πr*, one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$; a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth.

Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered pure numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word **radians** to denote these dimensionless units as needed. For instance, we say one revolution measures 2π radians; half of a revolution measures π radians, and so forth. As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write $\theta = \frac{\pi}{2}$, we mean 'θ is an angle which measures $\frac{\pi}{2}$ radians'.[10]

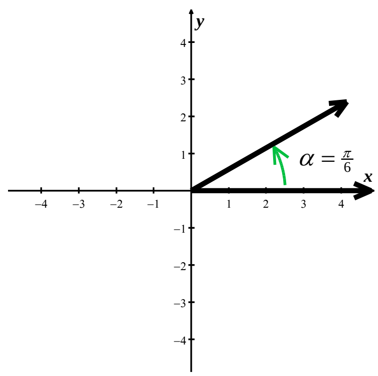
We extend radian measure to oriented angles, just as we did with degrees, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation.[11] Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

Example 1.2.1. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \frac{\pi}{6}$
2. $\beta = -\frac{4\pi}{3}$
3. $\gamma = \frac{9\pi}{4}$
4. $\varphi = -\frac{5\pi}{2}$

Solution.

1. The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle.



Coterminal angles θ are of the form $\theta = \alpha + 2\pi k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$.

[10] We are now identifying radians with real numbers. This will be justified shortly.
[11] This, in turn, endows the subtended arcs with an orientation as well. We address this in short order.

- Thus, when $k = 1$, we get

$$\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}.$$

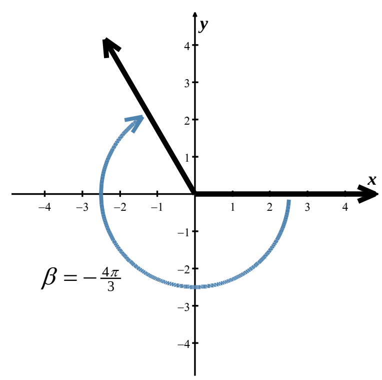
- Substituting $k = -1$ gives

$$\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}.$$

- When we let $k = 2$, we get

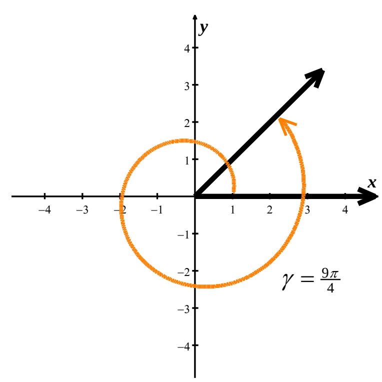
$$\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}.$$

2. Since $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle.



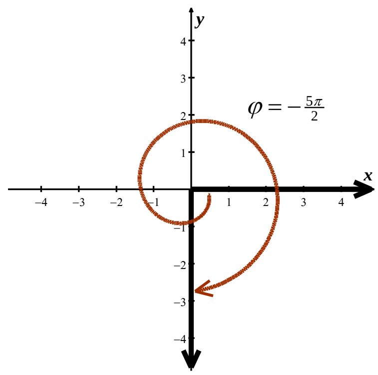
To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3}k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles.

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive *x*-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a Quadrant I angle.



All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4}k$, where k is an integer. Working through the arithmetic, we find coterminal angles of $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$.

4. To graph $\varphi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of φ lies on the negative y-axis, φ is a quadrantal angle.



To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2}k$ for a few integers k and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

□

It is worth mentioning that we could have plotted the angles in **Example 1.2.1** by first converting them to degree measure and following the procedure set forth in **Example 1.1.3**. While converting back and forth between degrees and radians is certainly a good skill to have, it is best that you learn to think in radians as well as you can think in degrees.

Converting Between Degrees and Radians

For converting between degrees and radians, since one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems.

Degree – Radian Conversion:

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$.
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$.

Example 1.2.2. Convert the following measures.

1. 60° to radians
2. $-\dfrac{5\pi}{6}$ radians to degrees
3. 1 radian to degrees

Solution.

1. $$60^\circ = 60^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)$$
$$= \frac{\pi}{3} \text{ radians}$$

2. $$-\frac{5\pi}{6} \text{ radians} = \left(-\frac{5\pi}{6} \text{ radians}\right)\left(\frac{180^\circ}{\pi \text{ radians}}\right)$$
$$= -150^\circ$$ [12]

3. $$1 \text{ radian} = (1 \text{ radian})\left(\frac{180^\circ}{\pi \text{ radians}}\right)$$
$$= \frac{180^\circ}{\pi}$$
$$\approx 57.2958^\circ$$

□

[12] Note that the negative sign indicates clockwise rotation in both systems, and so it is carried along accordingly.

1.2 Exercises

In Exercises 1 – 15, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

1. $\frac{5\pi}{6}$
2. $-\frac{11\pi}{3}$
3. $\frac{5\pi}{4}$
4. $\frac{3\pi}{4}$
5. $-\frac{\pi}{3}$
6. $\frac{7\pi}{2}$
7. $\frac{\pi}{4}$
8. $-\frac{\pi}{2}$
9. $\frac{7\pi}{6}$
10. $-\frac{5\pi}{3}$
11. 3π
12. -2π
13. $-\frac{\pi}{4}$
14. $\frac{15\pi}{4}$
15. $-\frac{13\pi}{6}$

In Exercises 16 – 23, convert the angle from degree measure into radian measure, giving the exact value in terms of π. These problems should be worked without the aid of a calculator.

16. 0°
17. 240°
18. 135°
19. -270°
20. -315°
21. 150°
22. 45°
23. -225°

In Exercises 24 – 31, convert the angle from radian measure into degree measure.

24. π
25. $-\frac{2\pi}{3}$
26. $\frac{7\pi}{6}$
27. $\frac{11\pi}{6}$
28. $\frac{\pi}{3}$
29. $\frac{5\pi}{3}$
30. $-\frac{\pi}{6}$
31. $\frac{\pi}{2}$

1.3 Applications of Radian Measure

Learning Objectives

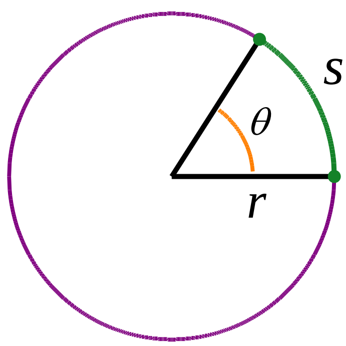
In this section you will:

- **Determine arc length.**
- **Determine area of a sector of a circle.**
- **Solve problems involving linear and angular velocity.**

Now that we have become familiar with radian measure, a whole world of applications awaits us.

Arc Length

We begin by using the radian measure of an angle, $\theta = \frac{s}{r}$, to determine circular **arc length**. Recall that θ is the central angle subtended by the arc of length s in a circle of radius r.



Length of a Circular Arc: If an arc of length s subtends an angle with non-negative **radian** measure θ, in a circle of radius r, then

$$s = r\theta$$

Note that in the above formula, θ is the radian measure of an angle. Since radians correlate with circumference but degrees do not, this arc length formula is only valid for angles in radians. As in the following example, when an angle is given in degree measure, we must first convert to radians.

Example 1.3.1. Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution. To determine the arc length, we must first convert the angle to radians.

$$\begin{aligned}215^\circ &= 215^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)\\ &= \frac{43\pi}{36} \text{ radians}\end{aligned}$$

We next use $r = 10$ units and $\theta = \dfrac{43\pi}{36}$ radians to determine the arc length.

$$\begin{aligned}\text{arc length} &= (10 \text{ units})\left(\frac{43\pi}{36} \text{ radians}\right)\\ &\approx 37.52 \text{ units}\end{aligned}$$

□

Example 1.3.2. Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (1) to determine the radian measure for Mercury's movement in one day.

Solution.

1. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned}C &= 2\pi r\\ &= 2\pi(36 \text{ million miles})\\ &\approx 226 \text{ million miles}\end{aligned}$$

   Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the approximate distance traveled in one day.

$$(0.0114)(226 \text{ million miles}) \approx 2.58 \text{ million miles}$$

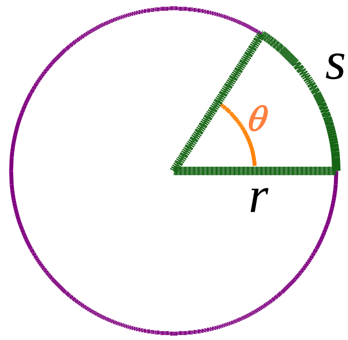
2. We use the arc length, which is the distance traveled, to determine the radian measure θ for Mercury's movement in one day.

$$\begin{aligned}\theta &= \frac{\text{arc length}}{\text{radius}}\\ &\approx \frac{2.58 \text{ million miles}}{36 \text{ million miles}}\\ &\approx 0.0717 \text{ radians}\end{aligned}$$

□

Area of a Sector

We next determine the **area of a sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc.



Consider the ratio of the area of the sector to the area of the circle. This ratio is equivalent to the ratio of the measure of the central angle of the sector to the measure of the central angle of the circle. For sector area A and central angle θ, we have the following.

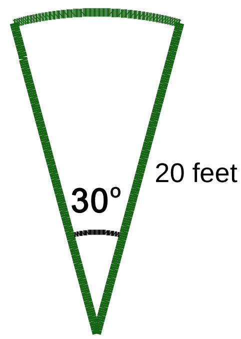
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$
$$A = \frac{\theta}{2\pi}\left(\pi r^2\right)$$
$$A = \frac{\theta r^2}{2}$$

Area of a Sector of a Circle: In a circle with radius r, any sector that is subtended by an angle with **radian** measure θ has area

$$A = \frac{1}{2}\theta r^2 \text{, for } 0 \le \theta \le 2\pi \text{ radians}$$

Be careful! As in the arc length formula, θ must be in radian measure when calculating the area of a sector of a circle.

Example 1.3.3. An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees. What is the area of the sector of grass the sprinkler waters?



Solution. We begin by converting the angle into radians.

$$30^\circ = 30^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)$$
$$= \frac{\pi}{6} \text{ radians}$$

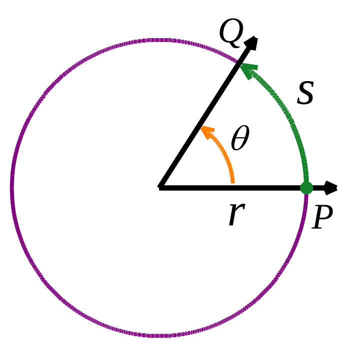
The area of the sector can then be determined using $\theta = \frac{\pi}{6}$ and $r = 20$.

$$\text{Area} = \frac{1}{2}\left(\frac{\pi}{6}\right)(20 \text{ feet})^2$$
$$\approx 104.72 \text{ ft}^2$$

□

Linear and Angular Velocity

We end Chapter 1 with applications involving circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.



Here, s represents a *displacement* so that $s > 0$ means the object is traveling in a counter-clockwise direction and $s < 0$ indicates motion in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation.

In Physics, the **average velocity** of the object, denoted $\overline{v}$ and read as 'v-bar', is defined as the average rate of change of the position of the object with respect to time. The displacement, a directed distance, is positive, negative, or zero, depending on the direction in which the object is traveling. As a result,

$$\overline{v} = \frac{\text{displacement}}{\text{time}}$$
$$\overline{v} = \frac{s}{t}.$$

The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas:

1. The direction in which the object is moving. The contribution of direction in the quantity $\overline{v}$ is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion).

2. How fast the position of the object is changing. The quantity $|\overline{v}|$ quantifies how fast the object is moving – it is the **speed** of the object.

Measuring θ in radians, we have $\theta = \frac{s}{r}$. Thus $s = r\theta$ and

$$\begin{aligned}\overline{v} &= \frac{s}{t}\\ &= \frac{r\theta}{t}\\ &= r \cdot \frac{\theta}{t}.\end{aligned}$$

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$.

If $\bar{\omega}$ is constant throughout the duration of the motion, then it can be shown[13] that the average velocities involved, namely $\bar{v}$ and $\bar{\omega}$, are the same as their instantaneous counterparts, v and ω, respectively. In this case, v is simply called the **velocity** of the object and is the instantaneous rate of change of the position of the object with respect to time. Similarly, ω is called the **angular velocity** and is the instantaneous rate of change of the angle with respect to time.

If the path of the object were uncurled from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity v is often called the **linear velocity** of the object in order to distinguish it from the angular velocity, ω.

Putting together the ideas of the previous paragraphs, we get the following.

Velocity for Circular Motion: For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by $v = r\omega$.

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units $\text{length} \cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}}$. The supposed contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length} \cdot \text{radians}}{\text{time}}$ reduce to $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example 1.3.4. Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle which completes one revolution in (approximately) 24 hours. The path traced out by the point during this 24 hour period is the latitude of that point. Salt Lake Community College is at 40.7608° North Latitude, and it can be shown[14] that the radius of the circle of revolution at this latitude is approximately 2999 miles. Find the linear velocity, in miles per hour, of Salt Lake Community College as the world turns.

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[13] By using calculus . . .

[14] We will discuss how we arrived at this approximation in **Section 2.5**.

Solution. To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The earth makes one revolution in 24 hours, and one revolution is 2π radians, so

$$\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}}.$$

Once again, we are using the fact that radians are real numbers and are dimensionless. (For simplicity's sake, we are also assuming that we are viewing the rotation of the earth as counterclockwise, so $\omega > 0$.) Hence, the linear velocity is

$$\begin{aligned} v &= 2999 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \\ &\approx 785 \text{ miles/hour.} \end{aligned}$$

□

It is worth noting that the quantity $\dfrac{1 \text{ revolution}}{24 \text{ hours}}$ in **Example 1.3.4** is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion.

On a related note, the quantity $T = \dfrac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of **Example 1.3.4**, the period of the motion is 24 hours, or one day.

The concept of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the **magnification factor** which relates ω and v.

While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: If an object is moving on a circular path of radius r with a fixed angular velocity (frequency) ω, what is the position of the object at time t? The answer to this question is the very heart of trigonometry and is answered in **Section 2.5**.

Linear and Angular Speed

We finish this section with a couple of examples that ask us to find speed rather than velocity. The two concepts are closely related, but speed does not include direction. **Linear speed** is the distance an object travels in a given time interval and, as mentioned earlier, is the absolute value of linear velocity. **Angular speed** is the object's angular rotation during a given time interval. In calculating angular speed, the radian measure of the angle θ is interpreted as being non-negative.

Example 1.3.5. An old vinyl record is played on a turntable rotating at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution. We find angular speed by dividing the total angular rotation by the time. We then convert to radians per second.

$$\begin{aligned}\text{angular speed} &= \frac{45 \text{ rotations}}{1 \text{ minute}} \\ &= \frac{45 \text{ rotations}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \text{ rotation}} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \\ &= \frac{3\pi}{2} \text{ radians/second}\end{aligned}$$

□

Example 1.3.6. A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution. Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road. The equation for velocity of circular motion, $v = r\omega$, allows us to find linear velocity given angular velocity. The same equation can be used for speed, keeping in mind that v and ω must be restricted to non-negative values: $|v|$ and $|\omega|$, respectively. We begin by converting angular speed from revolutions (rotations) per minute to radians per minute.

$$\frac{180 \text{ rotations}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \text{ rotation}} = 360\pi \frac{\text{radians}}{\text{minute}}$$

The diameter of the wheels is 28 inches, for a radius of 14 inches, and we have the following:

$$\begin{aligned}|v| &= r|\omega| \\ &= (14 \text{ inches})\left(360\pi \frac{\text{radians}}{\text{minute}}\right) \\ &= 5040\pi \ \frac{\text{inches}}{\text{minute}}.\end{aligned}$$

Since radians are a unitless measure, it is not necessary to include them. Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$\begin{aligned}|v| &= 5040\pi \ \frac{\text{inches}}{\text{minute}} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \\ &\approx 14.99 \text{ miles per hour}\end{aligned}$$

□

1.3 Exercises

In Exercises 1 – 6, round your answer to two decimal places.

1. Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

2. Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

3. Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

4. Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°.

5. Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220°.

6. Find the length of the arc of a circle of diameter 12 meters subtended by the central angle of 63°.

In Exercises 7 – 12, compute the areas of the circular sectors with the given central angles and radii. Round your answers to two decimal places.

7. $\theta = \frac{\pi}{6}$, $r = 12$

8. $\theta = \frac{5\pi}{4}$, $r = 100$

9. $\theta = 330^\circ$, $r = 9.3$

10. $\theta = \pi$, $r = 1$

11. $\theta = 240^\circ$, $r = 5$

12. $\theta = 1^\circ$, $r = 117$

13. A yo-yo which is 2.25 inches in diameter spins at a rate of 4500 revolutions per minute. How fast is the edge of the yo-yo spinning in miles per hour? Round your answer to two decimal places.

14. How many revolutions per minute would the yo-yo in **Exercise 13** have to complete if the edge of the yo-yo is to be spinning at a rate of 42 miles per hour? Round your answer to two decimal places.

15. In the yo-yo trick ‘Around the World’, the performer throws the yo-yo so it sweeps out a vertical circle whose radius is the yo-yo string. If the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle, compute the speed of the yo-yo in miles per hour. Round your answer to two decimal places.

16. A computer hard drive contains a circular disk with diameter 2.5 inches and spins at a rate of 7200 RPM (revolutions per minute). Find the liner speed of a point on the edge of the disk in miles per hour.

17. A rock got stuck in the tread of my tire and, while I was driving 70 miles per hour, the rock came loose and hit the inside of the wheel well of the car. How fast, in miles per hour, was the rock traveling when it came out of the tread? (The tire has a diameter of 23 inches.)

18. The Giant Wheel at Cedar Point Amusement Park is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming the riders are at the edge of the circle, how fast are they traveling in miles per hour?

19. A truck with 32-inch diameter wheels is traveling at 60 miles per hour. Find the angular speed of the wheels in radians per minute. How many revolutions per minute do the wheels make?

20. A CD has diameter 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

21. Imagine a rope tied around the Earth at the equator. Show that you need to add only 2π feet of length to the rope in order to lift it one foot above the ground around the entire equator. (You do NOT need to know the radius of the Earth to show this.)

CHAPTER 2
THE TRIGONOMETRIC FUNCTIONS

Chapter Outline

2.1 Right Triangle Trigonometry

2.2 Determining Cosine and Sine Values from the Unit Circle

2.3 The Six Circular Functions

2.4 Verifying Trigonometric Identities

2.5 Beyond the Unit Circle

Introduction

In this chapter we begin our venture into trigonometric functions by revisiting the geometric concept of similarity of triangles and defining the six trigonometric functions. Additionally, in Section 2.1, we note the values of trigonometric functions for some common angles, and we solve real-world application problems involving right triangles. In Section 2.2, we introduce the Unit Circle, and make use of this important tool in redefining the sine and cosine functions as circular functions. We add radian measure to the degree measure introduced in Section 2.1, and use reference angles and the Pythagorean Identity to determine circular function values. Section 2.3 advances the Unit Circle definition of circular functions to include the tangent, cosecant, secant and cotangent. In this section, reciprocal and quotient identities are introduced. We move on, in Section 2.4, to simplifying trigonometric expressions and proving that a trigonometric equation is an identity. Then, finally, Section 2.5 introduces definitions for circular functions of varying radii, along with applications.

2.1 Right Triangle Trigonometry

Learning Objectives

In this section you will:

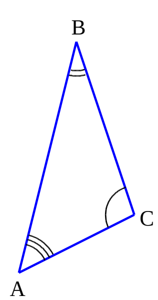
- **Identify the trigonometric functions.**
- **Learn the trigonometric function values for 30 degrees, 45 degrees and 60 degrees.**
- **Solve right triangles and related application problems.**

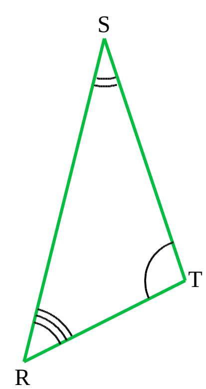
As we shall see in the sections to come, many applications in trigonometry involve finding the measures of the interior angles, and the lengths of the sides, of right triangles. Recall that a **right triangle** is a triangle containing one right angle and two acute angles. In this section, we will define a new group of functions known as trigonometric functions that will assist us in determining unknown angle measures and/or side lengths for right triangles. Noting that two right triangles are similar if they have one congruent acute angle, we will use properties of similar triangles to establish trigonometric function values for three special angles: 30°, 45° and 60°.

Similar Triangles

We begin with a definition from geometry. Recall that two triangles are **similar** if they have the same shape or, more specifically, if their corresponding angles are congruent. Additionally, two triangles are similar if and only if their corresponding sides are proportional.

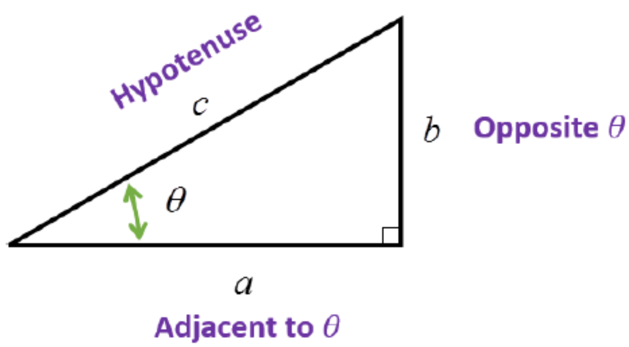
In the following triangles, $\angle A \cong \angle R$, $\angle B \cong \angle S$, and $\angle C \cong \angle T$. Thus, triangle *ABC* is similar to triangle *RST* and $\dfrac{AB}{RS} = \dfrac{BC}{ST} = \dfrac{CA}{TR}$.





Trigonometric Functions

We consider the generic right triangle below with acute angle θ. The side with length a is called the side of the triangle **adjacent** to θ; the side with length b is called the side of the triangle **opposite** θ; and the remaining side c (the side opposite the right angle) is called the **hypotenuse**.



The six commonly used trigonometric functions are defined below.

The Trigonometric Functions: Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a, the length of the side opposite θ is b, and the length of the hypotenuse is c, then

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{b}{c}$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{a}{c}$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{b}{a}$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{a}{b}$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{c}{a}$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{c}{b}$.

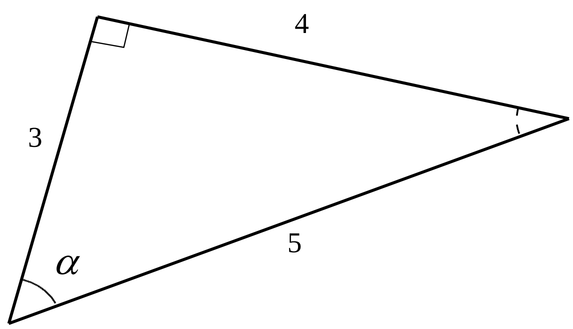
The following are important properties of the trigonometric functions.

1. For all right triangles with the same acute angle θ, because they are similar, the values of the resulting trigonometric functions of θ will be identical. This is a result of the property of equivalent proportions of corresponding sides within similar triangles.

2. Cosecant, secant and cotangent are **reciprocal functions** of sine, cosine and tangent, respectively. Thus, if we know the sine, cosine and tangent values for an angle, we can easily determine the remaining three trigonometric functions. In particular,

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Example 2.1.1. Use the following triangle to evaluate $\sin(\alpha)$, $\cos(\alpha)$, $\tan(\alpha)$, $\csc(\alpha)$, $\sec(\alpha)$ and $\cot(\alpha)$.



Solution. From the definitions of trigonometric functions,

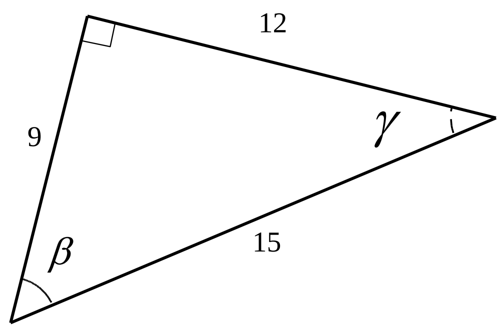
$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

The reciprocals of these three function values result in the remaining three trigonometric function values:

$$\csc(\alpha) = \frac{1}{\sin(\alpha)} = \frac{5}{4}$$
$$\sec(\alpha) = \frac{1}{\cos(\alpha)} = \frac{5}{3}$$
$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{3}{4}$$

□

Example 2.1.2. Verify that the following triangle is similar to the triangle in **Example 2.1.1**. Then evaluate the trigonometric function values for the angle corresponding to *α*.



Solution. The side lengths of the second triangle are proportional to the corresponding side lengths of the first triangle by a scale factor of 3:

$$\frac{9}{3} = \frac{12}{4} = \frac{15}{5} = 3.$$

Thus the triangles are similar, with the angle *β* being equal in measure to *α*. To evaluate the trigonometric function values for *β*, we save a bit of writing by using the abbreviations opp, adj and hyp in place of opposite, adjacent and hypotenuse, respectively. The trigonometric function values for this similar triangle will be

$$\sin(\beta) = \frac{\text{opp}}{\text{hyp}} = \frac{12}{15} = \frac{3\cdot 4}{3\cdot 5} = \frac{4}{5}$$
$$\cos(\beta) = \frac{\text{adj}}{\text{hyp}} = \frac{9}{15} = \frac{3\cdot 3}{3\cdot 5} = \frac{3}{5}$$
$$\tan(\beta) = \frac{\text{opp}}{\text{adj}} = \frac{12}{9} = \frac{3\cdot 4}{3\cdot 3} = \frac{4}{3}$$

Using reciprocal properties, the remaining three values are

$$\csc(\beta) = \frac{3\cdot 5}{3\cdot 4} = \frac{5}{4}$$
$$\sec(\beta) = \frac{3\cdot 5}{3\cdot 3} = \frac{5}{3}$$
$$\cot(\beta) = \frac{3\cdot 3}{3\cdot 4} = \frac{3}{4}$$

□

We note that the trigonometric function values are identical for the two similar triangles in **Example 2.1.1** and **Example 2.1.2**, and observe that trigonometric ratios are not affected by the value of the scale factor.

Pythagorean Theorem

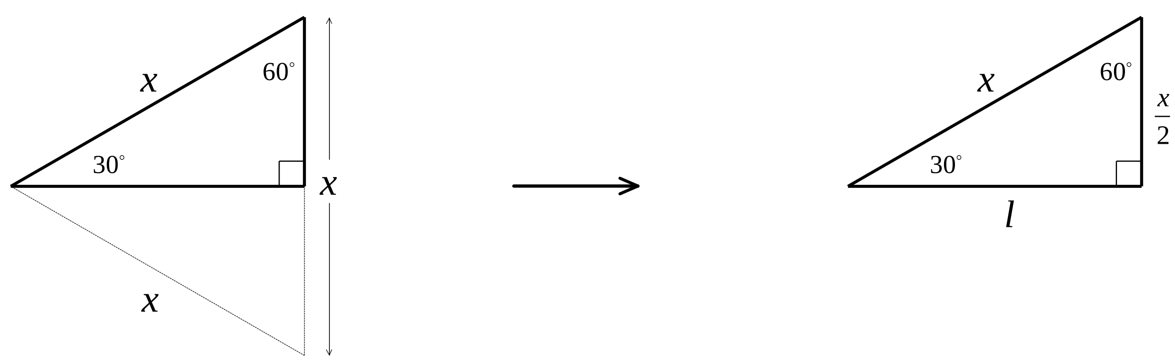
The Pythagorean Theorem will be useful in our next task: determining trigonometric function values for 30°, 45° and 60° angles.

The Pythagorean Theorem: The square of the hypotenuse in a right triangle is equal to the sum of the squares of the two shorter sides. In particular, in a right triangle with hypotenuse *c* and the shorter sides of lengths *a* and *b*,

$$c^2 = a^2 + b^2$$

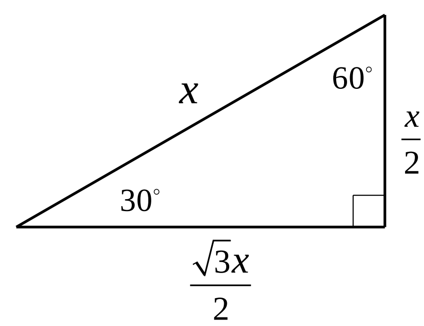
Trigonometric Functions of 30°, 60°, 90° Triangles

We begin by finding the values of trigonometric functions for 30°. We sketch a 30°, 60°, 90° right triangle with hypotenuse of length *x* and envision this triangle as being half of a 60°, 60°, 60° equilateral triangle with sides of length *x*.



Noting that the altitude of the equilateral triangle bisects its base, it follows that the shortest side of the 30°, 60°, 90° triangle has length $\frac{x}{2}$. We apply the Pythagorean Theorem to determine the length, *l*, of the third side of the 30°, 60°, 90° triangle in terms of *x*.

$$l^2 + \left(\frac{x}{2}\right)^2 = x^2$$
$$l^2 = x^2 - \frac{x^2}{4}$$
$$l^2 = \frac{3x^2}{4}$$
$$l = \frac{\sqrt{3}x}{2}$$



Using the resulting side lengths, along with the definitions of the trigonometric functions, we have

$$\sin(30^\circ) = \frac{(x/2)}{x} = \frac{1}{2}$$
$$\cos(30^\circ) = \frac{(\sqrt{3}x/2)}{x} = \frac{\sqrt{3}}{2}$$
$$\tan(30^\circ) = \frac{(x/2)}{(\sqrt{3}x/2)} = \frac{1}{\sqrt{3}}$$

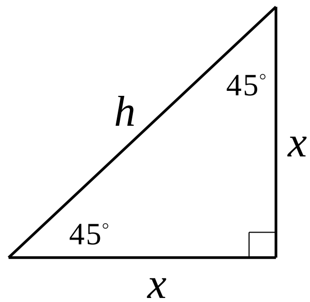
Taking the reciprocals of these three function values results in the remaining three trigonometric function values:

$$\csc(30^\circ) = \frac{1}{\sin(30^\circ)} = 2$$
$$\sec(30^\circ) = \frac{1}{\cos(30^\circ)} = \frac{2}{\sqrt{3}}$$
$$\cot(30^\circ) = \frac{1}{\tan(30^\circ)} = \sqrt{3}$$

We note that these trigonometric function values apply to any 30° angle. The reader is encouraged to determine the trigonometric function values for 60° angles.

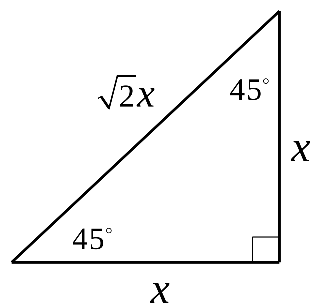
Trigonometric Functions of 45°, 45°, 90° Triangles

To find the values of the trigonometric functions for 45°, we sketch a 45°, 45°, 90° right isosceles triangle with hypotenuse *h* and remaining two sides each of length *x*.



Using the Pythagorean Theorem, the hypotenuse can be written in terms of *x* as follows.

$$x^2 + x^2 = h^2$$
$$2x^2 = h^2$$
$$h = \sqrt{2}x$$



The resulting trigonometric function values for 45° are

$$\sin(45^\circ) = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$$
$$\cos(45^\circ) = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$$
$$\tan(45^\circ) = \frac{x}{x} = 1$$
$$\csc(45^\circ) = \frac{1}{\sin(45^\circ)} = \sqrt{2}$$
$$\sec(45^\circ) = \frac{1}{\cos(45^\circ)} = \sqrt{2}$$
$$\cot(45^\circ) = \frac{1}{\tan(45^\circ)} = 1$$

After rationalizing denominators and adding trigonometric functions for 60°, we summarize the trigonometric function values for these special cases in the following table.

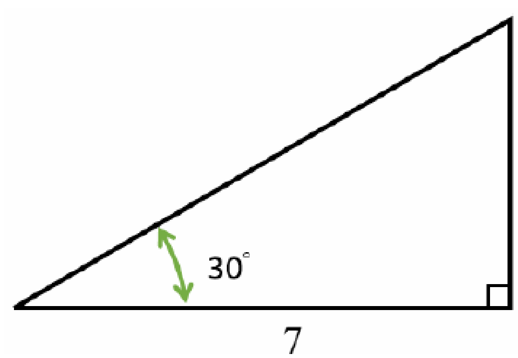
Trigonometric Function Values for 30°, 45° and 60°

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Solving Right Triangles

We will use these values in the next four examples to determine measures of missing angles and sides. This is sometimes referred to as **solving right triangles**.

Example 2.1.3. Find the measure of the missing angle and the lengths of the missing sides of:



Solution. The first and easiest task is to find the measure of the missing angle. Since the sum of the angles of a triangle is 180°, we know that the missing angle has measure $180^\circ - 30^\circ - 90^\circ = 60^\circ$. We now proceed to find the lengths of the remaining two sides of the triangle.

Let c denote the length of the hypotenuse of the triangle. From $\cos(30^\circ) = \frac{7}{c}$, we get

$$c = \frac{7}{\cos(30^\circ)}$$

$$c = 7 \cdot \frac{1}{\cos(30^\circ)}$$

$$c = 7 \cdot \sec(30^\circ).$$

Since $\sec(30^{\circ}) = \dfrac{2\sqrt{3}}{3}$, we arrive at the length of the hypotenuse: $c = \dfrac{14\sqrt{3}}{3}$.

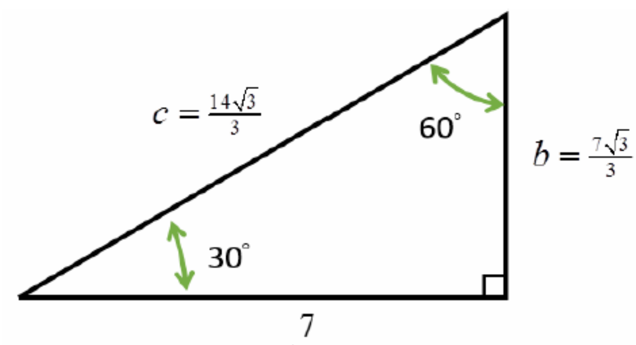
At this point, we have two ways to proceed to find the length of the side opposite the 30° angle, which we'll denote *b*. We know the length of the adjacent side is 7 and the length of the hypotenuse is $\dfrac{14\sqrt{3}}{3}$, so we could find the missing side by applying the Pythagorean Theorem and solving the following for *b*:

$$(7)^2 + b^2 = \left(\frac{14\sqrt{3}}{3}\right)^2.$$

Alternatively, we could use the definition, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$, to get $\tan(30^{\circ}) = \dfrac{b}{7}$. Choosing the latter, we find

$$\begin{aligned} b &= 7\tan(30^{\circ}) \\ &= 7 \cdot \frac{\sqrt{3}}{3} \\ &= \frac{7\sqrt{3}}{3}. \end{aligned}$$

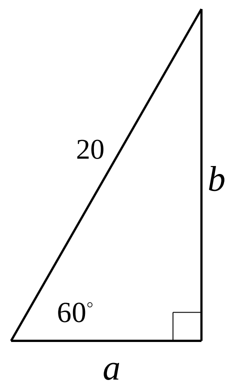
The triangle with all of its recorded data follows.



□

Example 2.1.4. A right triangle has one angle of 60° and a hypotenuse of 20. Find the unknown side lengths and missing angle measure.

Solution. Again, we begin with finding the measure of the missing angle. The sum of the angles of a triangle is 180°, from which it follows that the missing angle measure is $180^{\circ} - 60^{\circ} - 90^{\circ} = 30^{\circ}$. We assign the missing side lengths as *a*, for the side adjacent to 60°, and *b*, for the side opposite 60°.



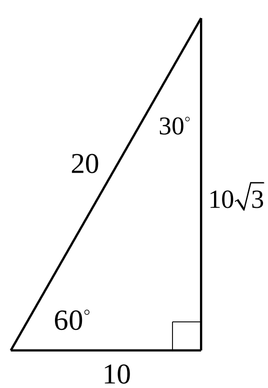
Since $\sin\left(60^\circ\right) = \frac{b}{20}$, we find

$$\begin{aligned} b &= 20\sin\left(60^\circ\right) \\ &= 20 \cdot \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3}. \end{aligned}$$

We find length *a* using the Pythagorean Theorem, although the same result could be achieved through solving $\cos\left(60^\circ\right) = \frac{a}{20}$ for *a*.

$$\begin{aligned} a^2 + \left(10\sqrt{3}\right)^2 &= 20^2 \\ a^2 &= 400 - 300 \\ a &= 10. \end{aligned}$$

The triangle with all of its data is recorded below.



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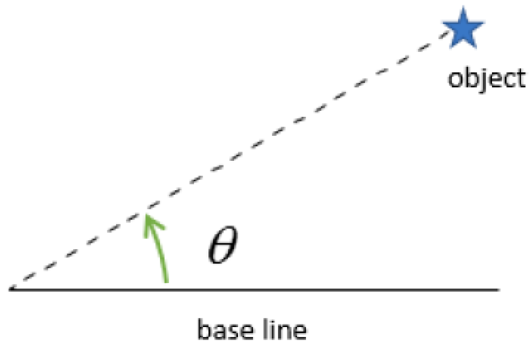
Solving Applied Problems

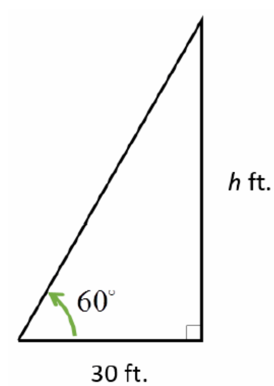
Right triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. The following example uses trigonometric functions as well as the concept of an 'angle of inclination'. The **angle of inclination**, commonly known as the **angle of elevation**, of an object refers to the angle whose initial side is some kind of horizontal base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. This is represented schematically below.



The angle of inclination (elevation) from the base line to the object is θ.

Example 2.1.5. The angle of inclination, from a point on the ground 30 feet away from the base of a water tower, to the top of the water tower, is 60°. Find the height of the water tower to the nearest foot.

Solution. We can represent the problem situation using a right triangle as shown. If we let h denote the height of the tower, then we have $\tan\left(60^\circ\right) = \dfrac{h}{30}$.



From this we get

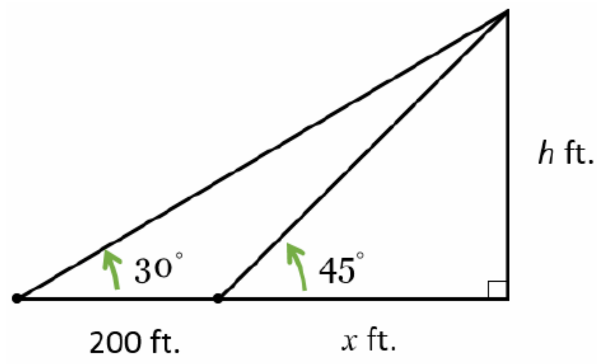
$$\begin{aligned} h &= 30\tan\left(60^\circ\right) \\ &= 30\sqrt{3} \\ &\approx 51.96 \end{aligned}$$

Hence, the water tower is approximately 52 feet tall.

□

Example 2.1.6. In order to determine the height of a California redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot.

Solution. Sketching the problem situation below, we find ourselves with two unknowns: the height *h* of the tree and the distance *x* from the base of the tree to the first observation point.



Using trigonometric functions, we get a pair of equations: $\tan(45^\circ) = \frac{h}{x}$ and $\tan(30^\circ) = \frac{h}{x+200}$. Since $\tan(45^\circ) = 1$, the first equation gives $\frac{h}{x} = 1$, or $x = h$. Substituting this into the second equation gives

$$\frac{h}{h+200} = \tan(30^\circ)$$
$$\text{or } \frac{h}{h+200} = \frac{\sqrt{3}}{3}.$$

Clearing fractions, we get $3h = (h+200)\sqrt{3}$. The result is a linear equation for *h*, so we proceed to expand the right hand side and gather all the terms involving *h* to one side.

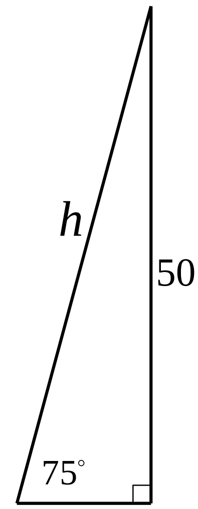
$$\begin{aligned} 3h &= (h+200)\sqrt{3} \\ 3h &= h\sqrt{3} + 200\sqrt{3} \\ 3h - h\sqrt{3} &= 200\sqrt{3} \\ (3-\sqrt{3})h &= 200\sqrt{3} \\ h &= \frac{200\sqrt{3}}{3-\sqrt{3}} \approx 273.21 \end{aligned}$$

Hence, the tree is approximately 273 feet tall.

□

Example 2.1.7. How long must a ladder be to reach a windowsill 50 feet above the ground if the ladder is resting against the building at an angle of 75° with the ground?

Solution. We know that the angle of inclination, or elevation, is 75° and that the opposite side is 50 feet in length. The length of the hypotenuse, *h*, will give us the necessary length for the ladder to reach a height of 50 feet. Using the trigonometric function for sine of 75°, we have



$$\sin\left(75^\circ\right) = \frac{50}{h}$$
$$h = \frac{50}{\sin\left(75^\circ\right)}$$
$$h \approx \frac{50}{0.9659258}$$
$$h \approx 51.76$$

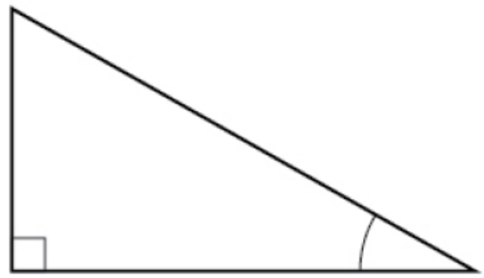
Noting that trigonometric function values for 75° are not included in the table for special cases, use of a calculator is necessary to find an approximate value for $\sin\left(75^\circ\right)$. It is good practice to verify that the calculator is set to the correct mode, in this case degrees, before proceeding with calculations. We have found that the height of the ladder is approximately 51.8 feet.

□

This section leads us to the Unit Circle and an alternate definition for trigonometric functions. Through this new definition we will expand the domain for trigonometric functions to include angle measures outside the interval $\left(0^\circ, 90^\circ\right)$.

2.1 Exercises

1. For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.

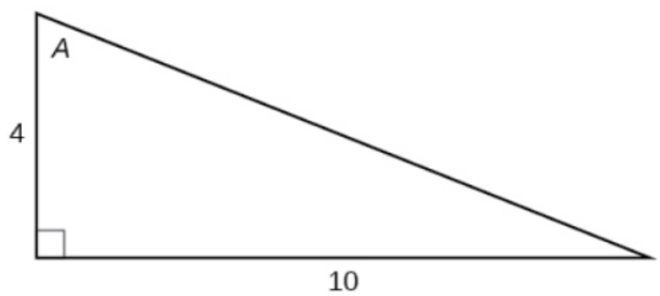


2. The tangent of an angle compares which sides of the right triangle?

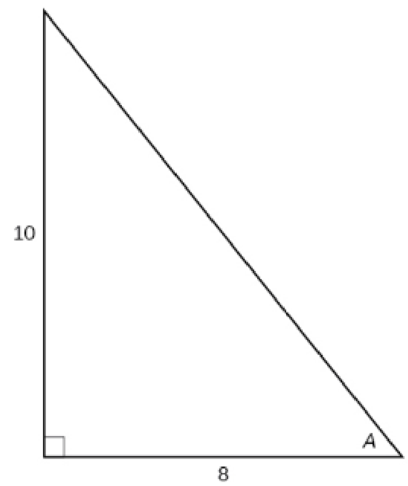
3. What is the relationship between the two acute angles in a right triangle?

In Exercises 4 and 5, use the given triangle to evaluate each trigonometric function of angle *A*.

4. Find $\sin(A)$, $\cos(A)$, $\tan(A)$, $\csc(A)$, $\sec(A)$ and $\cot(A)$.

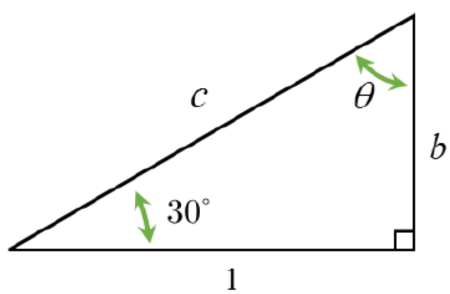


5. Find $\sin(A)$, $\cos(A)$, $\tan(A)$, $\csc(A)$, $\sec(A)$ and $\cot(A)$.

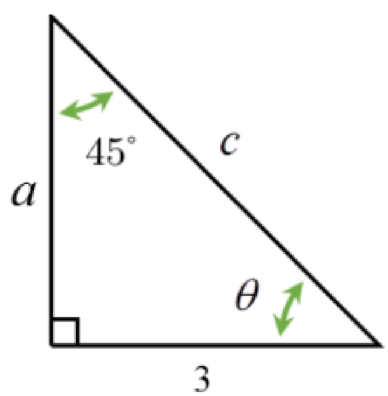


In Exercises 6 – 13, find the measurement of the missing angle and the lengths of the missing sides.

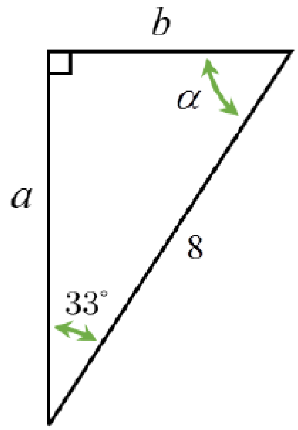
6. Find θ, b, and c.



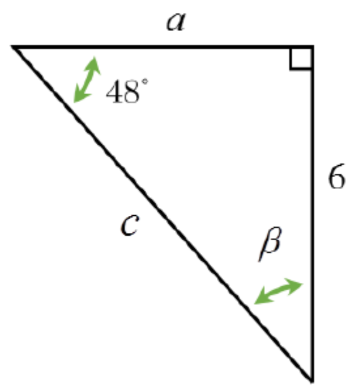
7. Find θ, a, and c.



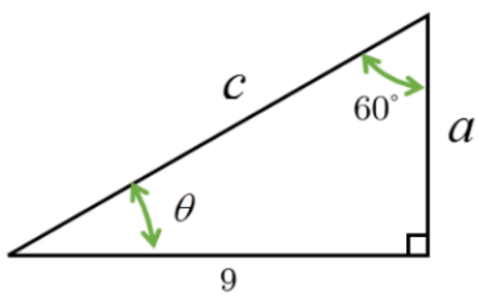
8. Find α, a, and b.



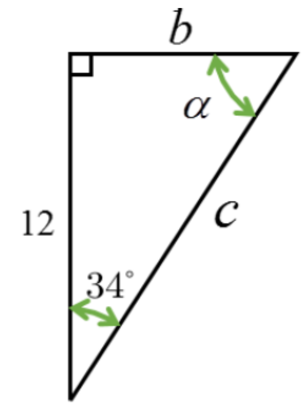
9. Find β, a, and c.



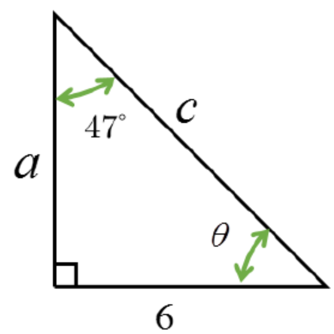
10. Find θ, a, and c.



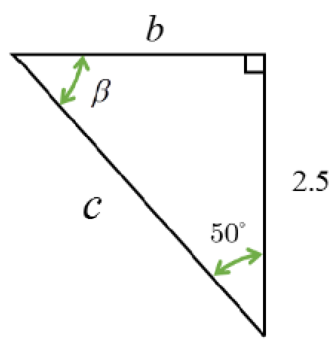
11. Find α, b, and c.



12. Find θ, a, and c.



13. Find β, b, and c.



In Exercises 14 – 25, assume that θ is an acute angle in a right triangle.

14. If $\theta = 12^\circ$ and the side adjacent to θ has length 4, how long is the hypotenuse?

15. If $\theta = 78.123^\circ$ and the hypotenuse has length 5280, how long is the side adjacent to θ?

16. If $\theta = 59^\circ$ and the side opposite θ has length 117.42, how long is the hypotenuse?

17. If $\theta = 5^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?

18. If $\theta = 5^\circ$ and the hypotenuse has length 10, how long is the side adjacent to θ?

19. If $\theta = 37.5^\circ$ and the side opposite θ has length 306, how long is the side adjacent to θ?

20. If $\theta = 30^\circ$ and the side opposite θ has length 4, how long is the side adjacent to θ?

21. If $\theta = 15^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?

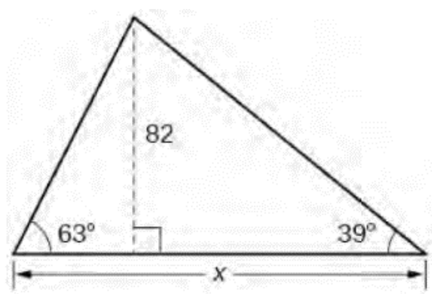
22. If $\theta = 87^\circ$ and the side adjacent to θ has length 2, how long is the side opposite θ?

23. If $\theta = 38.2^\circ$ and the side opposite θ has length 14, how long is the hypotenuse?

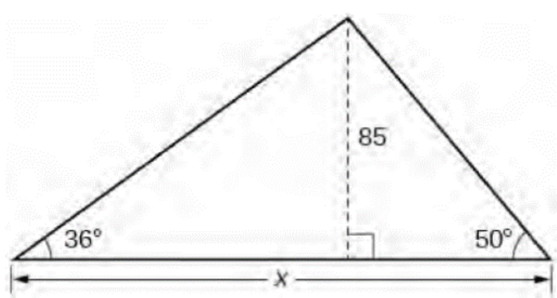
24. If $\theta = 2.05^\circ$ and the hypotenuse has length 3.98, how long is the side adjacent to θ?

25. If $\theta = 42^\circ$ and the side adjacent to θ has length 31, how long is the side opposite θ?

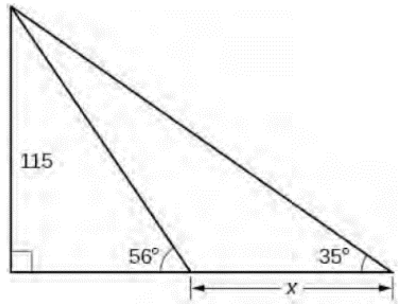
26. Find *x*.



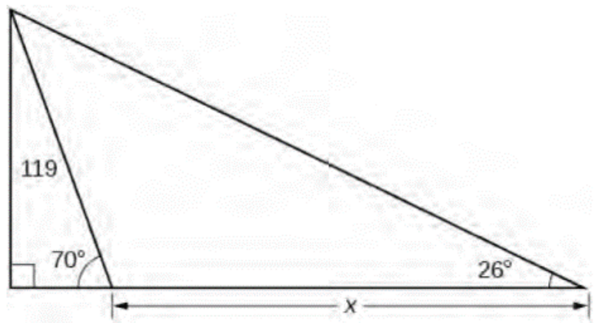
27. Find *x*.



28. Find *x*.



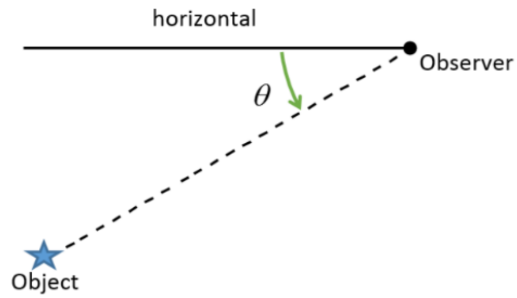
29. Find *x*.



30. A tree standing vertically on level ground casts a 120 foot long shadow. The angle of elevation from the end of the shadow to the top of the tree is 21.4°. Find the height of the tree to the nearest foot. With the help of your classmates, research the term *umbra versa* and see what it has to do with the shadow in this problem.

31. The broadcast tower for radio station WSAZ (Home of "Algebra in the Morning with Carl and Jeff") has two enormous flashing red lights on it: one at the very top and one a few feet below the top. From a point 5000 feet away from the base of the tower on level ground the angle of elevation to the top light is 7.970° and to the second light is 7.125°. Find the distance between the lights to the nearest foot.

32. In this section, we defined the angle of inclination (also known as the angle of elevation) and in this exercise we introduce a related angle – the angle of depression (also known as the angle of declination). The angle of depression of an object refers to the angle whose initial side is a horizontal line above the object and whose terminal side is the line-of-sight to the object below the horizontal. This is represented schematically below.



The angle of depression from the horizontal to the object is θ.

(a) Show that if the horizontal is above and parallel to level ground then the angle of depression (from observer to object) and the angle of inclination (from object to observer) will be congruent because they are alternate interior angles.

(b) From a fire tower 200 feet above level ground in the Sasquatch National Forest, a ranger spots a fire off in the distance. The angle of depression to the fire is 2.5°. How far away from the base of the tower is the fire?

(c) The ranger in part (b) sees a Sasquatch running directly from the fire towards the fire tower. The ranger takes two sightings. At the first sighting, the angle of depression from the tower to the Sasquatch is 6°. The second sighting, taken just 10 seconds later, gives the angle of depression as 6.5°. How far did the Sasquatch travel in those 10 seconds? Round your answer to the nearest foot. How fast is it running in miles per hour? Round your answer to the nearest mile per hour. If the Sasquatch keeps up this pace, how long will it take for the Sasquatch to reach the fire tower from his location at the second sighting? Round your answer to the nearest minute.

33. When I stand 30 feet away from a tree at home, the angle of elevation to the top of the tree is 50° and the angle of depression to the base of the tree is 10°. What is the height of the tree? Round your answer to the nearest foot.

34. From the observation deck of the lighthouse at Sasquatch Point 50 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2° and the second sighting had an angle of depression of 25.9°. How far had the boat traveled between the sightings?

35. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it makes a 43° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?

36. A 33 foot ladder leans against a building, so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

37. A 23 foot ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

38. The angle of elevation to the top of a building in New York City is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

39. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

40. Assuming that a 370 foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

41. Let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sin(\alpha) = \cos(\beta)$ and $\sin(\beta) = \cos(\alpha)$. The fact that co-functions of complementary angles are equal in this case is not an accident and a more general result will be given in Chapter 4.

42. Let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sec(\alpha) = \csc(\beta)$ and $\tan(\alpha) = \cot(\beta)$.

2.2 Determining Cosine and Sine Values from the Unit Circle

Learning Objectives

In this section you will:

- **Sketch oriented arcs on the Unit Circle.**
- **Determine the cosine and sine values of an angle from a point on the Unit Circle.**
- **Learn and apply the Pythagorean identity.**
- **Apply the Reference Angle Theorem.**
- **Learn the cosine and sine values for the common angles: 0°, 30°, 45°, 60° and 90°, or for their equivalent radian measures.**
- **Learn the signs of the cosine and sine functions in each quadrant.**

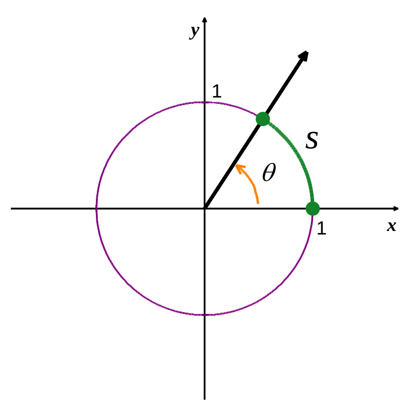
We have already defined the Trigonometric Functions as functions of acute angles within right triangles. In this section, we will expand upon that definition by redefining the cosine and sine functions using the Unit Circle. Thus, the new designation 'Circular Functions' will often be used in place of 'Trigonometric Functions' as we adopt this new definition.

The Unit Circle

Consider the **Unit Circle**, $x^2 + y^2 = 1$, as shown, with the angle θ in standard position and the corresponding arc measuring s units in length. By the definition established in **Section 1.2**, and the fact that the Unit Circle has radius 1, the radian measure of θ is

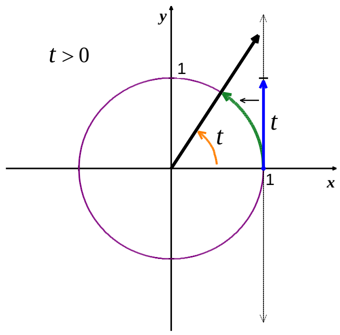
$$\frac{s}{r} = \frac{s}{1} = s$$

so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$.

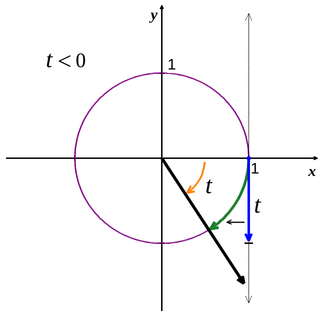


In order to identify real numbers with oriented angles, we make good use of this fact by essentially 'wrapping' the real number line around the Unit Circle and associating to each real number t an *oriented* arc on the Unit Circle with initial point $(1,0)$.

Given a real number $t > 0$ and the vertical line $x = 1$ containing the (vertical) interval $[0, t]$, we 'wrap' the interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of *t* units. Therefore, the corresponding angle has radian measure equal to *t*.



If $t < 0$, we wrap the interval $[t, 0]$ *clockwise* around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has the negative radian measure equal to *t*.



Note that if $t = 0$, we are at the point $(1, 0)$ on the *x*-axis which corresponds to an angle with radian measure 0.

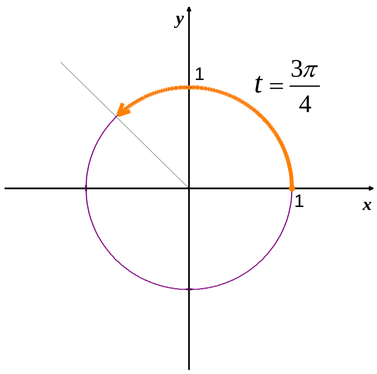
Thus, we identify each real number *t* with the corresponding angle having radian measure of *t*.

Example 2.2.1. Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

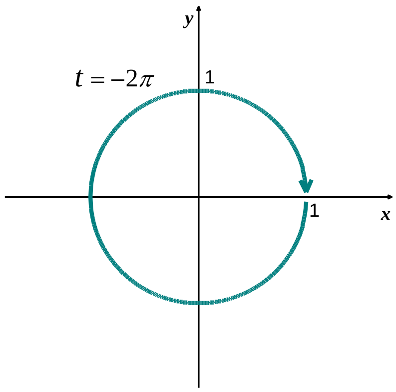
1. $t = \frac{3\pi}{4}$ 2. $t = -2\pi$ 3. $t = -2$ 4. $t = 117$

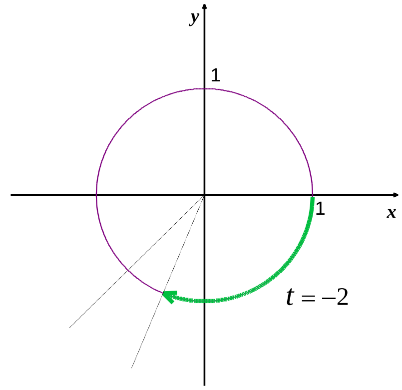
Solution.

1. The arc associated with $t = \frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1, 0)$ and proceeds counter-clockwise up to midway through Quadrant II.



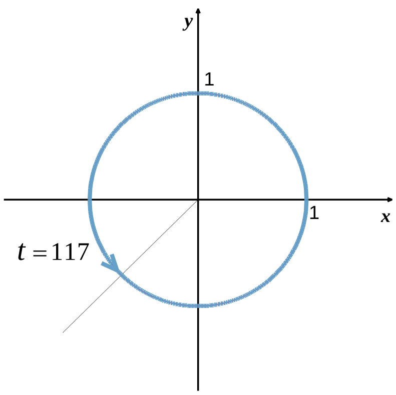
2. Since one revolution is 2π radians, and $t = -2\pi$ is negative, we graph the arc which begins at $(1, 0)$ and proceeds *clockwise* for one full revolution.





3. Like $t=-2\pi$, $t=-2$ is negative, so we begin our arc at $(1,0)$ and proceed clockwise around the Unit Circle. With $\frac{\pi}{2}\approx 1.57$ and $\pi\approx 3.14$, we find rotating 2 radians clockwise from the point $(1,0)$ lands us in Quadrant III between $-\frac{\pi}{2}$ and $-\pi$. To more accurately place the endpoint, we proceed as we did in **Example 1.1.3**, successively halving the angle measure until we find $-\frac{5\pi}{8}\approx -1.96$, which tells us our arc extends, clockwise, almost a quarter of the way into Quadrant III.

4. Since 117 is positive, the arc corresponding to $t=117$ begins at $(1,0)$ and proceeds counter-clockwise. As 117 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{117}{2\pi}$ as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of $\frac{5}{8}$ of a revolution, remaining. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III.



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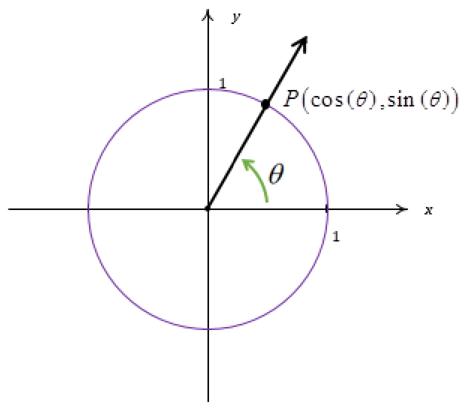
The Cosine and Sine as Circular Functions

This leads to our new definition for the cosine and sine functions. Consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle.



By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$, while the y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$.[15] The reader is encouraged to verify that these rules that match an angle with its cosine and sine satisfy the definition of a function: for each angle θ, there is only one associated value of $\cos(\theta)$ and only one associated value of $\sin(\theta)$.

It is important to note that any angle that is not labeled as being in degrees is, by default, assumed to be in radians. In the following example, the angles in number 2 and number 4 are radian measures: $\theta = -\pi$ radians and $\theta = \frac{\pi}{6}$ radians, respectively.

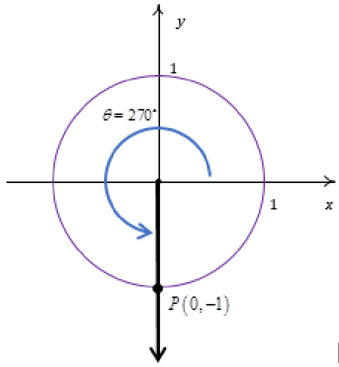
Example 2.2.2. Find the cosine and sine of the following angles.

1. $\theta = 270^\circ$ 2. $\theta = -\pi$ 3. $\theta = 45^\circ$ 4. $\theta = \frac{\pi}{6}$ 5. $\theta = 60^\circ$

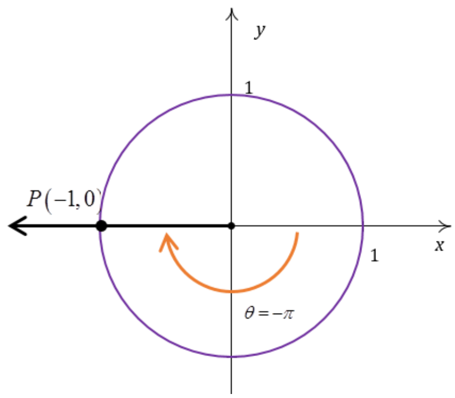
Solution.

1. To find $\cos(270^\circ)$ and $\sin(270^\circ)$, we plot the angle $\theta = 270^\circ$ in standard position and find the point on the terminal side of θ which lies on the Unit Circle. Since 270° represents $\frac{3}{4}$ of a counter-clockwise revolution, the terminal side of θ lies along the negative y-axis. Hence, the point we seek is $(0,-1)$ so that $\cos(270^\circ) = 0$ and $\sin(270^\circ) = -1$.

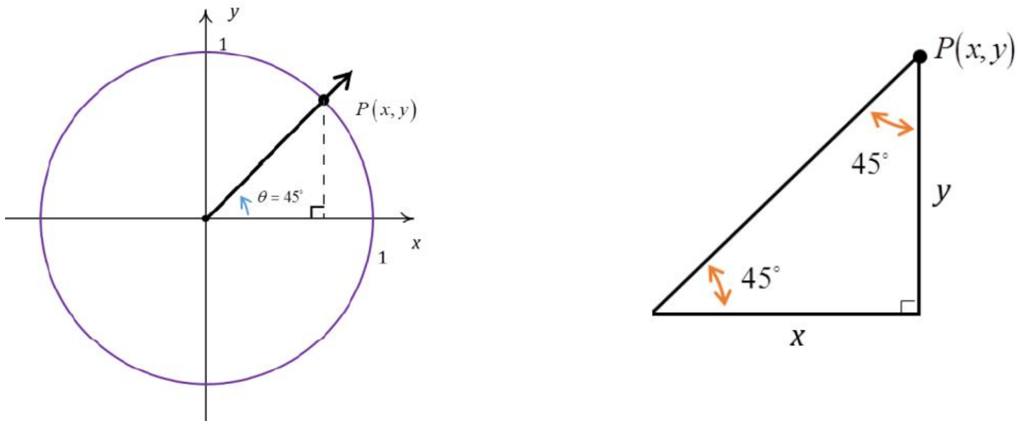
[15] The etymology of the name 'sine' is quite colorful, and the interested reader is invited to research it; the 'co' in 'cosine' is explained in Section 4.2.



2. The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative *x*-axis. The point on the Unit Circle that lies on the negative *x*-axis is $(-1,0)$, from which $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.



3. When we sketch $\theta = 45^\circ$ in standard position, we see that its terminal side does not lie along any of the coordinate axes. We let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. By definition, $x = \cos(45^\circ)$ and $y = \sin(45^\circ)$. If we drop a perpendicular line segment from *P* to the *x*-axis, we obtain a 45°, 45°, 90° right isosceles triangle whose legs have lengths *x* and *y* units. From the properties of isosceles triangles, it follows that $y = x$.

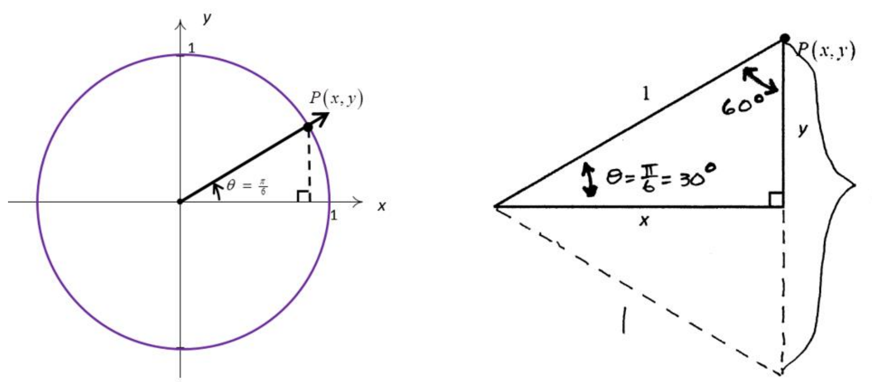


$P(x, y)$ lies on the Unit Circle, so $x^2 + y^2 = 1$. Substituting $y = x$ into this equation yields $2x^2 = 1$, or

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}.$$

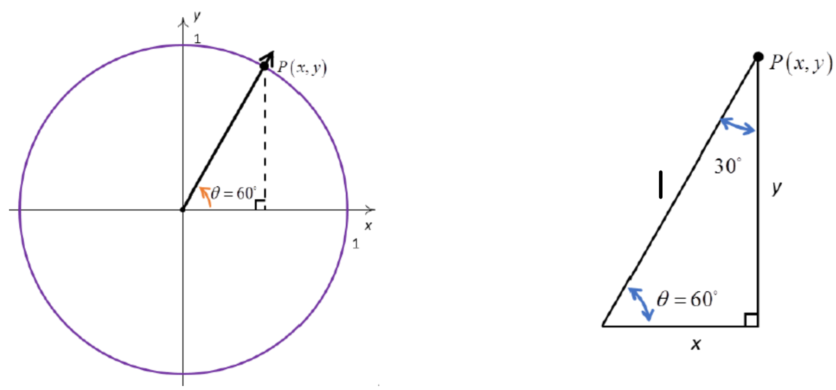
Now, $P(x, y)$ lies in the first quadrant where $x > 0$, so $x = \frac{\sqrt{2}}{2}$. Since $y = x$, we can also conclude that $y = \frac{\sqrt{2}}{2}$. Finally, we have $\cos(45^\circ) = \frac{\sqrt{2}}{2}$ and $\sin(45^\circ) = \frac{\sqrt{2}}{2}$.

4. As before, the terminal side does not lie on any of the coordinate axes so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle,we drop a perpendicular line segment from P to the x-axis to form a 30°, 60°, 90° right triangle.



Noting that we have half of an equilateral triangle with sides of length 1, we find $y = \frac{1}{2}$, so $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Since $P(x, y)$ lies on the Unit Circle, we substitute $y = \frac{1}{2}$ into $x^2 + y^2 = 1$ to get $x^2 = \frac{3}{4}$, or $x = \pm\frac{\sqrt{3}}{2}$. In the first quadrant $x > 0$, so $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

5. Plotting $\theta = 60^\circ$ in standard position, we find θ is not a quadrantal angle and set about using a triangle approach. Once again, we get a 30°, 60°, 90° right triangle and, after computations similar to part 4 of this example, we find $x = \cos(60^\circ) = \frac{1}{2}$ and $y = \sin(60^\circ) = \frac{\sqrt{3}}{2}$.



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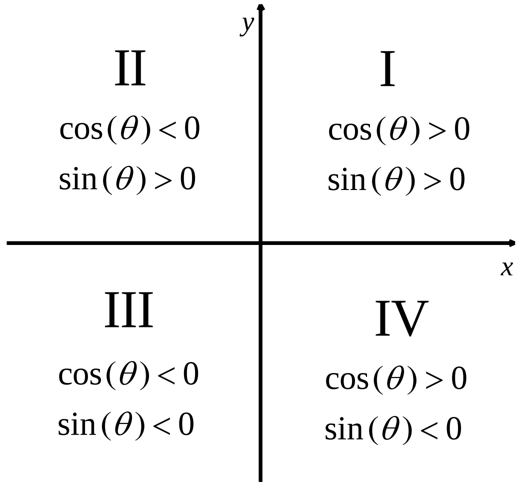
It is not by accident that the last three angles in **Example 2.2.2** are 30°, 45° and 60° (or $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$, respectively). In **Section 2.1** we used right triangles to obtain these same cosine and sine values for 30°, 45° and 60°. In this section, the Unit Circle approach to calculating trigonometric function values allows us to expand the domain imposed by acute angles within a right triangle to include negative angles, and other angles outside the interval $(0^\circ, 90^\circ)$.

Knowing which quadrant an angle θ terminates in will help us determine whether $\cos(\theta)$ and $\sin(\theta)$ are positive or negative, as indicated below.



Sign of Cosine and Sine in Each Quadrant

The Pythagorean Identity

In **Example 2.2.2**, it was quite easy to find the cosine and sine of the quadrantal angles, but for non-quadrantal angles, the task is more involved. In these latter cases, we made good use of the fact that the point $P(x,y)=(\cos(\theta),\sin(\theta))$ lies on the Unit Circle, $x^2+y^2=1$. If we substitute $x=\cos(\theta)$ and $y=\sin(\theta)$ into $x^2+y^2=1$, we get the identity $(\cos(\theta))^2+(\sin(\theta))^2=1$. An unfortunate convention, from a function notation perspective, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. We will follow this convention. Thus, our identity results in the following theorem, one of the most important results in trigonometry.

Theorem 2.1. The Pythagorean Identity: For any angle θ, $\cos^2(\theta)+\sin^2(\theta)=1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the distance formula and the equation for a circle are ultimately derived. The word 'identity' reminds us that, regardless of the angle θ, the equation in **Theorem 2.1** is always true. If one of $\cos(\theta)$ or $\sin(\theta)$ is known, **Theorem 2.1** can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, we can remove the ambiguity of the sign and completely determine the missing value as the next example illustrates.

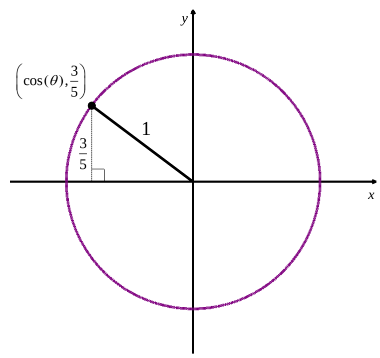
Example 2.2.3. Using the given information about θ, find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta)=\frac{3}{5}$, find $\cos(\theta)$.

2. If $\pi<\theta<\frac{3\pi}{2}$ with $\cos(\theta)=-\frac{\sqrt{5}}{5}$, find $\sin(\theta)$.

3. If $\sin(\theta)=1$, find $\cos(\theta)$.

Solution.



1. When we substitute $\sin(\theta)=\frac{3}{5}$ into the Pythagorean identity, $\cos^2(\theta)+\sin^2(\theta)=1$, we obtain $\cos^2(\theta)+\frac{9}{25}=1$. Solving, we find $\cos(\theta)=\pm\frac{4}{5}$. Since θ is a Quadrant II angle, its terminal side, when plotted in standard position, lies in Quadrant II. In Quadrant II, the x-coordinates are negative. Hence, $\cos(\theta)=-\frac{4}{5}$.

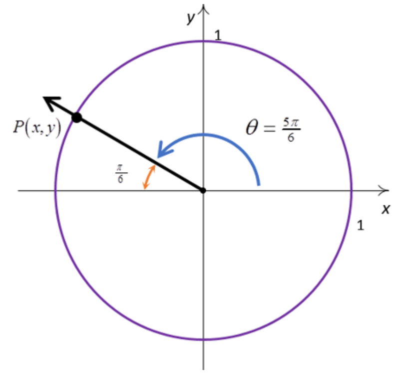
2. Substituting $\cos(\theta)=-\frac{\sqrt{5}}{5}$ into $\cos^2(\theta)+\sin^2(\theta)=1$ gives $\sin(\theta)=\pm\frac{2}{\sqrt{5}}=\pm\frac{2\sqrt{5}}{5}$. Since we are given that $\pi<\theta<\frac{3\pi}{2}$, we know θ is a Quadrant III angle. Since x and y are negative in Quadrant III, both sine and cosine are negative in Quadrant III. Hence, we conclude $\sin(\theta)=-\frac{2\sqrt{5}}{5}$.

3. When we substitute $\sin(\theta)=1$ into $\cos^2(\theta)+\sin^2(\theta)=1$, we find $\cos(\theta)=0$.

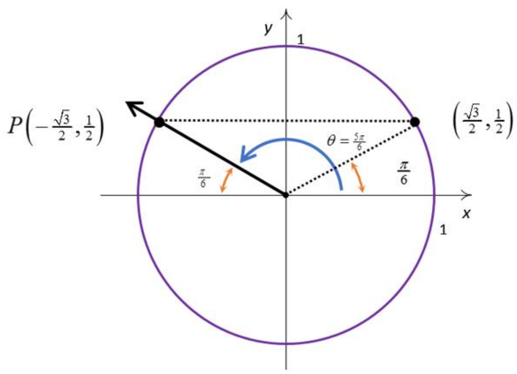
□

Symmetry

Another tool which helps immensely in determining cosines and sines of angles is the symmetry inherent in the Unit Circle. Suppose, for instance, we wish to know the cosine and sine of $\theta = \frac{5\pi}{6}$. We plot θ in standard position and, as usual, let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. Note that the terminal side of θ lies $\frac{\pi}{6}$ radians short of one half revolution.



In **Example 2.2.2**, we determined that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. This means that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



From the figure, it is clear that the point $P(x, y)$ can be obtained by reflecting the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ about the *y*-axis. Hence, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

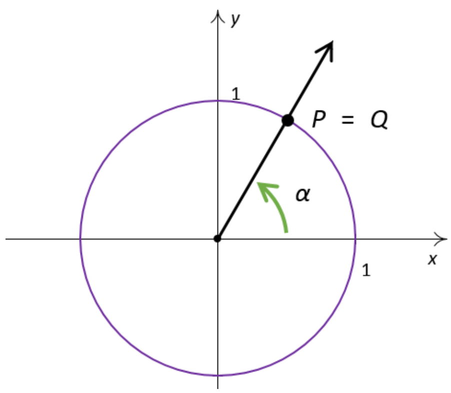
Reference Angles

In the above scenario, angle $\frac{\pi}{6}$ is called the reference angle for the angle $\frac{5\pi}{6}$.

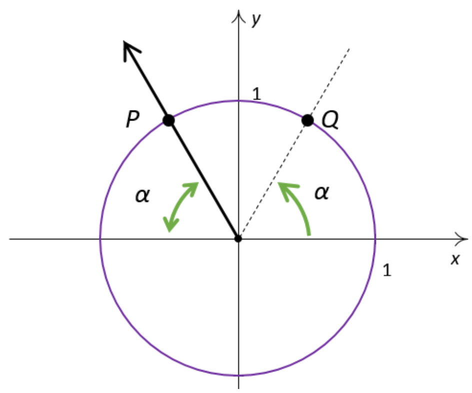
In general, for a non-quadrantal angle θ, the **reference angle** for θ (usually denoted α) is the *acute* angle made between the terminal side of θ and the x-axis.

- If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* x-axis.
- If θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* x-axis.

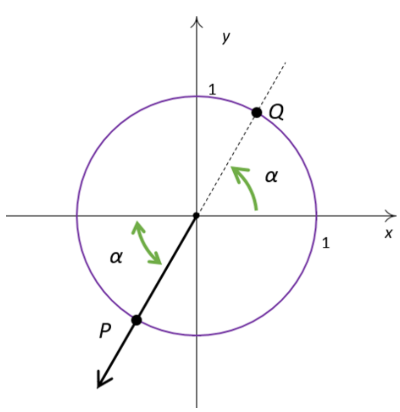
If we let P denote the point $(\cos(\theta), \sin(\theta))$, then P lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the x-axis, y-axis and origin, regardless of where the terminal side of θ lies, there is a point Q symmetric with P which determines θ's reference angle, α, as seen in the following illustration.



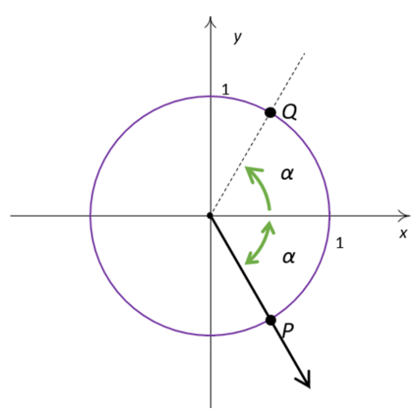
Reference angle α for a Quadrant I angle



Reference angle α for a Quadrant II angle



Reference angle α for a Quadrant III angle

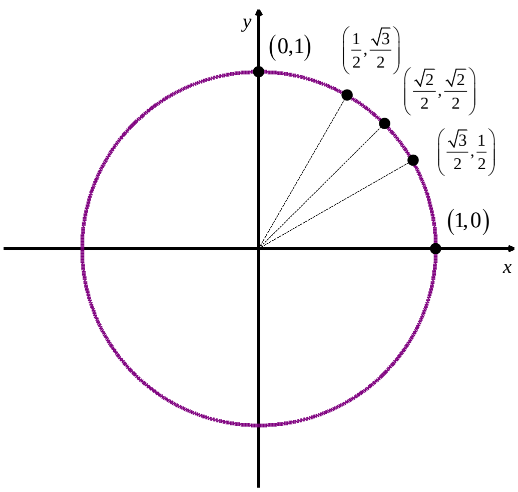


Reference angle α for a Quadrant IV angle

We have just outlined the proof of the following theorem.

Theorem 2.2. Reference Angle Theorem: Suppose α is the reference angle for θ. Then $\cos(\theta) = \pm\cos(\alpha)$ and $\sin(\theta) = \pm\sin(\alpha)$, where the sign, + or –, is determined by the quadrant in which the terminal side of θ lies.

In light of **Theorem 2.2**, it pays to know the cosine and sine values for certain common angles. In the following table, we summarize the values which we consider essential.



Cosine and Sine Values of Common Angles

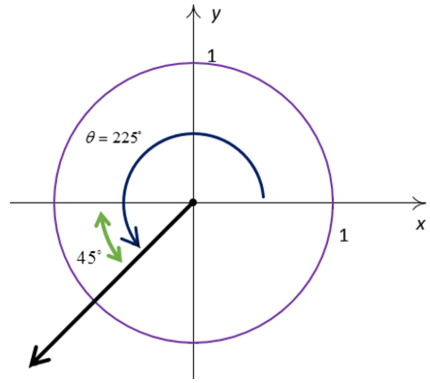
θ degrees	θ radians	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1

Example 2.2.4. Find the cosine and sine of the following angles.

1. $\theta = 225^\circ$
2. $\theta = \frac{11\pi}{6}$
3. $\theta = -\frac{5\pi}{4}$
4. $\theta = \frac{7\pi}{3}$

Solution.

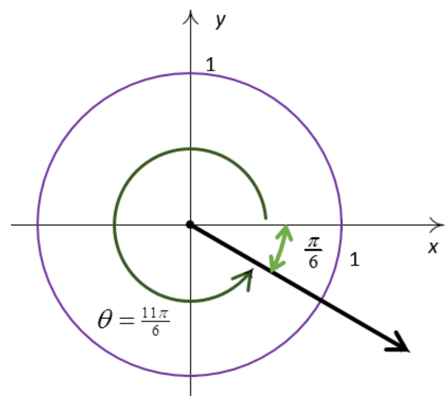
1. We begin by plotting $\theta = 225^\circ$ in standard position and find its terminal side overshoots the negative *x*-axis to land in Quadrant III. Hence, we obtain a reference angle α by subtracting:



$$\begin{aligned}\alpha &= \theta - 180^\circ \\ &= 225^\circ - 180^\circ \\ &= 45^\circ.\end{aligned}$$

Since θ is a Quadrant III angle, $\cos(\theta) < 0$ and $\sin(\theta) < 0$. The Reference Angle Theorem yields: $\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$ and $\sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$.

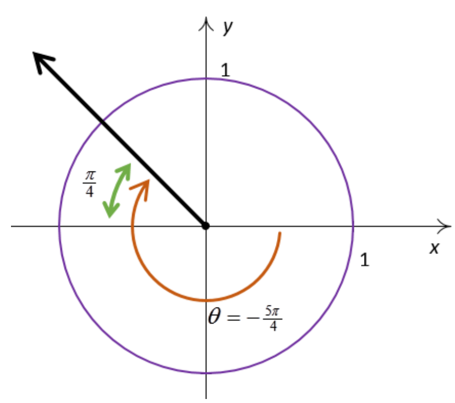
2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive *x*-axis. To find the reference angle α, we subtract:



$$\begin{aligned}\alpha &= 2\pi - \theta \\ &= 2\pi - \tfrac{11\pi}{6} \\ &= \tfrac{\pi}{6}.\end{aligned}$$

Since θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

3. To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive *x*-axis. The terminal side of θ, therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative *x*-

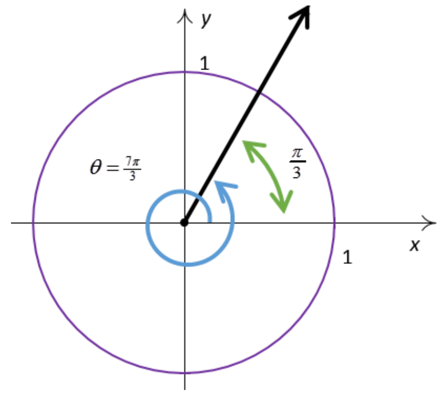


axis. Since θ is a Quadrant II angle, the Reference Angle Theorem gives:

$$\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \text{ and } \sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

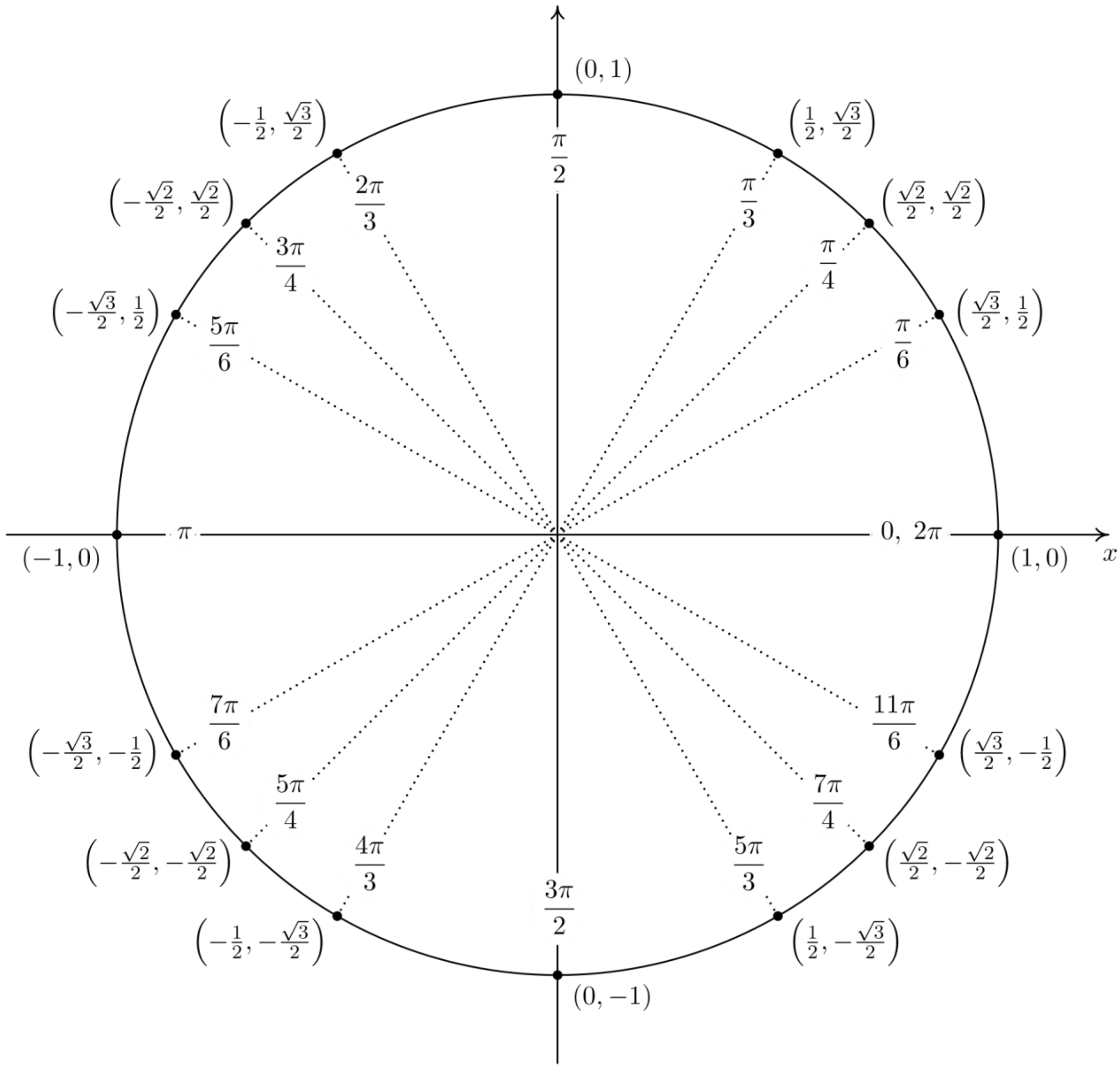
4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Since θ and α are coterminal,

$$\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } \sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$



□

The reader may have noticed that, when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle. Those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.



Important Points on the Unit Circle

The next example summarizes all of the important ideas discussed in this section.

Example 2.2.5. Suppose α is an acute angle with $\cos(\alpha) = \dfrac{5}{13}$.

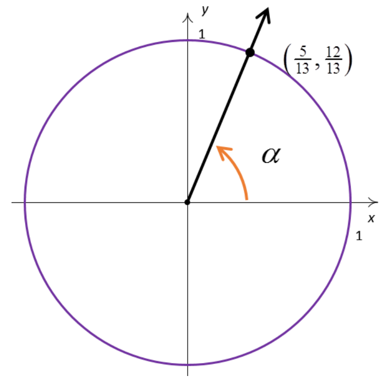
1. Find $\sin(\alpha)$ and use this to plot α in standard position.

2. Find the sine and cosine of the following angles.

(a) $\theta = \pi + \alpha$ (b) $\theta = 2\pi - \alpha$ (c) $\theta = 3\pi - \alpha$ (d) $\theta = \dfrac{\pi}{2} + \alpha$

Solution.

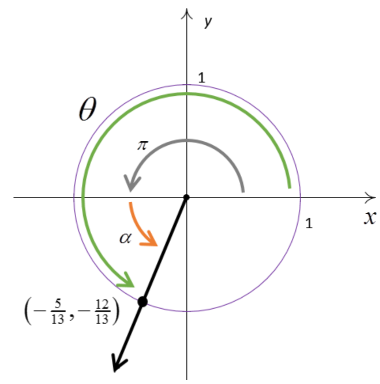
1. Proceeding as in Example 2.2.3, we substitute $\cos(\alpha) = \frac{5}{13}$ into $\cos^2(\alpha) + \sin^2(\alpha) = 1$ and find $\sin(\alpha) = \pm\frac{12}{13}$. Since α is an acute (and therefore Quadrant I) angle, $\sin(\alpha)$ is positive. Hence, $\sin(\alpha) = \frac{12}{13}$. To plot α in standard position, we begin our rotation from the positive x-axis to the ray which contains the point $(\cos(\alpha), \sin(\alpha)) = \left(\frac{5}{13}, \frac{12}{13}\right)$.



2. (a) To find the cosine and sine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotation: a rotation of π radians followed by a rotation of α radians.[16] We see that α is the reference angle for θ, so by the Reference Angle Theorem,

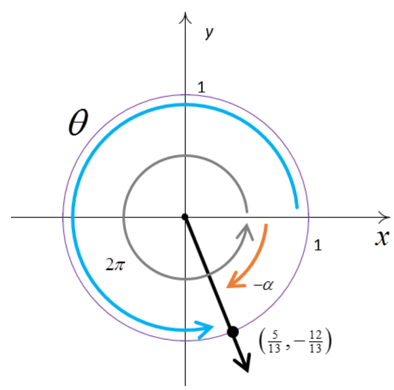
$$\cos(\theta) = \pm\cos(\alpha) = \pm\frac{5}{13} \text{ and } \sin(\theta) = \pm\sin(\alpha) = \pm\frac{12}{13}.$$

Since the terminal side of θ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative. Hence, $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.



(b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a *clockwise* revolution, or 'backing up' of α radians. We see that α is θ's reference angle, and since θ is a Quadrant IV angle, the Reference Angle Theorem gives:

$$\cos(\theta) = \frac{5}{13} \text{ and } \sin(\theta) = -\frac{12}{13}.$$



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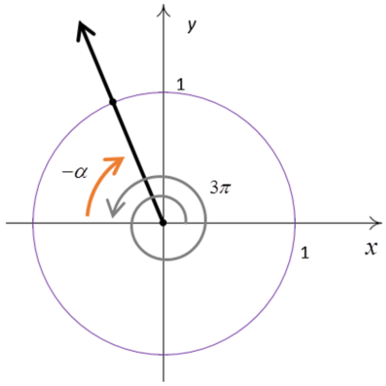
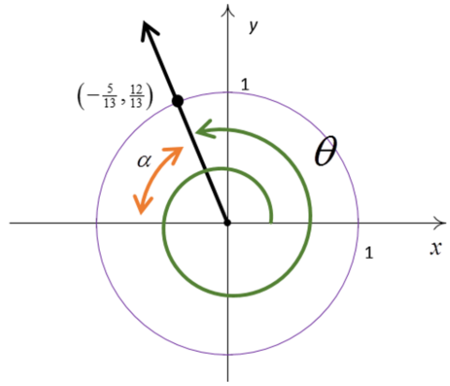
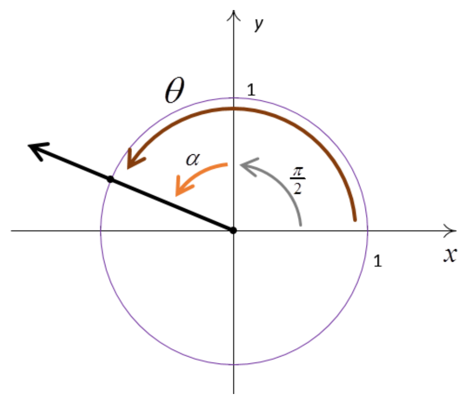
[16] Since $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations given here. Try it!

(c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so that when we 'back up' α radians, we end up in Quadrant II. Using the Reference Angle Theorem, we get $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = \frac{12}{13}$.

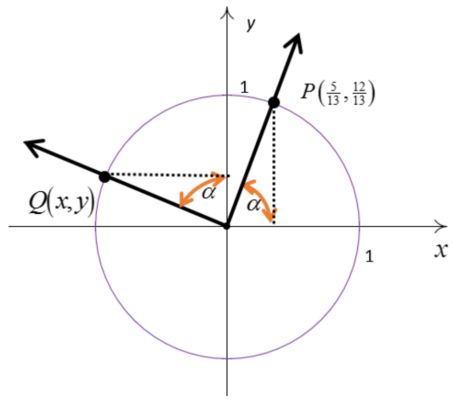




(d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is not α, so the Reference Angle Theorem is not immediately applicable. (It's important that you see why this is the case. Take a moment to think about this before reading on.)

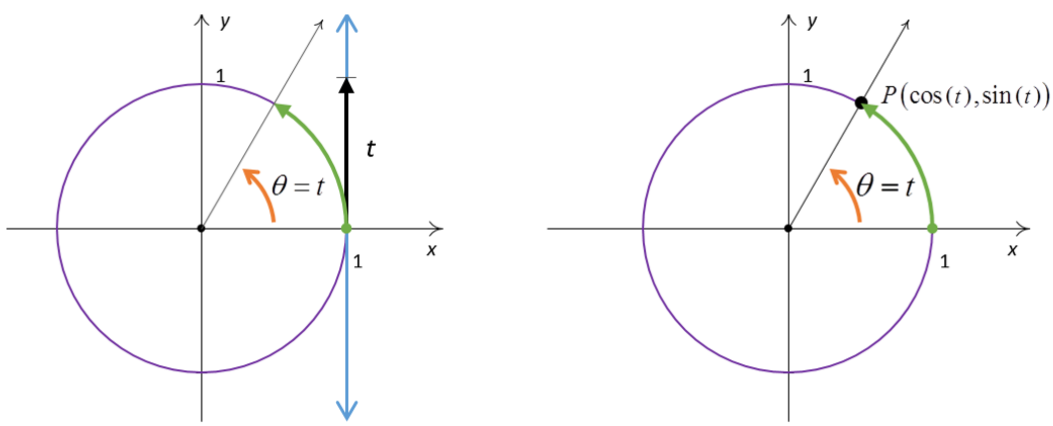


Let $Q(x, y)$ be the point on the terminal side of θ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph α in standard position, we use the fact that equal angles subtend equal chords to show that the dotted lines in the following figure are equal. Hence, $x = -\cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$.



□

We close this section by noting that we can easily extend the functions cosine and sine to real numbers by identifying a real number *t* with the angle $\theta = t$ radians. Using this identification, we define $\cos(t) = \cos(\theta)$ and $\sin(t) = \sin(\theta)$. In practice this means expressions like $\cos(\pi)$ and $\sin(2)$ can be found by regarding the inputs as angles in radian measure or real numbers; the choice is the reader's. If we trace the identification of real numbers *t* with angles θ in radian measure to its roots, as explained at the beginning of this section, we can spell out this correspondence more precisely. For each real number *t*, we associate an oriented arc *t* units in length with initial point $(1, 0)$ and endpoint $P(\cos(t), \sin(t))$.



2.2 Exercises

In Exercises 1 – 5, sketch the oriented arc on the Unit Circle which corresponds to the given real number.

1. $t = \frac{5\pi}{6}$ 2. $t = -\pi$ 3. $t = 6$ 4. $t = -2$ 5. $t = 12$

In Exercises 6 – 9, use the given sign of the cosine and sine functions to find the quadrant in which the terminal point determined by *t* lies.

6. $\cos(t) < 0$ and $\sin(t) < 0$ 7. $\cos(t) > 0$ and $\sin(t) > 0$

8. $\cos(t) < 0$ and $\sin(t) > 0$ 9. $\cos(t) > 0$ and $\sin(t) < 0$

10. Use the numbers 0, 1, 2, 3 and 4 to complete the following table of cosine and sine values for common angles. (This exercise serves as a memory tool for remembering these values.)

Θ	$\cos(\theta)$	$\sin(\theta)$
0	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$
$\frac{\pi}{2}$	$\frac{\sqrt{\square}}{2}$	$\frac{\sqrt{\square}}{2}$

In Exercises 11 – 30, find the exact value of the cosine and sine of the given angle.

11. $\theta = 0$ 12. $\theta = \frac{\pi}{4}$ 13. $\theta = \frac{\pi}{3}$ 14. $\theta = \frac{\pi}{2}$

15. $\theta = \frac{2\pi}{3}$ 16. $\theta = \frac{3\pi}{4}$ 17. $\theta = \pi$ 18. $\theta = \frac{7\pi}{6}$

19. $\theta = \frac{5\pi}{4}$ 20. $\theta = \frac{4\pi}{3}$ 21. $\theta = \frac{3\pi}{2}$ 22. $\theta = \frac{5\pi}{3}$

23. $\theta = \frac{7\pi}{4}$ 24. $\theta = \frac{23\pi}{6}$ 25. $\theta = -\frac{13\pi}{2}$ 26. $\theta = -\frac{43\pi}{6}$

27. $\theta = -\frac{3\pi}{4}$ 28. $\theta = -\frac{\pi}{6}$ 29. $\theta = \frac{10\pi}{3}$ 30. $\theta = 117\pi$

In Exercises 31 – 40, use the results developed throughout the section to find the requested value.

31. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?

32. If $\cos(\theta) = \frac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?

33. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?

34. If $\cos(\theta) = -\frac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?

35. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?

36. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?

37. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?

38. If $\cos(\theta) = \frac{\sqrt{10}}{10}$ and $2\pi < \theta < \frac{5\pi}{2}$, what is $\sin(\theta)$?

39. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, what is $\cos(\theta)$?

40. If $\cos(\theta) = -0.98$ and $\frac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

2.3 The Six Circular Functions

Learning Objectives

In this section you will:

- **Determine the values of the six circular functions from the coordinates of a point on the Unit Circle.**
- **Learn and apply the reciprocal and quotient identities.**
- **Learn and apply the Generalized Reference Angle Theorem.**
- **Find angles that satisfy circular function equations.**

In this section, we extend the definition of cosine and sine as points on the Unit Circle to include the remaining four circular functions: tangent, cotangent, secant and cosecant.

The Circular Functions

The Circular Functions: Suppose θ is an angle plotted in standard position and $P(x, y)$ is the point on the terminal side of θ which lies on the Unit Circle. The circular functions are defined as follows.

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \frac{y}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \frac{1}{y}$, provided $y \neq 0$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \frac{1}{x}$, provided $x \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \frac{x}{y}$, provided $y \neq 0$.

In Section 2.2, we defined $\cos(\theta)$ and $\sin(\theta)$ for angles θ using the coordinate values of points on the Unit Circle. As such, these functions earn the moniker **circular functions**.[17]

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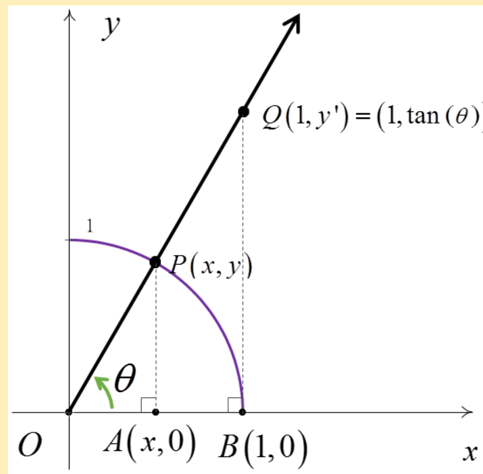
[17] In Section 2.1 we also showed cosine and sine to be functions of an angle residing in a right triangle so we could just as easily call them *trigonometric* functions. You will find that we do indeed use the phrase 'trigonometric function' interchangeably with 'circular function'.

Historical Note: While we left the history of the name 'sine' as an interesting research project in Section 2.2, the names 'tangent' and 'secant' can be explained using the diagram below. Consider the acute angle θ in standard position. Let $P(x, y)$ denote, as usual, the point on the terminal side of θ which lies on the Unit Circle and let $Q(1, y')$ denote the point on the terminal side of θ which lies on the vertical line $x = 1$.

The word 'tangent' comes from the Latin meaning 'to touch'. The line $x = 1$ is a *tangent* line to the Unit Circle since it intersects, or touches, the circle at only one point, namely $(1, 0)$. Dropping perpendiculars from P and Q creates the pair of similar triangles ΔOPA and ΔOQB. Thus $\frac{y'}{y} = \frac{1}{x}$ which gives $y' = \frac{y}{x} = \tan(\theta)$, where this last equality comes from the definition of the tangent of θ.



We have just shown that for acute angles θ, $\tan(\theta)$ is the y-coordinate of the point on the terminal side of θ which lies on the *tangent* line $x = 1$.

The word 'secant' means 'to cut'. A *secant* line is any line that cuts through a circle at two points. The line containing the terminal side of θ is a secant line that intersects the Unit Circle in Quadrants I and III. With the point P lying on the Unit Circle, the length of the hypotenuse of ΔOPA is 1. If we let h denote the length of the hypotenuse of ΔOQB, we have, from similar triangles, that $\frac{h}{1} = \frac{1}{x}$ or $h = \frac{1}{x} = \sec(\theta)$. Hence, for an acute angle θ, $\sec(\theta)$ is the length of the line segment which lies on the secant line determined by the terminal side of θ and 'cuts off' the tangent line $x = 1$.

Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for calculus which we'll explore in the Exercises.

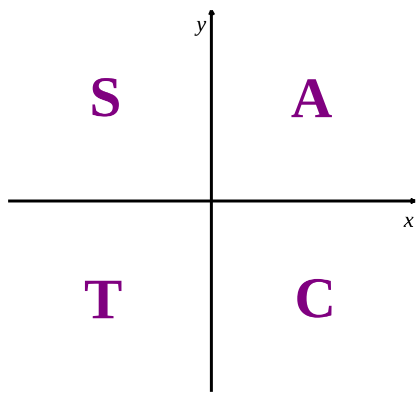
Of the six circular functions, only cosine and sine are defined for all angles. Since $\cos(\theta) = x$ and $\sin(\theta) = y$ in their definitions as circular functions, it is customary to rephrase the remaining four circular functions in terms of cosine and sine. The following theorem is a result of simply replacing x with $\cos(\theta)$ and y with $\sin(\theta)$ in the definitions presented at the beginning of this section.

Reciprocal and Quotient Identities

Theorem 2.3. Reciprocal and Quotient Identities:

- $\tan(\theta) = \dfrac{y}{x} = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$ then $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{x}{y} = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$ then $\cot(\theta)$ is undefined.
- $\sec(\theta) = \dfrac{1}{x} = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$ then $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{y} = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$ then $\csc(\theta)$ is undefined.

Before using **Theorem 2.3** in an example, the following mnemonic may help with remembering the signs of the trigonometric functions in each quadrant. We assign the first letter of each word in the phrase "**A**ll **S**tudents **T**ake **C**alculus" to Quadrants I, II, III and IV, respectively. Note that cosine, sine and tangent are **A**ll positive in Quadrant I, the **S**ine alone is positive in Quadrant II, then **T**angent alone is positive in Quadrant III and the **C**osine alone is positive in Quadrant IV.



It is high time for an example.

Example 2.3.1. Find the indicated value, if it exists.

1. $\sec(60^\circ)$
2. $\csc\left(\dfrac{7\pi}{4}\right)$
3. $\cot(3)$

4. $\tan(\theta)$, where θ is any angle coterminal with $\dfrac{3\pi}{2}$.

5. $\cos(\theta)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle.

6. $\sin(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \dfrac{3\pi}{2}$.

Solution.

1. From **Theorem 2.3**, the reciprocal identity for secant will help us out here.

$$\begin{aligned}\sec(60^\circ) &= \frac{1}{\cos(60^\circ)} \\ &= \frac{1}{\left(\frac{1}{2}\right)} \\ &= 2\end{aligned}$$

2. We apply the reciprocal identity for cosecant and note that $\sin\left(\dfrac{7\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$.

$$\begin{aligned}\csc\left(\frac{7\pi}{4}\right) &= \frac{1}{\sin\left(\frac{7\pi}{4}\right)} \\ &= \frac{1}{\left(-\frac{\sqrt{2}}{2}\right)} \\ &= -\sqrt{2}\end{aligned}$$

3. Since $\theta = 3$ radians is not one of the common angles from **Section 2.2**, we resort to the calculator for a decimal approximation. We use the quotient identity for cotangent and check that our calculator is in radian mode.

$$\begin{aligned}\cot(3) &= \frac{\cos(3)}{\sin(3)} \\ &\approx -7.015\end{aligned}$$

Noting that $\cot(\theta) = \dfrac{1}{\tan(\theta)}$, this problem could also be solved as follows.

$$\cot(3) = \frac{1}{\tan(3)}$$
$$\approx -7.015$$

4. If θ is coterminal with $\frac{3\pi}{2}$, then $\cos(\theta) = \cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$.

   Attempting to compute $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ results in $\frac{-1}{0}$, so $\tan(\theta)$ is undefined.

5. We are given that $\csc(\theta) = -\sqrt{5}$. From $\csc(\theta) = \frac{1}{\sin(\theta)}$, it follows that $\sin(\theta) = -\frac{1}{\sqrt{5}}$.

   Then we have the following.

$$\cos^2(\theta) + \left(-\frac{1}{\sqrt{5}}\right)^2 = 1 \qquad \textbf{Pythagorean identity}$$
$$\cos^2(\theta) = \frac{4}{5}$$
$$\cos(\theta) = \pm\frac{2}{\sqrt{5}}$$
$$\cos(\theta) = \frac{2}{\sqrt{5}} \qquad \textbf{cos}(\theta)\textbf{>0 in Quadrant IV}$$
$$\cos(\theta) = \frac{2\sqrt{5}}{5}$$

6. It is given that $\tan(\theta) = 3$. From the quotient identity for tangent, we know $\frac{\sin(\theta)}{\cos(\theta)} = 3$. Be careful! We can NOT assume any values for $\sin(\theta)$ and $\cos(\theta)$. We CAN assume that $\sin(\theta) = 3\cos(\theta)$.

$$\begin{aligned}
\sin(\theta) &= 3\cos(\theta) \\
\frac{1}{3}\sin(\theta) &= \cos(\theta) \\
\left(\frac{1}{3}\sin(\theta)\right)^2 + \sin^2(\theta) &= 1 \qquad \textbf{Pythagorean identity} \\
\frac{1}{9}\sin^2(\theta) + \sin^2(\theta) &= 1 \\
\frac{10}{9}\sin^2(\theta) &= 1 \\
\sin^2(\theta) &= \frac{9}{10} \\
\sin(\theta) &= \pm\frac{3}{\sqrt{10}} \\
\sin(\theta) &= -\frac{3}{\sqrt{10}} \qquad \textbf{sin}(\theta)<\textbf{0 in Quadrant III} \\
\sin(\theta) &= -\frac{3\sqrt{10}}{10}
\end{aligned}$$

□

While the reciprocal and quotient identities presented in **Theorem 2.3** allow us to always convert problems involving tangent, cotangent, secant and cosecant to problems involving cosine and sine, it is not always convenient to do so.[18] The tangent and cotangent values of the common angles are summarized in the following chart.

[18] As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to cosines and sines. However, when solving for tangent or cotangent, we usually stick with what we're dealt.

Tangent and Cotangent Values of Common Angles

θ degrees	θ radians	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$
$0°$	0	1	0	0	undefined
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	0	1	undefined	0

Finding Angles that Satisfy Cosine and Sine Equations

Our next example asks us to solve some very basic trigonometric equations.[19]

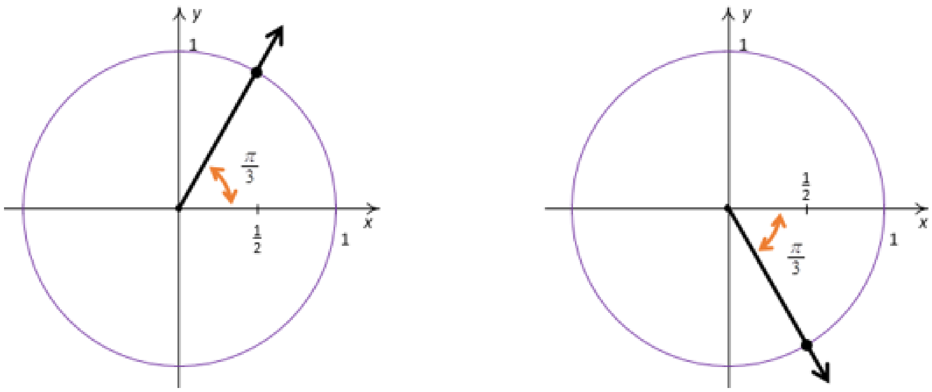
Example 2.3.2. Find all of the angles which satisfy the given equation.

1. $\cos(\theta)=\frac{1}{2}$
2. $\sin(\theta)=-\frac{1}{2}$
3. $\cos(\theta)=0$

Solution. Since there is no context in the problem to indicate whether to use degrees or radians, we will default to using radian measure in each of these problems. This choice will be justified later in the text when we study what is known as Analytic Trigonometry. In those sections to come, radian measure will be the *only* appropriate angle measure so it is worth the time to become fluent in radians now.

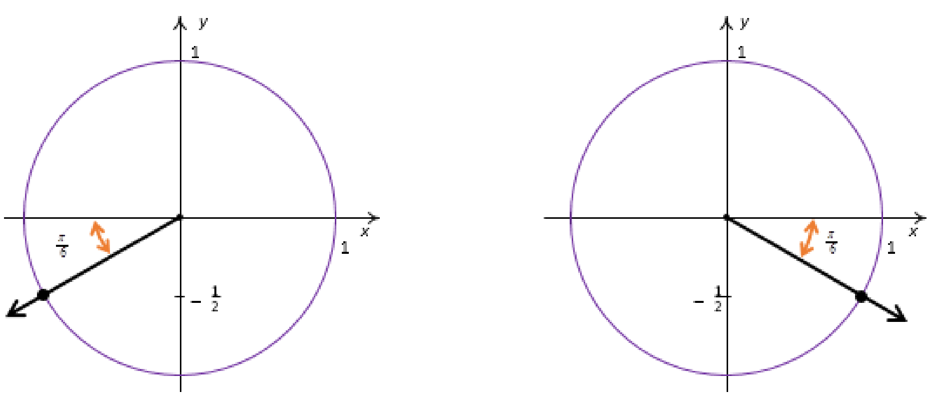
1. If $\cos(\theta)=\frac{1}{2}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x=\frac{1}{2}$. This means θ is a Quadrant I or IV angle with reference angle $\frac{\pi}{3}$.

[19] We will study trigonometric equations more formally in Chapter 6. Enjoy these relatively straightforward exercises while they last!



One solution in Quadrant I is $\theta = \frac{\pi}{3}$, and since all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$, we find $\theta = \frac{\pi}{3} + 2\pi k$ for integers k.[20] Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ is $\frac{5\pi}{3}$, so our answer in this quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

2. If $\sin(\theta) = -\frac{1}{2}$, then when θ is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine θ is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$.

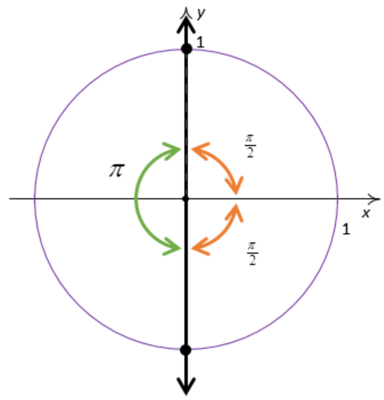


[20] Recall in Section 1.2, two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π. Hence to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k = 0, \pm 1, \pm 2, \cdots$.

In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π: $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k.

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the y-axis.

While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle, we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers k. While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers k. (Can you see why this works from the diagram?)



□

One of the key items to take from **Example 2.3.2** is that, in general, solutions to trigonometric equations consist of infinitely many answers. To get a feel for these answers, the reader is encouraged to follow our mantra 'When in doubt, write it out!' This is especially important when checking answers to the Exercises. For example, in number 2, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could have been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

Finding Angles that Satisfy Other Circular Function Equations

Before determining angles in equations of the other four circular functions, we introduce the Generalized Reference Angle Theorem. This theorem results from coupling the reciprocal and quotient identities, **Theorem 2.3**, with the Reference Angle Theorem, **Theorem 2.2**.

Theorem 2.4. Generalized Reference Angle Theorem: The values of the circular functions of an angle, if they exist, are the same, up to a sign, as the corresponding circular functions of the reference angle.

More specifically, if α is the reference angle for θ, then $\sin(\theta) = \pm\sin(\alpha)$, $\cos(\theta) = \pm\cos(\alpha)$, $\tan(\theta) = \pm\tan(\alpha)$, $\csc(\theta) = \pm\csc(\alpha)$, $\sec(\theta) = \pm\sec(\alpha)$ and $\cot(\theta) = \pm\cot(\alpha)$. The sign, + or –, is determined by the quadrant in which the terminal side of θ lies.

We put **Theorem 2.4** to good use in the following example.

Example 2.3.3. Find all angles which satisfy the given equation.

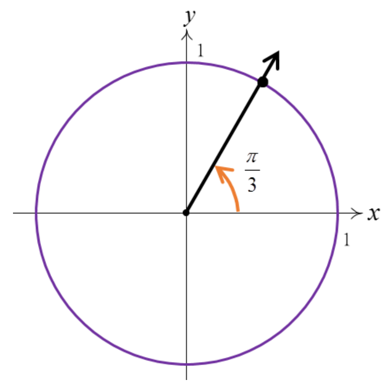
1. $\sec(\theta) = 2$
2. $\tan(\theta) = \sqrt{3}$
3. $\cot(\theta) = -1$

Solution.

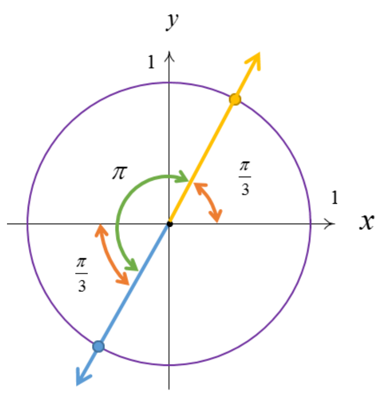
1. To solve $\sec(\theta) = 2$, we convert to a cosine and get $\dfrac{1}{\cos(\theta)} = 2$, or $\cos(\theta) = \dfrac{1}{2}$. This is the same equation we solved in **Example 2.3.2**, number 1, so we know the answer is $\theta = \dfrac{\pi}{3} + 2\pi k$ or $\theta = \dfrac{5\pi}{3} + 2\pi k$ for integers k.

2. Noting that $\tan\left(\dfrac{\pi}{3}\right) = \dfrac{\sin(\pi/3)}{\cos(\pi/3)}$, we see $\tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$. According to **Theorem 2.4**, we know the solutions to $\tan(\theta) = \sqrt{3}$ must, therefore, have a reference angle of $\dfrac{\pi}{3}$. Our next task is to determine in which quadrants the solutions to this equation lie.

Since tangent is defined as the ratio $\frac{y}{x}$ of points (x, y) on the Unit Circle with $x \neq 0$, tangent is positive when *x* and *y* have the same sign (i.e., when they are both positive or both negative.) This happens in Quadrants I and III. In Quadrant I we get the solutions $\theta = \frac{\pi}{3} + 2\pi k$ for integers *k*, and for Quadrant III we get $\theta = \frac{4\pi}{3} + 2\pi k$ for integers *k*.



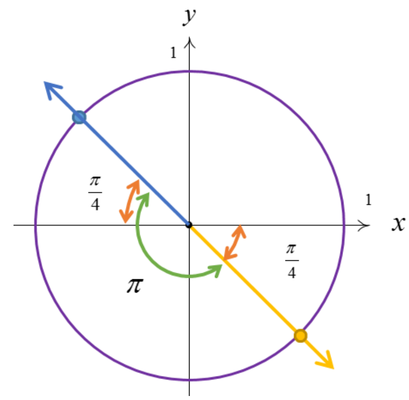
While these descriptions of the solutions are correct, they can be combined as $\theta = \frac{\pi}{3} + \pi k$ for integers *k*. The latter form of the solution is best understood looking at the geometry of the situation in the following diagram.[21]



3. We see that $\frac{\pi}{4}$ has a cotangent of 1, which means the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$.

[21] See Example 2.3.2, number 3, for another example of this kind of simplification of the solution.

To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point (x, y) on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then x and y must have different signs (i.e., one positive and one negative.) Hence, our solutions lie in Quadrants II and IV. Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV we get $\theta = \frac{7\pi}{4} + 2\pi k$, for integers k.



Can these be combined? Indeed they can! One such way to capture all the solutions is $\theta = \frac{3\pi}{4} + \pi k$ for integers k.

□

Suppose we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. As we have already mentioned, the distinction between t as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the exact same manner[22] as we did in **Example 2.3.2** number 2. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers k. As we progress in our study of the trigonometric functions, keep in mind that any properties developed which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

[22] Well, to be pedantic, we would be technically using reference numbers or reference arcs instead of reference angles, but the idea is the same.

2.3 Exercises

In Exercises 1 – 20, find the exact value or state that it is undefined.

1. $\tan\left(\frac{\pi}{4}\right)$
2. $\sec\left(\frac{\pi}{6}\right)$
3. $\csc\left(\frac{5\pi}{6}\right)$
4. $\cot\left(\frac{4\pi}{3}\right)$
5. $\tan\left(-\frac{11\pi}{6}\right)$
6. $\sec\left(-\frac{3\pi}{2}\right)$
7. $\csc\left(-\frac{\pi}{3}\right)$
8. $\cot\left(\frac{13\pi}{2}\right)$
9. $\tan(117\pi)$
10. $\sec\left(-\frac{5\pi}{3}\right)$
11. $\csc(3\pi)$
12. $\cot(-5\pi)$
13. $\tan\left(\frac{31\pi}{2}\right)$
14. $\sec\left(\frac{\pi}{4}\right)$
15. $\csc\left(-\frac{7\pi}{4}\right)$
16. $\cot\left(\frac{7\pi}{6}\right)$
17. $\tan\left(\frac{2\pi}{3}\right)$
18. $\sec(-7\pi)$
19. $\csc\left(\frac{\pi}{2}\right)$
20. $\cot\left(\frac{3\pi}{4}\right)$

In Exercises 21 – 44, find all angles which satisfy the given equation.

21. $\sin(\theta) = \frac{1}{2}$
22. $\cos(\theta) = -\frac{\sqrt{3}}{2}$
23. $\sin(\theta) = 0$
24. $\cos(\theta) = \frac{\sqrt{2}}{2}$
25. $\sin(\theta) = \frac{\sqrt{3}}{2}$
26. $\cos(\theta) = -1$
27. $\sin(\theta) = -1$
28. $\cos(\theta) = \frac{\sqrt{3}}{2}$
29. $\cos(\theta) = -1.001$
30. $\tan(\theta) = \sqrt{3}$
31. $\sec(\theta) = 2$
32. $\csc(\theta) = -1$
33. $\cot(\theta) = \frac{\sqrt{3}}{3}$
34. $\tan(\theta) = 0$
35. $\sec(\theta) = 1$
36. $\csc(\theta) = 2$
37. $\cot(\theta) = 0$
38. $\tan(\theta) = -1$
39. $\sec(\theta) = 0$
40. $\csc(\theta) = -\frac{1}{2}$
41. $\sec(\theta) = -1$
42. $\tan(\theta) = -\sqrt{3}$
43. $\csc(\theta) = -2$
44. $\cot(\theta) = -1$

In Exercises 45 – 61, solve the equation for *t*. Give exact values. (See the comments following Example 2.3.3.)

45. $\cos(t) = 0$

46. $\sin(t) = -\dfrac{\sqrt{2}}{2}$

47. $\cos(t) = 3$

48. $\sin(t) = -\dfrac{1}{2}$

49. $\cos(t) = \dfrac{1}{2}$

50. $\sin(t) = -2$

51. $\cos(t) = 1$

52. $\sin(t) = 1$

53. $\cos(t) = -\dfrac{\sqrt{2}}{2}$

54. $\cot(t) = 1$

55. $\tan(t) = \dfrac{\sqrt{3}}{3}$

56. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$

57. $\csc(t) = 0$

58. $\cot(t) = -\sqrt{3}$

59. $\tan(t) = -\dfrac{\sqrt{3}}{3}$

60. $\sec(t) = \dfrac{2\sqrt{3}}{3}$

61. $\csc(t) = \dfrac{2\sqrt{3}}{3}$

62. Explain why the fact that $\tan(\theta) = 3 = \frac{3}{1}$ does not necessarily mean $\sin(\theta) = 3$ and $\cos(\theta) = 1$. (See the solution to number 6 in Example 2.3.1.)

2.4 Verifying Trigonometric Identities

Learning Objectives

In this section you will:

- **Learn and apply the Pythagorean identities and conjugates.**
- **Simplify trigonometric expressions.**
- **Prove that a trigonometric equation is an identity.**

We have already seen the importance of identities in trigonometry. Our next task is to use the reciprocal and quotient identities found in **Theorem 2.3**, coupled with the Pythagorean identity found in **Theorem 2.1**, to derive the new Pythagorean-like identities for the remaining four circular functions.

The Pythagorean Identities

Theorem 2.1 states that, for any angle θ, $\cos^2(\theta)+\sin^2(\theta)=1$. Through manipulating this identity, we will obtain two alternate versions relating secant and tangent, followed by cosecant and cotangent.

To obtain an identity relating secant and tangent, we start with $\cos^2(\theta)+\sin^2(\theta)=1$ and, assuming $\cos(\theta)\neq 0$, divide both sides of the equation by $\cos^2(\theta)$.

$$\begin{aligned}
\cos^2(\theta)+\sin^2(\theta)&=1\\
\frac{\cos^2(\theta)}{\cos^2(\theta)}+\frac{\sin^2(\theta)}{\cos^2(\theta)}&=\frac{1}{\cos^2(\theta)}\\
1+\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2&=\left(\frac{1}{\cos(\theta)}\right)^2\\
1+\left(\tan(\theta)\right)^2&=\left(\sec(\theta)\right)^2 \quad \textbf{from quotient \& reciprocal identities}
\end{aligned}$$

The result is the Pythagorean identity $1+\tan^2(\theta)=\sec^2(\theta)$.

We next look for an identity relating cosecant and cotangent. We assume that $\sin(\theta)\neq 0$ and divide both sides of $\cos^2(\theta)+\sin^2(\theta)=1$ by $\sin^2(\theta)$.

$$\cos^2(\theta)+\sin^2(\theta)=1$$
$$\frac{\cos^2(\theta)}{\sin^2(\theta)}+\frac{\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)}$$
$$\left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2+1=\left(\frac{1}{\sin(\theta)}\right)^2$$
$$\left(\cot(\theta)\right)^2+1=\left(\csc(\theta)\right)^2$$ **from quotient & reciprocal identities**

Thus, we have $\cot^2(\theta)+1=\csc^2(\theta)$, our third Pythagorean Identity,.

The three Pythagorean Identities, along with some of their other common forms, are summarized in the following theorem.

Theorem 2.5. The Pythagorean Identities:

1. $\cos^2(\theta)+\sin^2(\theta)=1$

   Common Alternate Forms: $1-\sin^2(\theta)=\cos^2(\theta)$ and $1-\cos^2(\theta)=\sin^2(\theta)$

2. $1+\tan^2(\theta)=\sec^2(\theta)$, provided $\cos(\theta)\neq 0$

   Common Alternate Forms: $\sec^2(\theta)-\tan^2(\theta)=1$ and $\sec^2(\theta)-1=\tan^2(\theta)$

3. $1+\cot^2(\theta)=\csc^2(\theta)$, provided $\sin(\theta)\neq 0$

   Common Alternate Forms: $\csc^2(\theta)-\cot^2(\theta)=1$ and $\csc^2(\theta)-1=\cot^2(\theta)$

Trigonometric identities play an important role, both in trigonometry and calculus. We'll use them in this book to find the values of the circular functions of an angle and to solve equations. In calculus, they are needed to rewrite expressions in a format that enables or simplifies integration. In this next example, we make good use of **Theorem 2.3** and **Theorem 2.5**.

Example 2.4.1. Verify the following identities. Assume that all quantities are defined.

1. $\dfrac{1}{\csc(\theta)}=\sin(\theta)$
2. $\tan(\theta)=\sin(\theta)\sec(\theta)$
3. $\left(\sec(\theta)-\tan(\theta)\right)\left(\sec(\theta)+\tan(\theta)\right)=1$
4. $\dfrac{\sec(\theta)}{1-\tan(\theta)}=\dfrac{1}{\cos(\theta)-\sin(\theta)}$
5. $6\sec(\theta)\tan(\theta)=\dfrac{3}{1-\sin(\theta)}-\dfrac{3}{1+\sin(\theta)}$
6. $\dfrac{\sin(\theta)}{1-\cos(\theta)}=\dfrac{1+\cos(\theta)}{\sin(\theta)}$

Solution. In verifying identities, we typically start with the more complicated side of the equation and use known identities to transform it into the other side of the equation.

1. To verify $\frac{1}{\csc(\theta)} = \sin(\theta)$ we start with the left side, using the reciprocal identity for cosecant.

$$\begin{aligned} \frac{1}{\csc(\theta)} &= \frac{1}{\left(\frac{1}{\sin(\theta)}\right)} \\ &= \sin(\theta) \end{aligned}$$

2. We start with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$.

$$\begin{aligned} \sin(\theta)\sec(\theta) &= \sin(\theta)\frac{1}{\cos(\theta)} && \textbf{from reciprocal identity for secant} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta) && \textbf{from quotient identity for tangent} \end{aligned}$$

3. We begin with the left hand side of the equation.

$$\begin{aligned} &\left(\sec(\theta) - \tan(\theta)\right)\left(\sec(\theta) + \tan(\theta)\right) \\ &= \sec^2(\theta) + \sec(\theta)\tan(\theta) - \tan(\theta)\sec(\theta) - \tan^2(\theta) && \textbf{binomial multiplication} \\ &= \sec^2(\theta) - \tan^2(\theta) \\ &= 1 && \textbf{Pythagorean identity} \end{aligned}$$

4. While both sides of the equation contain fractions, the left side affords us more opportunities to use our identities.

$$\begin{aligned}
\frac{\sec(\theta)}{1-\tan(\theta)} &= \frac{\left(\dfrac{1}{\cos(\theta)}\right)}{1-\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)} \\
&= \frac{\left(\dfrac{1}{\cos(\theta)}\right)}{1-\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}\cdot\frac{\cos(\theta)}{\cos(\theta)} \\
&= \frac{\left(\dfrac{\cos(\theta)}{\cos(\theta)}\right)}{\cos(\theta)-\left(\dfrac{\sin(\theta)\cos(\theta)}{\cos(\theta)}\right)} \\
&= \frac{1}{\cos(\theta)-\sin(\theta)}
\end{aligned}$$

5. The right hand side of the equation seems to hold promise. We find a common denominator.

$$\begin{aligned}
\frac{3}{1-\sin(\theta)}-\frac{3}{1+\sin(\theta)} &= \frac{3(1+\sin(\theta))-3(1-\sin(\theta))}{(1-\sin(\theta))(1+\sin(\theta))} \\
&= \frac{3+3\sin(\theta)-3+3\sin(\theta)}{1-\sin^2(\theta)} \\
&= \frac{6\sin(\theta)}{1-\sin^2(\theta)} \\
&= \frac{6\sin(\theta)}{\cos^2(\theta)} && \textbf{Pythagorean identity} \\
&= 6\left(\frac{1}{\cos(\theta)}\right)\left(\frac{\sin(\theta)}{\cos(\theta)}\right) \\
&= 6\sec(\theta)\tan(\theta) && \textbf{reciprocal and quotient identities}
\end{aligned}$$

6. It is debatable which side of the equation is more complicated. One thing which stands out is that the denominator on the left hand side is $1-\cos(\theta)$, while the numerator on the right hand side is $1+\cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1+\cos(\theta)$.

$$\begin{aligned}\frac{\sin(\theta)}{1-\cos(\theta)} &= \frac{\sin(\theta)}{(1-\cos(\theta))}\cdot\frac{(1+\cos(\theta))}{(1+\cos(\theta))}\\ &= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)} && \textbf{binomial multiplication}\\ &= \frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)} && \textbf{Pythagorean identity}\\ &= \frac{\sin(\theta)(1+\cos(\theta))}{\sin(\theta)\sin(\theta)}\\ &= \frac{1+\cos(\theta)}{\sin(\theta)}\end{aligned}$$

□

In the preceding example, number 6, we see that multiplying $1-\cos(\theta)$ by $1+\cos(\theta)$ produces a difference of squares that can be simplified to one term using **Theorem 2.5**. This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1-\sqrt{2}$ by $1+\sqrt{2}$. For this reason, the quantities $(1-\cos(\theta))$ and $(1+\cos(\theta))$ are called 'Pythagorean conjugates'. The following list includes other Pythagorean conjugates.

Pythagorean Conjugates

- $1-\cos(\theta)$ and $1+\cos(\theta)$: $(1-\cos(\theta))(1+\cos(\theta))=1-\cos^2(\theta)=\sin^2(\theta)$
- $1-\sin(\theta)$ and $1+\sin(\theta)$: $(1-\sin(\theta))(1+\sin(\theta))=1-\sin^2(\theta)=\cos^2(\theta)$
- $\sec(\theta)-1$ and $\sec(\theta)+1$: $(\sec(\theta)-1)(\sec(\theta)+1)=\sec^2(\theta)-1=\tan^2(\theta)$
- $\sec(\theta)-\tan(\theta)$ and $\sec(\theta)+\tan(\theta)$:

  $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta))=\sec^2(\theta)-\tan^2(\theta)=1$
- $\csc(\theta)-1$ and $\csc(\theta)+1$: $(\csc(\theta)-1)(\csc(\theta)+1)=\csc^2(\theta)-1=\cot^2(\theta)$
- $\csc(\theta)-\cot(\theta)$ and $\csc(\theta)+\cot(\theta)$:

  $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta))=\csc^2(\theta)-\cot^2(\theta)=1$

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling just to become proficient in the basics. Like many things in life, there is no short-cut here. There is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) follows and ample practice is provided for you in the Exercises.

Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in **Theorem 2.3** to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify any resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean identities in **Theorem 2.5** to exchange sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator and denominator by Pythagorean conjugates in order to take advantage of the Pythagorean identities in **Theorem 2.5**.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Most importantly, keep in mind that we are not solving equations. To verify identities, we choose one side of the identity and work with that side until it matches the other side. Verifying identities is an important skill and we will work with identities again in **Chapter 4**, as more tools become available. Time spent now in developing some proficiency will be useful throughout the course.

2.4 Exercises

In Exercises 1 – 47, verify the identity. Assume that all quantities are defined.

1. $\cos(\theta)\sec(\theta)=1$
2. $\tan(\theta)\cos(\theta)=\sin(\theta)$
3. $\sin(\theta)\csc(\theta)=1$
4. $\tan(\theta)\cot(\theta)=1$
5. $\csc(\theta)\cos(\theta)=\cot(\theta)$
6. $\dfrac{\sin(\theta)}{\cos^2(\theta)}=\sec(\theta)\tan(\theta)$
7. $\dfrac{\cos(\theta)}{\sin^2(\theta)}=\csc(\theta)\cot(\theta)$
8. $\dfrac{1+\sin(\theta)}{\cos(\theta)}=\sec(\theta)+\tan(\theta)$
9. $\dfrac{1-\cos(\theta)}{\sin(\theta)}=\csc(\theta)-\cot(\theta)$
10. $\dfrac{\cos(\theta)}{1-\sin^2(\theta)}=\sec(\theta)$
11. $\dfrac{\sin(\theta)}{1-\cos^2(\theta)}=\csc(\theta)$
12. $\dfrac{\sec(\theta)}{1+\tan^2(\theta)}=\cos(\theta)$
13. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)}=\sin(\theta)$
14. $\dfrac{\tan(\theta)}{\sec^2(\theta)-1}=\cot(\theta)$
15. $\dfrac{\cot(\theta)}{\csc^2(\theta)-1}=\tan(\theta)$
16. $4\cos^2(\theta)+4\sin^2(\theta)=4$
17. $9-\cos^2(\theta)-\sin^2(\theta)=8$
18. $\tan^3(\theta)=\tan(\theta)\sec^2(\theta)-\tan(\theta)$
19. $\sin^5(\theta)=\left(1-\cos^2(\theta)\right)^2\sin(\theta)$
20. $\sec^{10}(\theta)=\left(1+\tan^2(\theta)\right)^4\sec^2(\theta)$
21. $\cos^2(\theta)\tan^3(\theta)=\tan(\theta)-\sin(\theta)\cos(\theta)$
22. $\sec^4(\theta)-\sec^2(\theta)=\tan^2(\theta)+\tan^4(\theta)$
23. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1}=\dfrac{1+\sec(\theta)}{1-\sec(\theta)}$
24. $\dfrac{\sin(\theta)+1}{\sin(\theta)-1}=\dfrac{1+\csc(\theta)}{1-\csc(\theta)}$
25. $\dfrac{1-\cot(\theta)}{1+\cot(\theta)}=\dfrac{\tan(\theta)-1}{\tan(\theta)+1}$
26. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)}=\dfrac{\cos(\theta)-\sin(\theta)}{\cos(\theta)+\sin(\theta)}$
27. $\tan(\theta)+\cot(\theta)=\sec(\theta)\csc(\theta)$
28. $\csc(\theta)-\sin(\theta)=\cot(\theta)\cos(\theta)$
29. $\cos(\theta)-\sec(\theta)=-\tan(\theta)\sin(\theta)$
30. $\cos(\theta)\left(\tan(\theta)+\cot(\theta)\right)=\csc(\theta)$

31. $\sin(\theta)\left(\tan(\theta)+\cot(\theta)\right)=\sec(\theta)$

32. $\dfrac{1}{1-\cos(\theta)}+\dfrac{1}{1+\cos(\theta)}=2\csc^2(\theta)$

33. $\dfrac{1}{\sec(\theta)+1}+\dfrac{1}{\sec(\theta)-1}=2\csc(\theta)\cot(\theta)$

34. $\dfrac{1}{\csc(\theta)+1}+\dfrac{1}{\csc(\theta)-1}=2\sec(\theta)\tan(\theta)$

35. $\dfrac{1}{\csc(\theta)-\cot(\theta)}-\dfrac{1}{\csc(\theta)+\cot(\theta)}=2\cot(\theta)$

36. $\dfrac{\cos(\theta)}{1-\tan(\theta)}+\dfrac{\sin(\theta)}{1-\cot(\theta)}=\sin(\theta)+\cos(\theta)$

37. $\dfrac{1}{\sec(\theta)+\tan(\theta)}=\sec(\theta)-\tan(\theta)$

38. $\dfrac{1}{\sec(\theta)-\tan(\theta)}=\sec(\theta)+\tan(\theta)$

39. $\dfrac{1}{\csc(\theta)-\cot(\theta)}=\csc(\theta)+\cot(\theta)$

40. $\dfrac{1}{\csc(\theta)+\cot(\theta)}=\csc(\theta)-\cot(\theta)$

41. $\dfrac{1}{1-\sin(\theta)}=\sec^2(\theta)+\sec(\theta)\tan(\theta)$

42. $\dfrac{1}{1+\sin(\theta)}=\sec^2(\theta)-\sec(\theta)\tan(\theta)$

43. $\dfrac{1}{1-\cos(\theta)}=\csc^2(\theta)+\csc(\theta)\cot(\theta)$

44. $\dfrac{1}{1+\cos(\theta)}=\csc^2(\theta)-\csc(\theta)\cot(\theta)$

45. $\dfrac{\cos(\theta)}{1+\sin(\theta)}=\dfrac{1-\sin(\theta)}{\cos(\theta)}$

46. $\csc(\theta)-\cot(\theta)=\dfrac{\sin(\theta)}{1+\cos(\theta)}$

47. $\dfrac{1-\sin(\theta)}{1+\sin(\theta)}=\left(\sec(\theta)-\tan(\theta)\right)^2$

In Exercises 48 – 51, verify the identity. You may need to review the properties of absolute value and logarithms before proceeding.

48. $\ln|\sec(\theta)|=-\ln|\cos(\theta)|$

49. $-\ln|\csc(\theta)|=\ln|\sin(\theta)|$

50. $-\ln|\sec(\theta)-\tan(\theta)|=\ln|\sec(\theta)+\tan(\theta)|$

51. $-\ln|\csc(\theta)+\cot(\theta)|=\ln|\csc(\theta)-\cot(\theta)|$

2.5 Beyond the Unit Circle

Learning Objectives

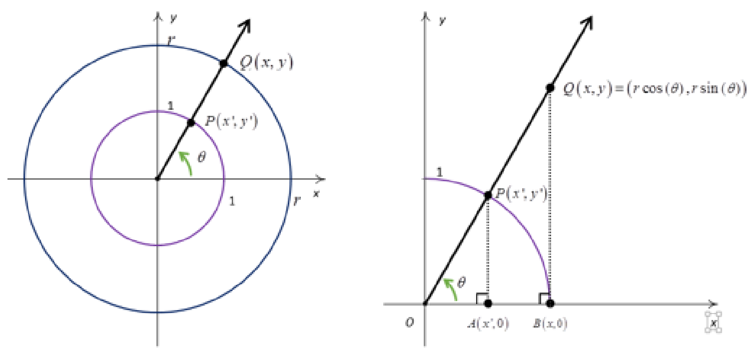
In this section you will:

- **Determine the values of the six circular functions from the coordinates of a point on a circle, centered at the origin, with any radius r.**
- **Solve related application problems.**
- **Describe the position of a particle experiencing circular motion.**

Recall that in defining the cosine and sine functions in **Section 2.2**, we assigned to each angle a position on the Unit Circle. Here we broaden our scope to include circles of radius r centered at the origin.

Determining Cosine and Sine

Consider for the moment the *acute* angle θ drawn below in standard position.



Let $Q(x, y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two triangles, ΔOPA and ΔOQB. These triangles are similar[23]. Thus, it follows that $\frac{x}{x'} = \frac{r}{1} = r$, from which $x = rx'$. We similarly find $y = ry'$. Since, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of Q to be $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By reflecting these points through the x-axis, y-axis and origin, we obtain the result for all non-quadrantal angles θ, and we leave it to the reader to verify these formulas hold for the quadrantal angles.

[23] Do you remember why?

Not only can we describe the coordinates of Q in terms of $\cos(\theta)$ and $\sin(\theta)$, but since the radius of the circle is $r=\sqrt{x^2+y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of Q. Throughout this textbook, by convention, the radius r of a circle is treated as positive as it relates to solving for trigonometric values.

These results are summarized in the following theorem.

Theorem 2.6. If $Q(x,y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2+y^2=r^2$ then $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Moreover,

$$\cos(\theta)=\frac{x}{r}=\frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin(\theta)=\frac{y}{r}=\frac{y}{\sqrt{x^2+y^2}}.$$

Note that in the case of the Unit Circle we have $r=\sqrt{x^2+y^2}=1$, so **Theorem 2.6** reduces to our Unit Circle definitions of $\cos(\theta)$ and $\sin(\theta)$.

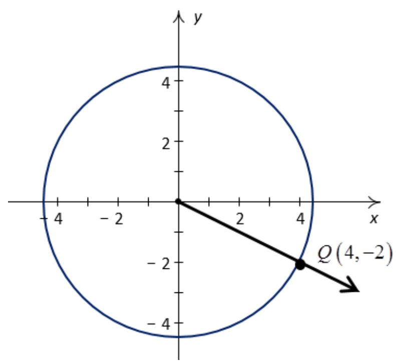
Example 2.5.1

1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4,-2)$. Find $\cos(\theta)$ and $\sin(\theta)$.
2. In **Example 1.3.4** in **Section 1.3**, we approximated the radius of the circle of revolution at 40.7608° North Latitude on Earth to be 2999 miles. Justify this approximation if the radius of the circle of revolution at the Equator is approximately 3960 miles.

Solution.

1. Using **Theorem 2.6** with $x=4$ and $y=-2$, we find

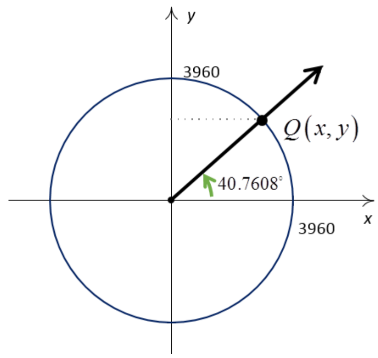
$$\begin{aligned} r &= \sqrt{(4)^2+(-2)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$



so that

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$
$$\cos(\theta) = \frac{4}{2\sqrt{5}} \qquad \sin(\theta) = \frac{-2}{2\sqrt{5}}$$
$$\cos(\theta) = \frac{2\sqrt{5}}{5} \qquad \sin(\theta) = -\frac{\sqrt{5}}{5}.$$

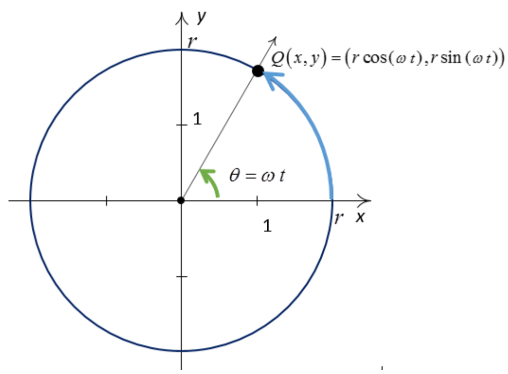
2. Assuming the Earth is a sphere, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the *x*-axis, the value we seek is the *x*-coordinate of the point $Q(x, y)$ indicated in the figure. Using **Theorem 2.6**, we get $x = 3960\cos(40.7608^\circ)$. Using a calculator in **degree** mode, we find $3960\cos(40.7608^\circ) \approx 2999$. Hence, the radius of the circle of revolution at North Latitude 40.7608° is approximately 2999 miles.



□

Position of a Particle in Circular Motion

Theorem 2.6 gives us what we need to describe the position of an object traveling in a circular path of radius *r* with constant angular velocity ω. Suppose that at time *t*, the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r, 0)$ when $t = 0$, the angle θ is in standard position.



By definition, $\omega = \frac{\theta}{t}$, which we rewrite as $\theta = \omega \cdot t$.

According to **Theorem 2.6**, the location of the object $Q(x, y)$ on the circle is found using the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$. With $\theta = \omega \cdot t$, we have $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$. Hence, at time *t*, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$.

We have just argued the following.

Equations for Circular Motion: Suppose an object is traveling on a circular path of radius r centered at the origin with constant angular velocity ω. If $t = 0$ corresponds to the point $(r, 0)$, then the x and y coordinates of the object are functions of t and are given by $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$. Here, $\omega > 0$ indicates a clockwise direction.

Example 2.5.2. Suppose we are in the situation of **Example 1.3.4**. Find the equations of motion of Salt Lake Community College as the Earth rotates.

Solution. From **Example 1.3.4**, we take $r = 2999$ miles and $\omega = \dfrac{\pi}{12 \text{ hours}}$. Hence, using $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$, the equations of motion are

$$x = 2999\cos\left(\frac{\pi}{12}t\right) \text{ and } y = 2999\sin\left(\frac{\pi}{12}t\right).$$

Note that x and y are measured in miles and t is measured in hours.

□

Determining the Other Four Circular Functions

We have generalized the cosine and sine functions from coordinates on the Unit Circle to coordinates on circles of radius r. Using **Theorem 2.6** in conjunction with **Theorem 2.3**, we generalize the remaining circular functions in kind.

Theorem 2.7. Suppose $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$. Then the circle has radius r and

- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2 + y^2}}{y}$, provided $y \neq 0$.
- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2 + y^2}}{x}$, provided $x \neq 0$.
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

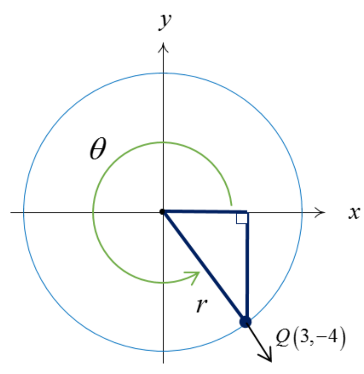
Example 2.5.3.

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3, -4)$. Find the values of the six circular functions of θ.
2. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of θ.

Solution.

1. The radius of the circle containing the point $Q(3, -4)$ is

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(3)^2 + (-4)^2} \\ &= 5. \end{aligned}$$

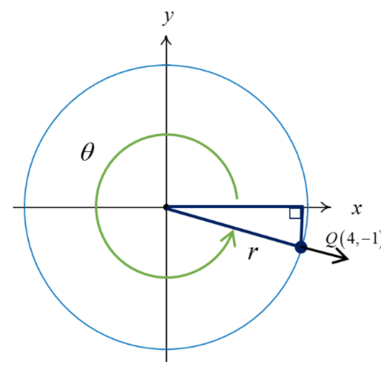


With $x = 3$, $y = -4$ and $r = 5$, we apply **Theorems 2.6** and **2.7** to find the values of the six circular functions of θ.[24]

[24] For convenience, the sketch shows $0 < \theta < 2\pi$. In reality, θ may be any angle, plotted in standard position, which contains the point Q (3,–4) on its terminal side.

$$\sin(\theta) = \frac{y}{r} = -\frac{4}{5} \qquad \csc(\theta) = \frac{r}{y} = -\frac{5}{4}$$
$$\cos(\theta) = \frac{x}{r} = \frac{3}{5} \qquad \sec(\theta) = \frac{r}{x} = \frac{5}{3}$$
$$\tan(\theta) = \frac{y}{x} = -\frac{4}{3} \qquad \cot(\theta) = \frac{x}{y} = -\frac{3}{4}.$$

2. We look for a point $Q(x, y)$ which lies on the terminal side of θ[25], when θ is plotted in standard position. We are given that θ is a Quadrant IV angle, so we know $x > 0$ and $y < 0$. Also, $\cot(\theta) = -4 = \frac{x}{y}$. Since $-4 = \frac{4}{-1}$, we may choose[26] $x = 4$ and $y = -1$, from which



$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(4)^2 + (-1)^2} \\ &= \sqrt{17}. \end{aligned}$$

The five remaining circular function values follow.

$$\sin(\theta) = \frac{y}{r} = \frac{-1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$$
$$\cos(\theta) = \frac{x}{r} = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$$
$$\tan(\theta) = \frac{y}{x} = -\frac{1}{4}$$
$$\csc(\theta) = \frac{r}{y} = \frac{\sqrt{17}}{-1} = -\sqrt{17}$$
$$\sec(\theta) = \frac{r}{x} = \frac{\sqrt{17}}{4}$$

□

[25] Again, θ may be any angle, plotted in standard position, with Q on its terminal side.

[26] We may choose *any* values x and y so long as $x > 0$, $y < 0$ and $x/y = -4$. For example, we could choose $x = 8$ and $y = -2$. The fact that all such points lie on the terminal side of θ is a consequence of the fact that the terminal side of θ is the portion of the line with slope $-1/4$ which extends from the origin into Quadrant IV.

We close this section by noting that we have not yet discussed the domains and ranges of the circular functions. In **Chapter 3**, we will graph the circular functions. This will provide a visual platform for the introduction of the domain and range of each circular function.

2.5 Exercises

In Exercises 1 – 8, let θ be the angle in standard position whose terminal side contains the given point. Find the values of the six circular functions of θ.

1. $A(1,5)$ 2. $B(3,-1)$ 3. $C(-6,-2)$ 4. $D(-10,12)$

5. $P(-7,24)$ 6. $Q(3,4)$ 7. $R(5,-9)$ 8. $T(-2,-11)$

In Exercises 9 – 10, determine the radius of the circle of revolution at the given latitude on Earth. Assume that the radius of the circle of revolution at the Equator is approximately 3960 miles.

9. 55.39° North Latitude 10. 44.29° South Latitude

In Exercises 11 – 12, determine the radius of the circle of revolution at the latitude corresponding to the given position on Earth. You may use the Internet for determining latitudes. Assume that the radius of the circle of revolution at the Equator is approximately 3960 miles.

11. Sydney, Australia 12. Nome, Alaska

In Exercises 13 – 16, find the equations of motion for the given scenario. Assume that the center of the motion is the origin, the motion is counter-clockwise and that $t = 0$ corresponds to a position along the positive *x*-axis.

13. A point on the edge of a yo-yo which is 2.25 inches in diameter and spins at 4500 revolutions per minute.

14. A point on the edge of a yo-yo used in the trick 'Around the World' in which the performer throws the yo-yo so it sweeps out a vertical circle whose radius is the yo-yo string. Assume the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle.

15. A point on the edge of the circular disk in a computer hard drive. The circular disk has diameter 2.5 inches and spins at a rate of 7200 RPM (revolutions per minute).

16. A passenger on the Giant Wheel at Cedar Point Amusement Part. The Giant Wheel is a circle with diameter 128 feet. It completes two revolutions in 2 minutes and 7 seconds.

In Exercises 17 – 30, use the given information to find the exact values of the remaining circular functions of θ.

17. $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II.

18. $\tan(\theta) = \frac{12}{5}$ with θ in Quadrant III.

19. $\csc(\theta) = \frac{25}{24}$ with θ in Quadrant I.

20. $\sec(\theta) = 7$ with θ in Quadrant IV.

21. $\csc(\theta) = -\frac{10\sqrt{91}}{91}$ with θ in Quadrant III.

22. $\cot(\theta) = -23$ with θ in Quadrant II.

23. $\tan(\theta) = -2$ with θ in Quadrant IV.

24. $\sec(\theta) = -4$ with θ in Quadrant II.

25. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.

26. $\cos(\theta) = \frac{1}{3}$ with θ in Quadrant I.

27. $\cot(\theta) = 2$ with $0 < \theta < \frac{\pi}{2}$.

28. $\csc(\theta) = 5$ with $\frac{\pi}{2} < \theta < \pi$.

29. $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \frac{3\pi}{2}$.

30. $\sec(\theta) = 2\sqrt{5}$ with $\frac{3\pi}{2} < \theta < 2\pi$.

31. In deriving the equations for circular motion, we assumed that at $t = 0$ the object was at the point $(r, 0)$. If this is not the case, we can adjust the equations of motion by introducing a time delay. If $t_0 > 0$ is the first time the object passes through the point $(r, 0)$, show, with the help of your classmates, the equations of motion are $x = r\cos(\omega(t - t_0))$ and $y = r\sin(\omega(t - t_0))$.

CHAPTER 3
GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Chapter Outline

3.1 Graphs of the Cosine and Sine Functions

3.2 Properties of the Graphs of Sinusoids

3.3 Graphs of the Tangent and Cotangent Functions

3.4 Graphs of the Secant and Cosecant Functions

Introduction

In Chapter 3, we graph the six trigonometric functions and learn about important properties of each function such as domain, range, period, and whether the function is even or odd. We begin with graphs of the cosine and sine functions in Section 3.1, which lead into further discussion of their designation as sinusoids in Section 3.2. We notice the similarities between sine and cosine graphs, along with horizontal shifts that will turn a sine graph into a cosine graph, or vice versa. Graphs of sinusoids and their applications, including harmonic motion, in Section 3.2 are followed by graphs and properties of tangent and cotangent functions in Section 3.3. We end with Section 3.4, graphing secant and cosecant functions and observing properties of each of these functions, as well as the relationship between the two.

Throughout Chapter 3, applications are introduced. Particular attention is paid to the graphing techniques that allow us to graph transformations of each function. This is an essential chapter in our textbook as it provides many of the tools we will need in applying identities and formulas, as well as solving trigonometric equations, in future chapters.

3.1 Graphs of the Cosine and Sine Functions

Learning Objectives

In this section you will:

- **Graph the cosine and sine functions and their transformations. Identify the period.**
- **Learn the properties of the cosine and sine functions, including domain and range.**
- **Determine whether a function is even or odd.**

We return to our discussion of the circular (trigonometric) functions as functions of real numbers and turn our attention to graphing the cosine and sine functions in the Cartesian Plane. Before proceeding with graphs of these trigonometric functions, we note that the graphs of both functions are continuous and smooth. Geometrically this means the graphs of the cosine and sine functions have no jumps, gaps, holes, vertical asymptotes, corners or cusps. As we shall see, the graphs of both functions meander nicely and don't cause any trouble.

Graph of the Cosine Function

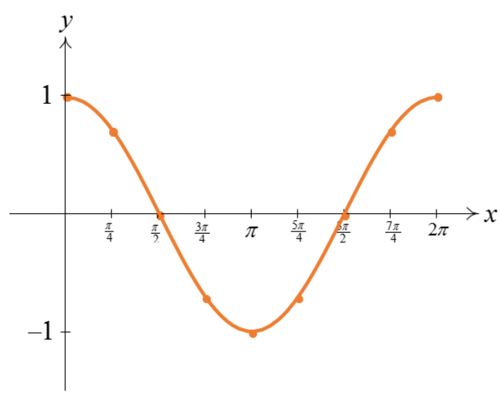
To graph the cosine function, we use x as the independent variable and y as the dependent variable.[27] This allows us to turn our attention to graphing the cosine and sine functions in the Cartesian Plane. We graph $y = \cos(x)$ by making a table using some of the common values of x in the interval $[0, 2\pi]$. This generates a portion of the cosine graph, which we call the **fundamental cycle** of $y = \cos(x)$.

[27] The use of x and y in this context is not to be confused with the x- and y-coordinates of points on the Unit Circle which define cosine and sine. Using the term 'trigonometric' function as opposed to 'circular' function may help to avoid confusion.

x	$y=\cos(x)$	$(x,\cos(x))$
0	1	$(0,1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2},0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4},-\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi,-1)$

x	$y=\cos(x)$	$(x,\cos(x))$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4},-\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2},0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4},\frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi,1)$

Noting that $y=\cos(x)$ is defined for all real numbers x, we plot the points $(x,\cos(x))$ from the table to guide us in sketching the graph of $y=\cos(x)$ on the interval $[0,2\pi]$.

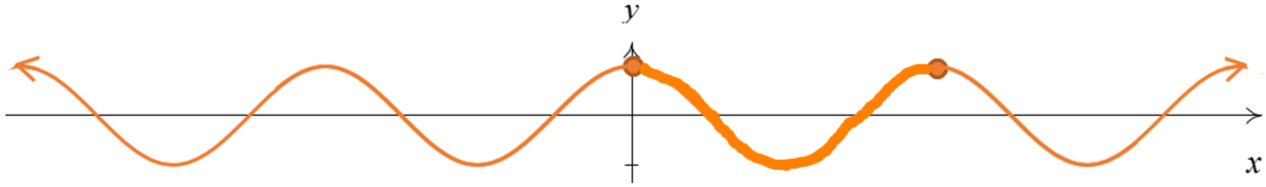


The fundamental cycle of $y=\cos(x)$.

A few things about the graph above are worth mentioning:

1. This graph represents only part of the graph of $y=\cos(x)$. To get the entire graph, imagine copying and pasting this graph end to end infinitely in both directions (left and right) along the x-axis.
2. The vertical scale here has been greatly exaggerated for clarity and aesthetics. Below is an accurate to-scale graph of $y=\cos(x)$ showing several cycles with the fundamental cycle plotted thicker than the others.

The graph of $y=\cos(x)$ is usually described as 'wavelike' and, indeed, many of the applications involving the cosine and sine functions feature the modeling of wavelike phenomena.



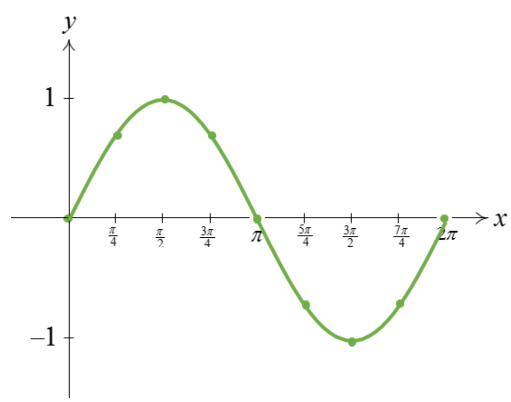
An accurately scaled graph of $y=\cos(x)$.

Graph of the Sine Function

We can plot the fundamental cycle of the graph of $y=\sin(x)$ similarly.

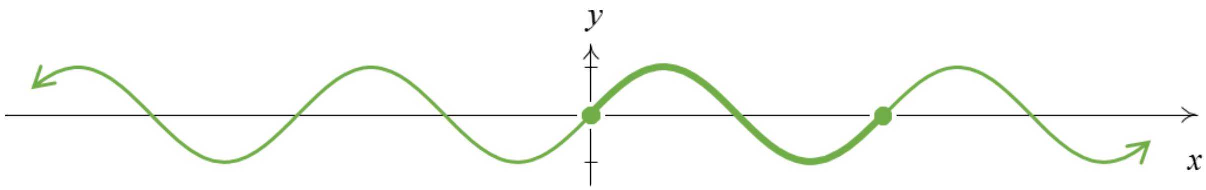
x	$y=\sin(x)$	$(x,\sin(x))$
0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2},1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4},\frac{\sqrt{2}}{2}\right)$
π	0	$(\pi,0)$

x	$y=\sin(x)$	$(x,\sin(x))$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4},-\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2},-1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4},-\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi,0)$



The fundamental cycle of $y=\sin(x)$.

As with the graph of $y = \cos(x)$, we can provide an accurately scaled graph of $y = \sin(x)$ with the fundamental cycle highlighted.



An accurately scaled graph of $y = \sin(x)$.

It is no accident that the graphs of $y = \cos(x)$ and $y = \sin(x)$ are so similar. The graph of $y = \cos(x)$ is a result of the graph of $y = \sin(x)$ being shifted $\frac{\pi}{2}$ units to the left. Try it!

Period of the Cosine and Sine Functions

Not only can we obtain a graph of the sine function by shifting the graph of the cosine function $\frac{\pi}{2}$ units to the left, we can shift the graph of $y = \cos(x)$ by 2π units to the left and obtain a graph that is equivalent to the original graph of $y = \cos(x)$. The same can be said for shifts of 4π, 6π, 8π, $\cdots$ units to the left. We say that the cosine function is periodic, as defined below.

Periodic Functions: A function *f* is said to be **periodic** if there is a real number *p* so that $f(x+p) = f(x)$ for all real numbers *x* in the domain of *f*. The smallest positive number *p*, if it exists, is called the **period** of *f*.

We see that by the definition of periodic functions, $f(x) = \cos(x)$ is periodic since $\cos(x + 2\pi k) = \cos(x)$ for any integer *k*. To determine the period of *f*, we need to find the smallest positive real number *p* so that $f(x+p) = f(x)$ for all real numbers *x* or, said differently, the smallest positive real number *p* such that $\cos(x+p) = \cos(x)$ for all real numbers *x*.

We know that $\cos(x + 2\pi) = \cos(x)$ for all real numbers *x* but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\cos(x+p) = \cos(x)$ for all real numbers *x*. Then, in particular, $\cos(0+p)=\cos(0)$ so that $\cos(p)=1$. From this we know that p is a multiple of 2π and, since the smallest positive multiple of 2π is 2π itself, we have the result.

Similarly, we can show $g(x)=\sin(x)$ is periodic with 2π as its period. Having period 2π essentially means that we can completely understand everything about the functions $f(x)=\cos(x)$ and $g(x)=\sin(x)$ by studying one interval of length 2π, say $[0,2\pi]$.

Even/Odd Properties of the Cosine and Sine Functions

While we will explore the even and odd properties of the cosine and sine functions further in **Section 4.1**, for now it is worth revisiting the graphical test for symmetry as introduced in college algebra. Recall that we refer to a function as **even** if its graph is symmetric about the *y*-axis and a function as **odd** if its graph is symmetric about the origin.

Observe the symmetry of the 'accurately scaled' graphs of the cosine and sine functions from earlier in this section. The graph of $y=\cos(x)$ is symmetric about the *y*-axis. As will be proved algebraically in **Section 4.1**, the cosine function is an even function. The graph of $y=\sin(x)$ is symmetric about the origin. The sine function is an odd function.

From a previous algebra class, you may recall that, for an even function *f*, $f(-x)=f(x)$. If *f* is odd then $f(-x)=-f(x)$. This is true for all values of *x*, and applies to the trigonometric functions as well. For the cosine function, an even function, $\cos(-x)=\cos(x)$. The sine function is odd, and thus $\sin(-x)=-\sin(x)$.

Domain and Range of the Cosine and Sine Functions

The two functions $f(x)=\cos(x)$ and $g(x)=\sin(x)$, both trigonometric functions of *x*, are defined for all real values of *x*. This is evident in the preceding graphs of cosine and sine. Or, going back to our Unit Circle definitions for cosine and sine, whether we think of identifying the real number *t* with the angle $\theta=t$ radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both cosine and sine are defined for any real input number *t*. Thus, the domain of both $f(t)=\cos(t)$ and $g(t)=\sin(t)$ is $(-\infty,\infty)$.

Looking at the graphs of the cosine and sine functions, we see that the range includes all real numbers between –1 and 1, inclusive. Revisiting the Unit Circle, $f(t)=\cos(t)$ and $g(t)=\sin(t)$ represent *x*- and *y*-coordinates, respectively, of points on the Unit Circle. As points on the Unit Circle, they take on all of the values between –1 and 1, inclusive. In other words, the range of both $f(t)=\cos(t)$ and $g(t)=\sin(t)$ is the interval $[-1,1]$. Following is a summary of properties of both functions.

Theorem 3.1. Properties of the Cosine and Sine Functions:

The function $f(t)=\cos(t)$

- has domain $(-\infty,\infty)$
- has range $[-1,1]$
- is continuous and smooth
- is even
- has period 2π

The function $g(t)=\sin(t)$

- has domain $(-\infty,\infty)$
- has range $[-1,1]$
- is continuous and smooth
- is odd
- has period 2π

Graphs of Transformations of the Cosine and Sine Functions

Now that we know the basic shapes of the graphs of $y=\cos(x)$ and $y=\sin(x)$, we can use transformations to graph more complicated curves. The following theorem, borrowed from college algebra, will guide us in transformations of graphs of cosine and sine functions.

Theorem 3.2. Transformations of Periodic Functions: Suppose *f* is a periodic function. If $A \neq 0$ and $\omega > 0$, then to graph

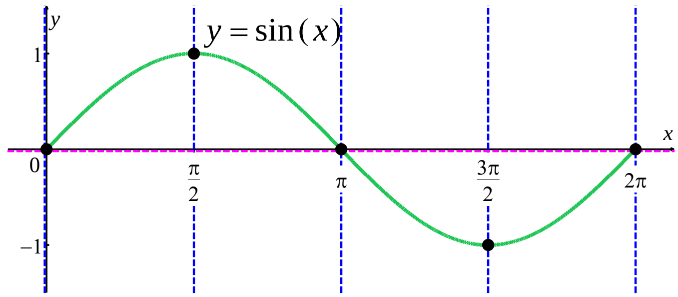
$$g(x) = A \cdot f(\omega x + \phi) + B$$

1. Divide the period of the function $f(x)$ by ω to determine the period of the transformed function *g*. For sine and cosine, the period is 2π, so $\frac{2\pi}{\omega}$ is the period of *g*. Note that ω is the number of times the cycle of *g* repeats on $[0, 2\pi)$.
2. Find the vertical shift by determining the value of *B*. The graph will be shifted up by *B* if $B > 0$ and down by $|B|$ if $B < 0$. The line $y = B$ will be the 'baseline'.
3. Determine the vertical scaling, $|A|$, also known as the amplitude. If $A < 0$, note that the graph will be reflected about the baseline $y = B$.
4. The horizontal shift $-\frac{\phi}{\omega}$, also known as the phase shift, can be determined by rewriting $\omega x + \phi$ as $\omega\left[x - \left(-\frac{\phi}{\omega}\right)\right]$. This results in a horizontal shift to the left if $-\frac{\phi}{\omega} < 0$ or right if $-\frac{\phi}{\omega} > 0$.

In transforming one cycle of $y = \cos(x)$ or $y = \sin(x)$, we need to keep track of the movement of some key points. We choose to track the points with *x*-values 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π. These **quarter marks** correspond to quadrantal angles, and as such, mark the location of the zeros and the local extrema of these functions over exactly one period.

Example 3.1.1. Graph one cycle of the function $f(x) = 2\sin(3x) + 1$.

Solution. We start with the following graph of the fundamental cycle of $y = \sin(x)$. The locations of quarter marks are denoted with dashed blue lines, and intersect the graph at the **key points** $(0,0)$, $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, -1\right)$ and $(2\pi, 0)$. The baseline of $y = 0$ is marked by a dashed pink line.



The steps in **Theorem 3.2** help us determine transformations for graphing $f(x)=2\sin(3x)+1$.

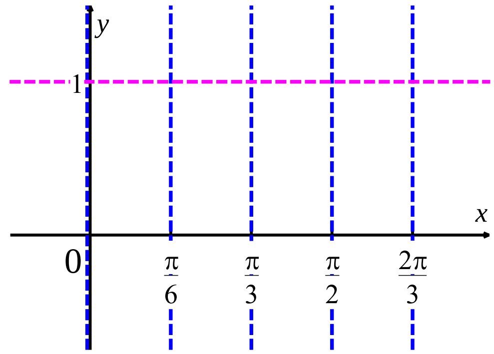
Step 1: We first determine the period. The period for $y=\sin(x)$ is 2π. For $f(x)=2\sin(3x)+1$, the coefficient of x is 3, so the graph of the function will cycle three times as fast as $y=\sin(x)$. In other words, it will repeat itself three times on the interval $[0,2\pi)$, with one cycle being completed in a period of $\frac{2\pi}{3}$.

Step 2: We next determine the vertical shift. The graph of $f(x)=2\sin(3x)+1$ has a vertical shift of 1. By shifting the baseline for the graph of $y=\sin(x)$ up by 1, we have a baseline of $y=1$ for the graph of $f(x)=2\sin(3x)+1$.

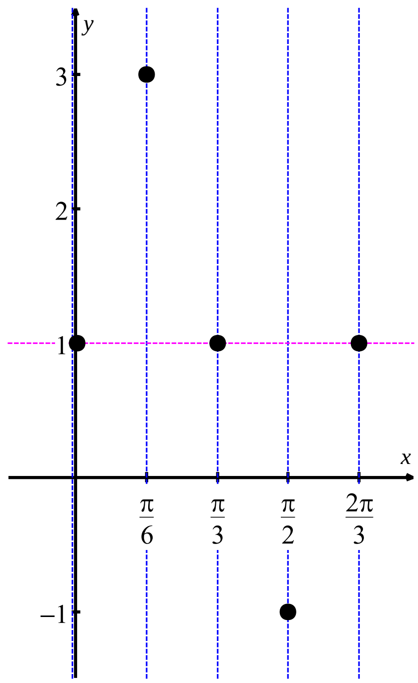
Step 3: The vertical scaling, or amplitude, of $y=\sin(x)$ is 1. For the function $f(x)=2\sin(3x)+1$, the sine function is multiplied by 2 so the function values are twice as much. Thus the amplitude is 2.

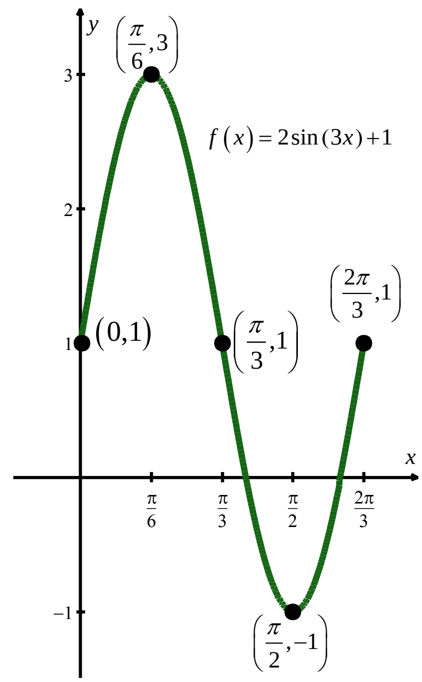
Step 4: The horizontal shift is 0.

Finally, we are ready to graph one cycle of $f(x)=2\sin(3x)+1$. Since the horizontal shift is 0, we can leave the first quarter mark at $x=0$. We then divide the period of $\frac{2\pi}{3}$ by 4 to determine a distance of $\frac{\pi}{6}$ between quarter marks. The positions for quarter marks and the baseline of $y=1$ are used as a guide in graphing $f(x)=2\sin(3x)+1$.



The graph of $f(x)=2\sin(3x)+1$ will maintain the shape of the sine function. With an amplitude of 2, the local maximums and minimums will occur 2 units above and 2 units below the baseline, respectively. We can plot 5 key points of this function using the quarter mark locations and baseline as a guide, and finish the graph by connecting the points with a continuous smooth curve.

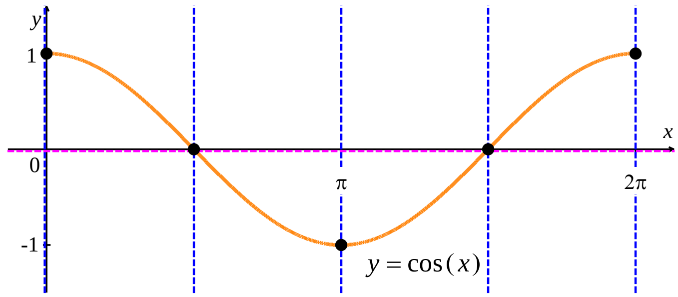




□

Example 3.1.2. Graph one cycle of the function $f(x)=-3\cos(2x-\pi)$.

Solution. We begin with a graph of the cosine function, noting the key points of $(0,1)$, $\left(\frac{\pi}{2},0\right)$, $(\pi,-1)$, $\left(\frac{3\pi}{2},0\right)$ and $(2\pi,1)$.



Step 1: The period of $y = \cos(x)$ is 2π. For $f(x) = -3\cos(2x - \pi)$, the coefficient of x is 2 so the

period is $\frac{2\pi}{2} = \pi$

Step 2: The vertical shift of $f(x) = -3\cos(2x - \pi)$ is 0. Thus, the baseline will remain at $y = 0$.

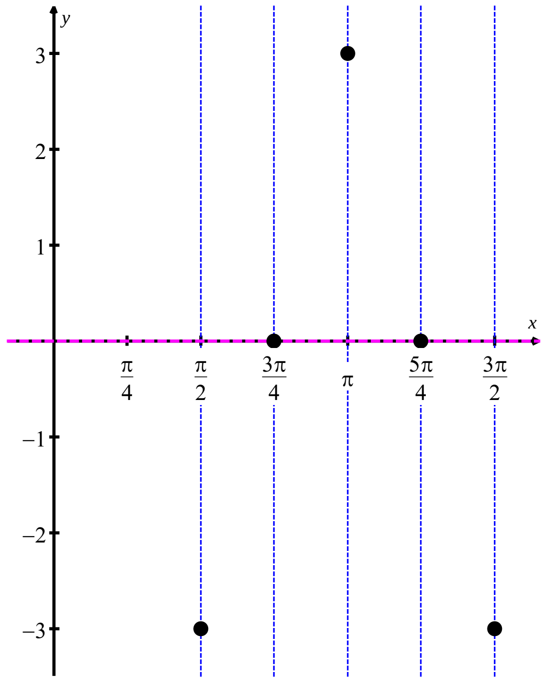
Step 3: The amplitude is $|-3| = 3$. Since $-3 < 0$, the graph will be reflected about the baseline $y = 0$.

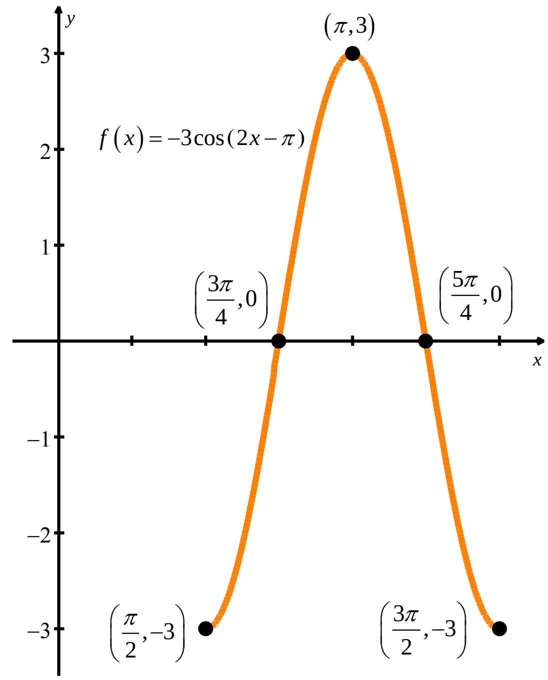
Step 4: To determine the horizontal shift, the function can be rewritten as

$$
\begin{aligned}
f(x) &= -3\cos(2x - \pi) \\
&= -3\cos\left[2\left(x - \frac{\pi}{2}\right)\right]
\end{aligned}
$$

The horizontal shift is $\frac{\pi}{2}$, indicating a shift to the right of $\frac{\pi}{2}$ units.

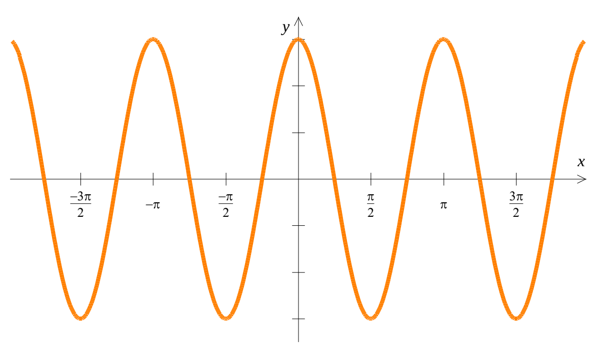
Now we are ready to graph this transformation of $y = \cos(x)$. From the horizontal shift of $\frac{\pi}{2}$, the first quarter mark will be at $x = \frac{\pi}{2}$. The period is π, so the distance between quarter marks will be $\frac{\pi}{4}$. We denote the positions of the quarter marks and the baseline $y = 0$. The amplitude is 3. While the basic shape is that of the cosine, the graph will be reflected about the baseline $y = 0$, resulting in reversed positions for the local maximum and minimum values. After determining the 5 key points for one cycle of $f(x) = -3\cos(2x - \pi)$, we connect the points with a continuous smooth curve, replicating the shape of $y = \cos(x)$.





□

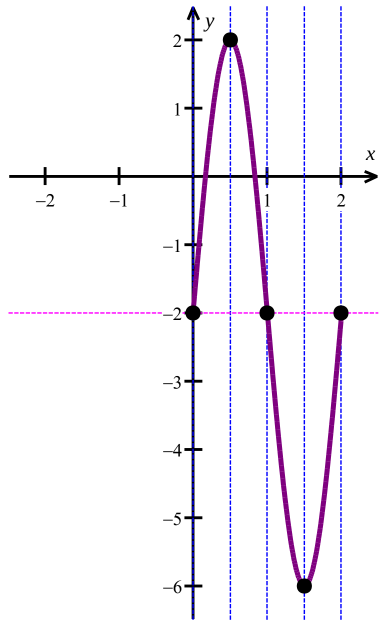
Above, we have graphed one complete period of the function $f(x) = -3\cos(2x - \pi)$. This graph could easily be extended in either direction.



In the next example we sketch the graph of a sine function over the larger interval of two periods.

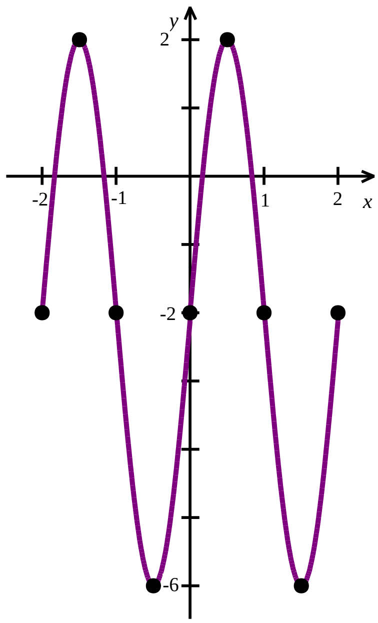
Example 3.1.3. Graph two full periods of the function $g(x) = 4\sin(\pi x) - 2$ and state the period. Identify the maximum and minimum *y*-values and their corresponding *x*-values.

Solution. To sketch one period of the function $g(x) = 4\sin(\pi x) - 2$ through transformations of the the fundamental cycle of $y = \sin(x)$, we note the following.



1. The period is $\frac{2\pi}{\pi} = 2$, so the distance between quarter marks will be $\frac{2}{4} = \frac{1}{2}$.
2. There is a vertical shift of –2, resulting in a baseline of $y = -2$.
3. The amplitude is 4.
4. There is no horizontal shift, so the first quarter mark is at $x = 0$.

To sketch two periods, we extend the graph for one period, as follows.



$y = g(x) = 4\sin(\pi x) - 2$

The period of 2 appears twice in the preceding graph. A maximum *y*-value of 2 corresponds with *x*-values of $-\frac{3}{2}$ and $\frac{1}{2}$. The minimum *y*-value is -6 and corresponds with *x*-values of $-\frac{1}{2}$ and $\frac{3}{2}$.

□

The functions in this section are examples of sinusoids. Roughly speaking, a sinusoid is the result of performing transformations to the basic graph of $f(x) = \cos(x)$ or $g(x) = \sin(x)$. Sinusoids will be looked at extensively in **Section 3.2**.

3.1 Exercises

1. Why are the cosine and sine functions called periodic functions?

2. How does the graph of $y = \sin(x)$ compare with the graph of $y = \cos(x)$? Explain how you could horizontally translate the graph of $y = \sin(x)$ to obtain the graph of $y = \cos(x)$.

3. For the function $f(x) = A\cos(Bx + C) + D$, what constants affect the range and how do they affect the range?

In Exercises 4 – 15, graph one cycle of the given function. State the period of the function.

4. $y = 3\sin(x)$
5. $y = \sin(3x)$
6. $y = -2\cos(x)$
7. $y = \cos\left(x - \frac{\pi}{2}\right)$
8. $y = -\sin\left(x + \frac{\pi}{3}\right)$
9. $y = \sin(2x - \pi)$
10. $y = -\frac{1}{3}\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)$
11. $y = \cos(3x - 2\pi) + 4$
12. $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$
13. $y = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4x\right) + 1$
14. $y = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}$
15. $y = 4\sin(-2\pi x + \pi)$

In Exercises 16 – 27, graph two full periods of each function and state the period. Identify the maximum and minimum *y*-values and their corresponding *x*-values.

16. $f(x) = 2\sin(x)$
17. $f(x) = \frac{2}{3}\cos(x)$
18. $f(x) = -3\sin(x)$
19. $f(x) = 4\sin(x)$
20. $f(x) = 2\cos(x)$
21. $f(x) = \cos(2x)$
22. $f(x) = 2\sin\left(\frac{1}{2}x\right)$
23. $f(x) = 4\cos(\pi x)$
24. $f(x) = 3\cos\left(\frac{6}{5}x\right)$
25. $y = 3\sin\left[8(x+4)\right] + 5$
26. $y = 2\sin(3x - 21) + 4$
27. $y = 5\sin(5x + 20) - 2$

28. Show that a constant function *f* is periodic by showing that $f\left(x+117\right)=f\left(x\right)$ for all real numbers *x*. Then show that *f* has no period by showing that you cannot find a smallest number *p* such that $f\left(x+p\right)=f\left(x\right)$ for all real numbers *x*. Said another way, show that $f\left(x+p\right)=f\left(x\right)$ for all real numbers *x* for ALL values of $p>0$, so no smallest value exists to satisfy the definition of period.

3.2 Properties of the Graphs of Sinusoids

Learning Objectives

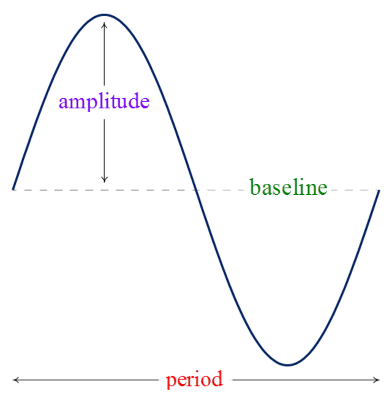
In this section you will:

- **Learn the properties of graphs of sinusoidal functions, including period, phase shift, amplitude and vertical shift.**
- **Use properties to graph sinusoidal functions.**
- **Write an equation of the form $C(x) = A\cos(\omega x + \phi) + B$ or $S(x) = A\sin(\omega x + \phi) + B$ from the graph of a sinusoidal function.**
- **Solve applications of sinusoids, including harmonic motion.**

Sinusoids can be characterized by four properties: period, amplitude, phase shift, and vertical shift.

1. We have already discussed **period**, that is, how long it takes for the sinusoid to complete one cycle. The standard period of both $f(x) = \cos(x)$ and $g(x) = \sin(x)$ is 2π, but horizontal scalings will change the period of the resulting sinusoid.

2. The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.



3. The **phase shift** of the sinusoid is the horizontal shift experienced by the fundamental cycle. We have seen that a phase (horizontal) shift of $\frac{\pi}{2}$ to the right takes $f(x) = \cos(x)$ to $g(x) = \sin(x)$ since $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. As the reader can verify, a phase shift of $\frac{\pi}{2}$ to the left takes $g(x) = \sin(x)$ to $f(x) = \cos(x)$.

4. The **vertical shift** of a sinusoid is assumed to be 0, but we will state the more general case.

The following theorem shows how to find these four fundamental quantities from the formula of a given sinusoid.

Theorem 3.3. For $\omega > 0$, the functions

$$C(x) = A\cos(\omega x + \phi) + B \text{ and } S(x) = A\sin(\omega x + \phi) + B$$

- have period $\frac{2\pi}{\omega}$
- have phase shift $-\frac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift B

We note that in some scientific and engineering circles, the quantity ϕ mentioned in **Theorem 3.3** is called the **phase angle** of the sinusoid. Since our interest in this book is primarily with *graphing* sinusoids, we focus our attention on the horizontal shift $-\frac{\phi}{\omega}$ induced by ϕ.

Graphs of Sinusoids

The proof of **Theorem 3.3** is left to the reader. The parameter ω, which is stipulated to be positive, is called the (**angular**) **frequency** of the sinusoid and is the number of cycles the sinusoid completes over a 2π interval. We can always ensure $\omega > 0$ using the even property of the cosine function, or the odd property of the sine function, from which $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. We now test out **Theorem 3.3** using the functions $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$ and $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$ in the following two examples.

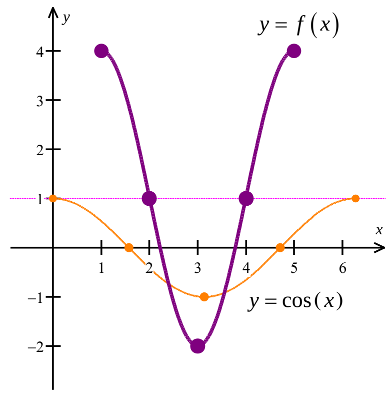
Example 3.2.1. Graph $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$.

Solution. Using **Theorem 3.3**, we first write the function f in the form prescribed in the theorem.

$$\begin{aligned} f(x) &= 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1 \\ &= 3\cos\left[\frac{\pi}{2}x + \left(-\frac{\pi}{2}\right)\right] + 1 \end{aligned}$$

From **Theorem 3.3**, $A = 3$, $\omega = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$ and $B = 1$, resulting in the following.

- The period of *f* is $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{(\pi/2)} = 4$.
- The amplitude is $|A| = |3| = 3$.
- The phase shift is $-\dfrac{\phi}{\omega} = -\dfrac{(-\pi/2)}{(\pi/2)} = 1$, indicating a shift to the *right* 1 unit.
- The vertical shift is $B = 1$, indicating a shift *up* 1 unit, and a baseline of $y = 1$.



The graph shows one cycle of $f(x) = 3\cos\left(\dfrac{\pi x - \pi}{2}\right) + 1$. Using the period, amplitude, phase shift and vertical shift, in conjunction with techniques from **Section 3.1**, the graph of $y = f(x)$ is sketched as a transformation of $y = \cos(x)$. Key points are emphasized on each graph. The baseline for $y = f(x)$ is shown as a dashed line.

□

Example 3.2.2. Graph $g(x) = \dfrac{1}{2}\sin(\pi - 2x) + \dfrac{3}{2}$.

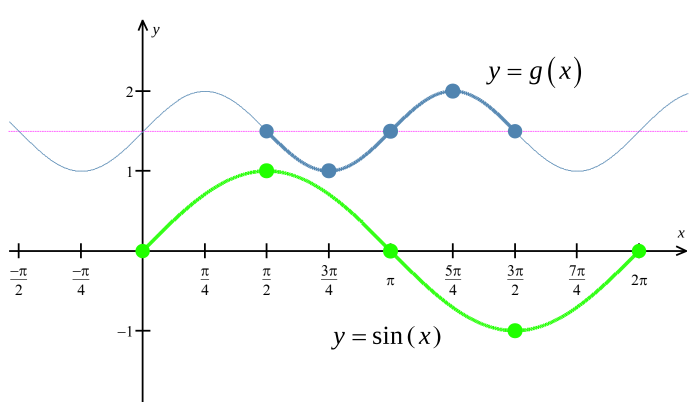
Solution. Before applying **Theorem 3.3**, we use the odd property of the sine function to write $g(x)$ in the required form.

$$\begin{aligned} g(x) &= \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2} \\ &= \frac{1}{2}\sin\left[-(2x - \pi)\right] + \frac{3}{2} \\ &= -\frac{1}{2}\sin(2x - \pi) + \frac{3}{2} \qquad \textbf{from odd property of sine} \\ &= -\frac{1}{2}\sin\left[2x + (-\pi)\right] + \frac{3}{2} \end{aligned}$$

We next determine that $A = -\frac{1}{2}$, $\omega = 2$, $\phi = -\pi$ and $B = \frac{3}{2}$. The properties follow from **Theorem 3.3**.

- The period of g is $\frac{2\pi}{2} = \pi$.
- The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.
- The phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$, indicating a shift *right* $\frac{\pi}{2}$ units.
- The vertical shift is *up* $\frac{3}{2}$ units.

We graph $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$ via transformations of the sine function, using the above properties as a guide. Before graphing, we note that $A = -\frac{1}{2}$. Since $A < 0$, the graph of $y = g(x)$ must be reflected about the baseline.



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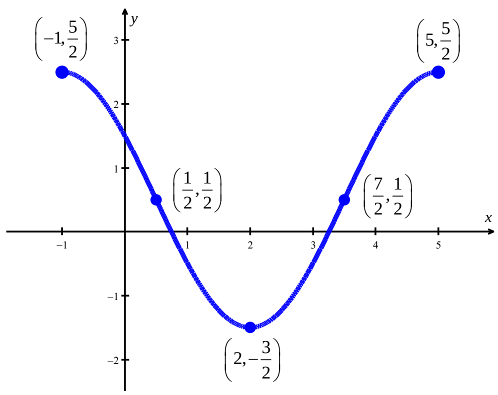
Remember that the cycle graphed through transformations of the sine function in **Example 3.1.2** is only one portion of the graph of $y = g(x)$. Indeed, another complete cycle begins at $x = -\frac{\pi}{2}$, and a third at $x = \frac{3\pi}{2}$. Note that whatever cycle we choose is sufficient to completely determine the sinusoid.

Determining an Equation from the Graph of a Sinusoid

Example 3.2.3. Below is the graph of one complete cycle of a sinusoid $y = f(x)$.



One cycle of $y = f(x)$.

1. Find a cosine function whose graph matches the graph of $y = f(x)$.

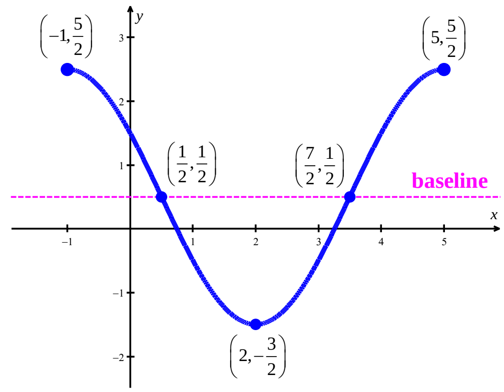
2. Find a sine function whose graph matches the graph of $y = f(x)$.

Solution.

1. We fit the data to a function of the form $C(x) = A\cos(\omega x + \phi) + B$ by determining A, ω, ϕ and B.

   - Since one cycle is graphed over the interval $[-1,5]$, its period is $5-(-1)=6$. According to **Theorem 3.3**, $6 = \frac{2\pi}{\omega}$, so that $\omega = \frac{\pi}{3}$.

- To find the amplitude, we note that the range of the sinusoid is $\left[-\frac{3}{2},\frac{5}{2}\right]$. The midpoint of the range is $\frac{1}{2}$, indicating a baseline of $y=\frac{1}{2}$. After marking the graph with the baseline, we see that the amplitude is $A=\frac{5}{2}-\frac{1}{2}=2$.



- Next, we see that the phase shift is -1, so we have $-\frac{\phi}{\omega}=-1$ or $\phi=\omega=\frac{\pi}{3}$.

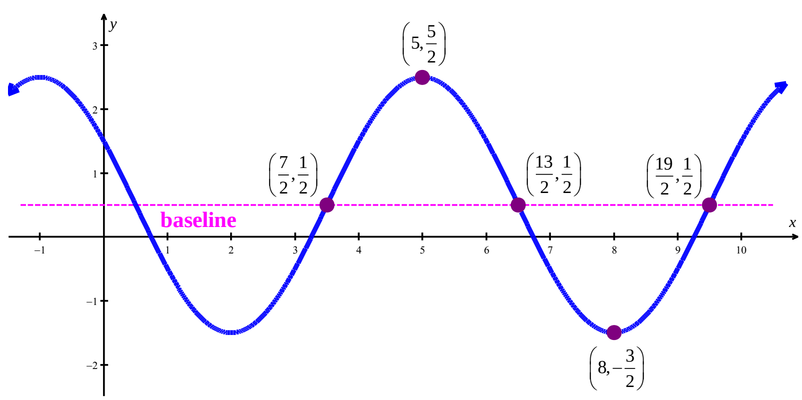
- Finally, we refer to the baseline to verify a vertical shift of $B=\frac{1}{2}$.

Our final answer is $C(x)=2\cos\left(\frac{\pi}{3}x+\frac{\pi}{3}\right)+\frac{1}{2}$.

2. Most of the work to fit the data to a function of the form $S(x)=A\sin(\omega x+\phi)+B$ is done.

- The period, amplitude and vertical shift are the same as part 1. Thus, $\omega=\frac{\pi}{3}$, $A=2$ and $B=\frac{1}{2}$.

- The trickier part is finding the phase shift. To that end, we imagine extending the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at $\left(\frac{7}{2},\frac{1}{2}\right)$.



Taking the phase shift to be $\frac{7}{2}$, we get $-\frac{\phi}{\omega}=\frac{7}{2}$, or

$$\begin{aligned}\phi&=-\frac{7}{2}\omega\\&=-\frac{7}{2}\left(\frac{\pi}{3}\right)\\&=-\frac{7\pi}{6}.\end{aligned}$$

Hence, our answer is $S(x)=2\sin\left(\frac{\pi}{3}x-\frac{7\pi}{6}\right)+\frac{1}{2}$.

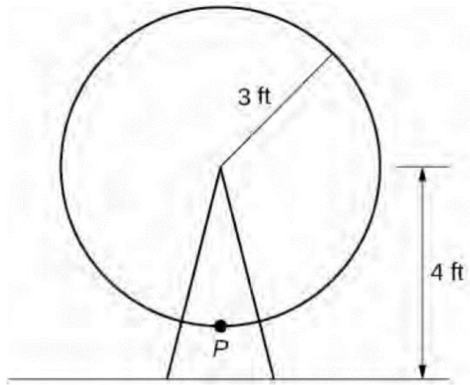
□

Note that each of the answers given in **Example 3.2.3** is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at $\left(\frac{1}{2},\frac{1}{2}\right)$ taking $A=-2$. In this case, the phase shift is $\frac{1}{2}$ so $\phi=-\frac{\pi}{6}$ for an answer of $S(x)=-2\sin\left(\frac{\pi}{3}x-\frac{\pi}{6}\right)+\frac{1}{2}$. Alternately, we could have extended the graph of $y=f(x)$ to the left and considered a sine function starting at $\left(-\frac{5}{2},\frac{1}{2}\right)$, and so on. Each of these formulas determine the same sinusoidal curve and the formulas are equivalent using identities.

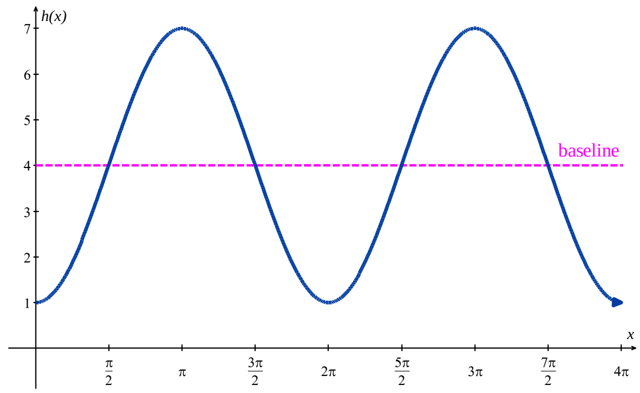
Applications of Sinusoids

Sinusoids are used to model a fair number of behaviors that possess a wavelike motion, such as sound, voltage, and spring action. The following examples look at circular motion that can be expressed as a sinusoidal function.

Example 3.2.4. A circle with radius 3 feet is mounted with its center 4 feet off the ground. The point closest to the ground is labeled *P*, as shown below. Sketch a graph of the height above the ground of the point *P* as the circle is rotated. Find a function *h* that gives the height in terms of the angle *x* of rotation.



Solution. Sketching the height, we note that it will start 1 foot above the ground, then increase up to 7 feet above the ground, and continue to oscillate 3 feet above and below the center value of 4 feet.



Although we could use a transformation of either the cosine or sine function, we start by looking for characteristics that would make one function easier to use than the other. Since this graph starts at its lowest value, when $x = 0$, using the cosine function would not require a horizontal shift. Thus, we choose to model the graph with a cosine function. We note that a standard cosine graph starts at the highest value so we do need to incorporate a vertical reflection.

We see that the graph oscillates 3 feet above and below the horizontal center of the graph. The basic cosine graph has an amplitude of 1, so this graph has been vertically stretched by 3.

Finally, to move the center of the circle up to a height of 4 feet, the graph has been vertically shifted up by 4. Putting these transformations together, we find that $h(x) = -3\cos(x) + 4$.

□

Example 3.2.5. The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution. The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve: $H(t) = A\cos(\omega t + \phi) + B$, for time *t* and height *H*. A phase shift is not required.

With a diameter of 135 meters, the wheel has a radius of 67.5 meters. The height will oscillate with amplitude 67.5 meters above and below the horizontal center of the wheel.

Passengers board 2 meters above the ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ meters above ground level. The horizontal midline of the oscillation will be at 69.5 meters.

Putting this all together, we have

- Period: $30 = \dfrac{2\pi}{\omega} \Rightarrow \omega = \dfrac{\pi}{15}$
- Amplitude: $|A| = 67.5$; $A = -67.5$ (due to the vertical reflection of the cosine curve)
- Phase Shift: $0 = -\dfrac{\phi}{\omega} \Rightarrow \phi = 0$
- Vertical Shift: $B = 69.5$

An equation for the rider's height, with *t* in minutes and *H* in meters, is

$$H(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

□

Harmonic Motion

One of the major applictions of sinusoids in Science and Engineering is the study of **harmonic motion**. The equations for harmonic motion can be used to describe a wide range of phenomena, from the motion of an object on a spring, to the response of an electronic circuit. Here, we restrict our attention to modeling a simple spring system.

Before we jump into the mathematics, there are some Physics terms and concepts we need to discuss.

- In Physics, 'mass' is defined as a measure of an object's resistance to straight-line motion whereas 'weight' is the amount of force (pull) gravity exerts on an object. An object's mass cannot change,[28] while its weight could change. An object which weighs 6 pounds on the surface of the Earth would weigh 1 pound on the surface of the Moon, but its mass is the same in both places.
- In the English system of units, 'pounds' (lbs.) is a measure of force (weight), and the corresponding unit of mass is the 'slug'. In the SI system, the unit of force is 'Newtons' (N) and the associated unit of mass is the 'kilogram' (kg).
- We convert between mass and weight using the formula[29] $w = mg$. Here, *w* is the weight of the object, *m* is the mass and *g* is the acceleration due to gravity. In the English system, $g = 32\,\text{feet}/\text{second}^2$ and in the SI system, $g = 9.8\ \text{meters}/\text{second}^2$. Hence, on Earth a *mass* of 1 slug *weighs* 32 lbs. and a *mass* of 1 kg *weighs* 9.8 N.[30]

Suppose we attach an object with mass *m* to a spring as detected below. The weight of the object will stretch the spring. The system is said to be in 'equilibrim' when the weight of the object is perfectly balanced with the restorative force of the spring. How far the spring stretches to reach equilibrium depends on the spring's 'spring constant'. Usually denoted by the letter *k*, the spring constant relates the force *F* applied to the spring to the amount *d* the spring stretches in accordance with Hooke's Law $F = kd$.

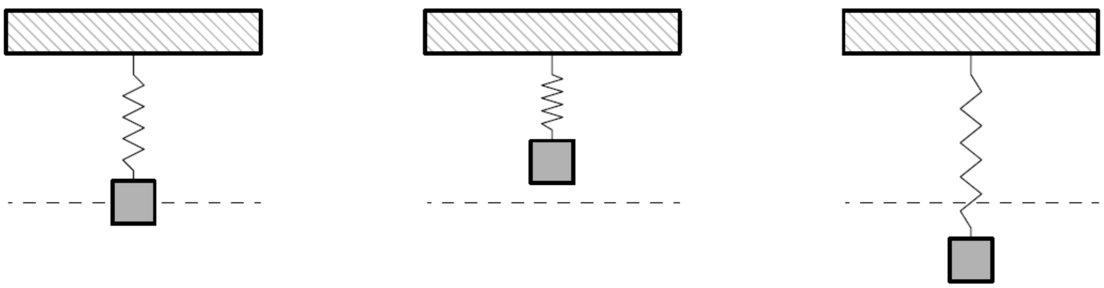
If the object is released above or below the equilibrium position, or if the object is released with an upward or downward velocity, the object will bounce up and down on the end of the spring until some

[28] Well, assuming the object isn't subjected to relativistic speeds . . .
[29] This is a consequence of Newton's Second Law of Motion *F*=*ma* where *F* is force, *m* is mass and *a* is acceleration. In our present setting, the force involved is weight which is caused by the acceleration due to gravity.
[30] Note that 1 pound = 1 slug foot / second^2 and 1 Newton = 1 kg meter / second^2.

external force stops it. If we let $x(t)$ denote the object's displacement from the equilibrium position at time *t*, then

- $x(t)=0$ means the object is at the equilibrium position,
- $x(t)<0$ means the object is *above* the equilibrium position,
- $x(t)>0$ means the object is *below* the equilibrium position.

The function $x(t)$ is called the 'equation of motion' of the object.[31]



$x(t)=0$ at the equilibrium position | $x(t)<0$ above the equilibrium position | $x(t)>0$ below the equilibrium position

If we ignore all other influences on the system except gravity and the spring force, then Physics tells us that gravity and the spring force will battle each other forever and the object will oscillate indefinitely. In this case, we describe the motion as 'free' (meaning there is no external force causing the motion) and 'undamped' (meaning we ignore friction caused by surrounding medium, which in our case is air).

In the following theorem, which comes from Differential Equations,

- $x(t)$ is a function of the mass *m* of the object, the spring constant *k*, the initial displacement x_0 of the object and initial velocity v_0 of the object.
- $x_0=0$ means the object is released from the equilibrium position, $x_0<0$ means the object is released *above* the equilibrium position and $x_0>0$ means the object is released *below* the equilibrium position.

[31] To keep units compatible, if we are using the English system, we use feet (ft.) to measure displacement. If we are in the SI system, we measure displacement in meters (m). Time is always measured in seconds (s).

- $v_0 = 0$ means the object is released from rest, $v_0 < 0$ means the object is heading *upwards* and $v_0 > 0$ means the object is heading *downwards*.[32]

> **Theorem 3.4. Equation for Free Undamped Harmonic Motion:** Suppose an object of mass *m* is suspended from a spring with spring constant *k*. If the initial displacement from the equilibrium position is x_0 and the initial velocity of the object is v_0, then the displacement *x* from the equilibrium position at time *t* is given by $x(t) = A\sin(\omega t + \phi)$ where
>
> - $\omega = \sqrt{\dfrac{k}{m}}$ and $A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}$
> - $A\sin(\phi) = x_0$ and $A\omega\cos(\phi) = v_0$.

It is a great exercise in 'dimensional analysis' to verify that the formulas in **Theorem 3.4** work out so that ω has units $\dfrac{1}{s}$ and *A* has units ft. or m, depending on which system we choose.

Example 3.2.6. Suppose an object weighing 64 pounds stretches a spring 8 feet.

1. If an object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, $x(t)$. When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?
2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, $x(t)$. What is the longest distance the object travels *above* the equilibrium position? When does this first happen?

Solution. In order to use the formulas in **Theorem 3.4**, we first need to determine the spring constant *k* and the mass *m* of the object. We know the object weighs 64 lbs. and stretches the spring 8 ft. Using Hooke's Law with $F = 64$ and $d = 8$, we get

[32] The sign conventions here are carried over from Physics. If not for the spring, the object would fall towards the ground, which is the natural or positive direction. Since the spring force acts in direct oppostition to gravity, any movement upwards is considered negative.

$$
\begin{aligned}
F &= kd \quad \textbf{Hooke's Law}\\
64 &= k\cdot 8\\
k &= 8\,\frac{\text{lbs.}}{\text{ft.}}
\end{aligned}
$$

To find *m*, we use $w = mg$ with $w = 64$ lbs. and $g = 32\frac{ft.}{s^2}$. We get $m = 2$ slugs. We can now proceed to apply **Theorem 3..4**.

1. To find the equation of motion, $x(t) = A\sin(\omega t + \phi)$, we must determine values for *A*, ω and ϕ. We begin with ω.

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} \quad \textbf{from Theorem 3.4}\\
&= \sqrt{\frac{8}{2}} \quad \textbf{since } k = 8\,\textbf{lbs./ft. and } m = 2\,\textbf{slugs}\\
&= 2
\end{aligned}
$$

Now, since the object is released 3 feet *below* the equilibrium position, from rest, $x_0 = 3$ and $v_0 = 0$. We use these values, along with $\omega = 2$, to find *A*.

$$
\begin{aligned}
A &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \quad \textbf{from Theorem 3.4}\\
&= \sqrt{3^2 + \left(\frac{0}{2}\right)^2}\\
&= 3
\end{aligned}
$$

To determine the phase ϕ, we use $x_0 = 3$ and $A = 3$, along with a formula from **Theorem 3.4**.

$$
\begin{aligned}
A\sin(\phi) &= x_0 \quad \textbf{from Theorem 3.4}\\
3\sin(\phi) &= 3\\
\sin(\phi) &= 1\\
\phi &= \frac{\pi}{2}
\end{aligned}
$$

Hence, the equation of motion is $x(t) = 3\sin\left(2t + \frac{\pi}{2}\right)$.

Next, to find when the object passes through the equilibrium position, we solve $x(t)=0$.

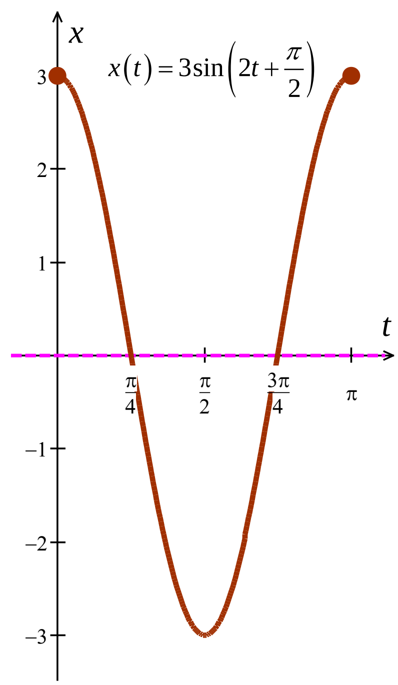
$$3\sin\left(2t+\frac{\pi}{2}\right)=0$$
$$\sin\left(2t+\frac{\pi}{2}\right)=0$$
$$t=-\frac{\pi}{4}+\frac{\pi}{2}k \text{ for integers } k \quad \textbf{after the usual analysis}$$

The object first passes through the equilibrium position at the smallest positive t value, which in this case is $t=\frac{\pi}{4}\approx 0.78$ seconds after the start of the motion.

Common sense suggests that if we release the object below the equilibrium position, the object should be traveling upwards when it first passes through it.

To check this answer, we graph one cycle of $x(t)$. Since our applied domain in this situation is $t\geq 0$, and the period of $x(t)$ is $\frac{2\pi}{\omega}=\frac{2\pi}{2}=\pi$, we graph $x(t)$ over the interval $[0,\pi]$.

Remembering that $x(t)>0$ means the object is below the equilibrium position and $x(t)<0$ means the object is above the equilibrium position, the fact our graph is crossing through the t-axis at $t=\frac{\pi}{4}$ confirms our answer.



2. The only difference between this problem and the previous problem is that we now release the object with an upward velocity of 8 ft./s. We still have $\omega=2$ and $x_0=3$, but now we have $v_0=-8$, the negative indicating the velocity is directed upwards. Here, we get

$$\begin{aligned} A &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \\ &= \sqrt{3^2 + \left(\frac{-8}{2}\right)^2} \\ &= 5. \end{aligned}$$

We use $A = 5$ and $x_0 = 3$ to determine ϕ.

$$\begin{aligned} A\sin(\phi) &= x_0 \quad \textbf{from Theorem 3.4} \\ 5\sin(\phi) &= 3 \\ \sin(\phi) &= \frac{3}{5} \end{aligned}$$

We will need to identify ϕ using the arcsine since $\frac{3}{5}$ is not a common angle. With the sine being positive, ϕ is in Quadrant I or Quadrant II. Knowing whether the cosine is positive or negative will determine which of those quadrants ϕ resides in. Since we know $A = 5$, $\omega = 2$ and $v_0 = -8$, the formula for v_0 from **Theorem 3.4**. will help us find the cosine.

$$\begin{aligned} A\omega\cos(\phi) &= v_0 \\ (5)(2)\cos(\phi) &= -8 \\ \cos(\phi) &= -\frac{4}{5} \end{aligned}$$

This tells us that ϕ is in Quadrant II, so we have $\phi = \pi - \arcsin\left(\frac{3}{5}\right)$.

Hence, $x(t) = 5\sin\left(2t + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right)$.

Now, since the amplitude is 5, the object will travel at most 5 feet above the equilibrium position. This happens when $x(t) = -5$, the negative sign once again signifying that the object is *above* the equilibrium position.

$$\begin{aligned} 5\sin\left(2t + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right) &= -5 \\ \sin\left(2t + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right) &= -1 \end{aligned}$$

Going through the usual machinations, we get $t = \frac{1}{2}\arcsin\left(\frac{3}{5}\right) + \frac{\pi}{4} + \pi k$ for integers *k*. The smallest of these values is when $k = 0$, that is, $t = \frac{1}{2}\arcsin\left(\frac{3}{5}\right) + \frac{\pi}{4} \approx 1.107$ seconds after the start of the motion.

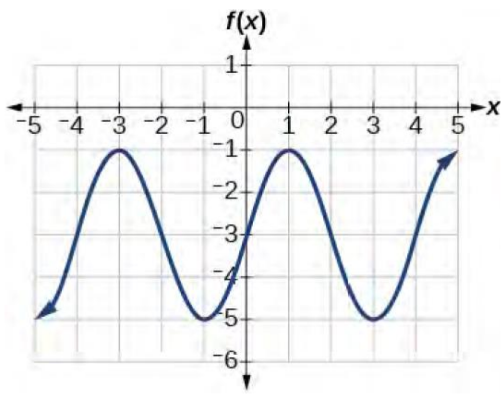
□

3.2 Exercises

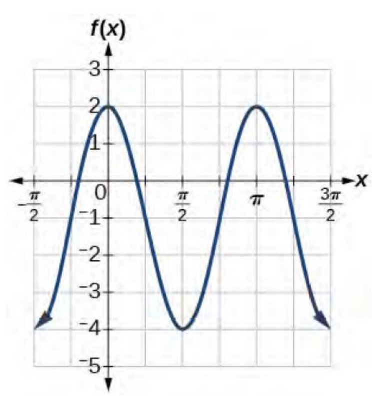
In Exercises 1 – 12, state the period, phase shift, amplitude and vertical shift of the given function. Graph one cycle of the function.

1. $y = 3\sin(x)$
2. $y = \sin(3x)$
3. $y = -2\cos(x)$
4. $y = \cos\left(x - \frac{\pi}{2}\right)$
5. $y = -\sin\left(x + \frac{\pi}{3}\right)$
6. $y = \sin(2x - \pi)$
7. $y = -\frac{1}{3}\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)$
8. $y = \cos(3x - 2\pi) + 4$
9. $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$
10. $y = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4x\right) + 1$
11. $y = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) - \frac{1}{2}$
12. $y = 4\sin(-2\pi x + \pi)$

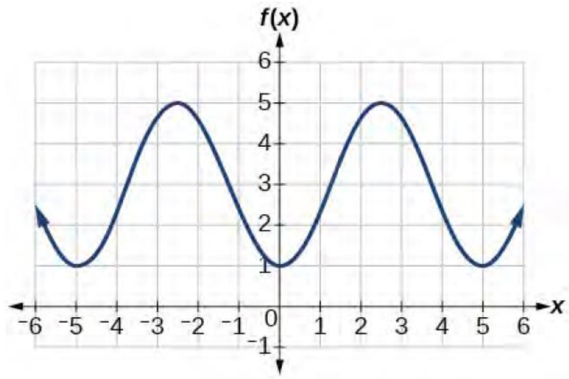
13. Write an equation of the form $S(x) = A\sin(\omega x + \phi) + B$ for the sine function whose graph is shown below.



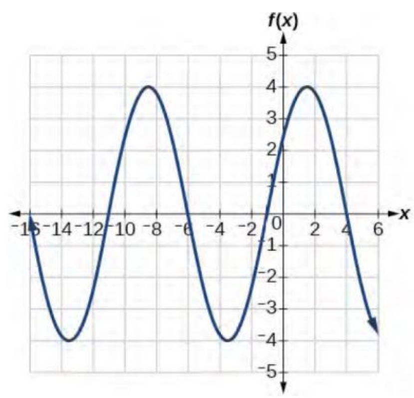
14. Write an equation of the form $C(x) = A\cos(\omega x + \phi) + B$ for the cosine function whose graph is shown below.



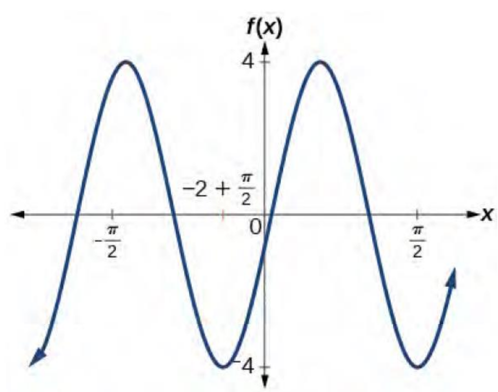
15. Write an equation of the form $C(x) = A\cos(\omega x + \phi) + B$ for the cosine function whose graph is shown below.



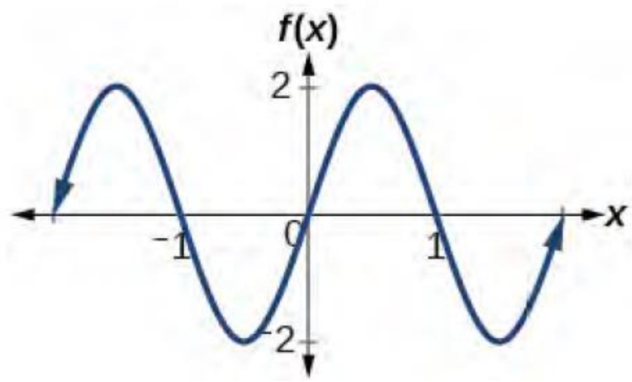
16. Write an equation of the form $S(x) = A\sin(\omega x + \phi) + B$ for the sine function whose graph is shown below.



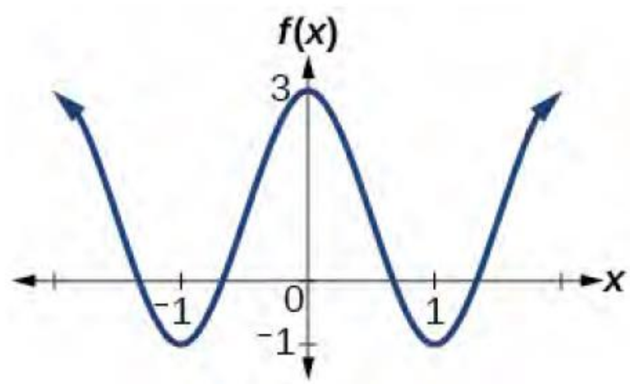
17. Write an equation of the form $C(x) = A\cos(\omega x + \phi) + B$ for the cosine function whose graph is shown below.



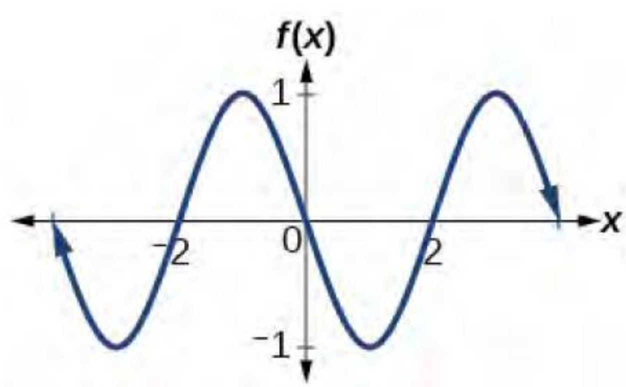
18. Write an equation of the form $S(x) = A\sin(\omega x + \phi) + B$ for the sine function whose graph is shown below.



19. Write an equation of the form $C(x) = A\cos(\omega x + \phi) + B$ for the cosine function whose graph is shown below.



20. Write an equation of the form $S(x) = A\sin(\omega x + \phi) + B$ for the sine function whose graph is shown below.



In Exercises 21 – 22, verify the identity by using technology to graph the right and left hand sides.

21. $\cos^2(x)+\sin^2(x)=1$

22. $\cos(x)=\sin\left(\frac{\pi}{2}-x\right)$

In Exercises 23 – 26, graph the function with the help of technology and discuss the given questions with your classmates.

23. $f(x)=\cos(3x)+\sin(x)$. Is this function periodic? If so, what is the period?

24. $f(x)=\frac{\sin(x)}{x}$. What appears to be the horizontal asymptote of the graph?

25. $f(x)=x\sin(x)$. Graph $y=\pm x$ on the same set of axes and describe the behavior of *f*.

26. $f(x)=\sin\left(\frac{1}{x}\right)$. What's happening as $x\to 0$?

27. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground *t* minutes after the wheel begins to turn.

   a. Find the period, amplitude and vertical shift of $h(t)$.

   b. Find a formula for the height function $h(t)$.

   c. How high off the ground is a person after 5 minutes?

28. Suppose an object weighing 10 pounds is suspended from the ceiling by a spring which stretches 2 feet to its equilibrium position when the object is attached.

   a. Find the spring constant *k* in $\frac{\text{lbs.}}{\text{ft.}}$ and the mass of the object in slugs.

   b. Find the equation of motion of the object if it is released from 1 foot *below* the equilibrium position from rest. When is the first time the object passes through the equilibrium position? In which direction is it heading?

c. Find the equation of motion of the object if it is released from 6 inches *above* the equilibrium position with a *downward* velocity of 2 feet per second. Find when the object passes through the equilibrium position heading downwards for the third time.

3.3 Graphs of the Tangent and Cotangent Functions

Learning Objectives

In this section you will:

- **Graph the tangent and cotangent functions and their transformations. Identify the period and vertical asymptotes.**
- **Learn the properties of the tangent and cotangent functions, including domain and range; determine whether a function is even or odd.**

We now turn our attention to the graphs of the tangent and cotangent functions.

Graph of the Tangent Function

When constructing a table of values for the tangent function, we recall that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

We can use our common values for the fundamental cycles of $y = \cos(x)$ and $y = \sin(x)$ in determining values of the tangent. It follows that the tangent function, $y = \tan(x)$, is undefined at $x = \pi/2$ and $x = 3\pi/2$, both points at which $\cos(x) = 0$.

x	$\cos(x)$	$\sin(x)$	$\tan(x)=\dfrac{\sin(x)}{\cos(x)}$	$(x,\tan(x))$
0	1	0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\left(\frac{\pi}{4},1\right)$
$\frac{\pi}{2}$	0	1	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$\left(\frac{3\pi}{4},-1\right)$
π	-1	0	0	$(\pi,0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$\left(\frac{5\pi}{4},1\right)$
$\frac{3\pi}{2}$	0	-1	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\left(\frac{7\pi}{4},-1\right)$
2π	1	0	0	$(2\pi,0)$

To determine the behavior of the graph of $y = \tan(x)$ when *x* is close to $\pi/2$ or $3\pi/2$, we will look at some values for $\tan(x)$ on both sides of $x = \pi/2$ and on both sides of $x = 3\pi/2$.

- The following chart shows some values for $\tan(x)$ when *x* is less than, but close to $\pi/2$. We note that $\pi/2 \approx 1.571$ and include approximate values of the tangent for the indicated radian measures of *x*. We note that values of $\tan(x)$ are positive, getting larger and larger, as *x* approaches $\pi/2$ from the left. The result is a vertical asymptote at $x = \pi/2$.

x	1.5	1.55	1.56	1.57	$\cdots$	$\frac{\pi}{2} \approx 1.571$
$\tan(x)$	14	48	93	1256	$\cdots$	undefined

Mathematical notation:

$$\text{As } x \to \frac{\pi}{2}^{-}, \ \tan(x) \to \infty.$$

- When *x* is greater than, but close to $\pi/2$, as *x* approaches $\pi/2$ from the right, the values of $\tan(x)$ get smaller and smaller, approaching negative infinity:

$$\text{As } x \to \frac{\pi}{2}^{+}, \ \tan(x) \to -\infty.$$

x	$\frac{\pi}{2} \approx 1.571$	$\cdots$	1.58	1.59	1.6	1.7
$\tan(x)$	undefined	$\cdots$	–109	–52	–34	–8

Noting that $3\pi/2 \approx 4.712$, we look at values for $\tan(x)$ when *x* is close to $3\pi/2$.

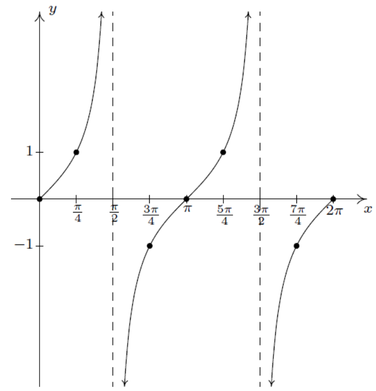
x	4.6	4.65	4.7	4.71	$\cdots$	$\frac{3\pi}{2} \approx 4.712$
$\tan(x)$	9	16	81	419	$\cdots$	undefined

$$\text{As } x \to \frac{3\pi}{2}^{-}, \ \tan(x) \to \infty.$$

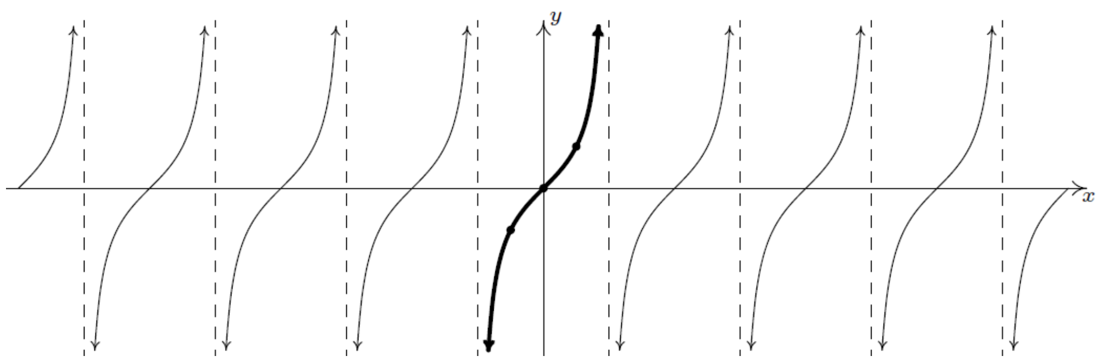
x	$\frac{3\pi}{2} \approx 4.712$	$\cdots$	4.72	4.73	4.75	4.8
$\tan(x)$	undefined	$\cdots$	–131	–57	–27	–11

$$\text{As } x \to \frac{3\pi}{2}^{+},\ \tan(x) \to -\infty$$

Thus we have vertical asymptotes at $x = \pi/2$ and at $x = 3\pi/2$. We also know something about the behavior of the graph as it approaches these vertical asymptotes from each side. Plotting this information followed by the usual 'copy and paste' produces the following two graphs.



The graph of $y = \tan(x)$ over $[0, 2\pi]$.



The graph of $y = \tan(x)$, with fundamental cycle highlighted.

From the graph, it appears the tangent function is periodic with period π. This is, in fact, the case as we will prove in **Section 4.2**, following the introduction of the sum identity for tangent. We take as our fundamental cycle for $y = \tan(x)$ the interval $(-\pi/2, \pi/2)$.

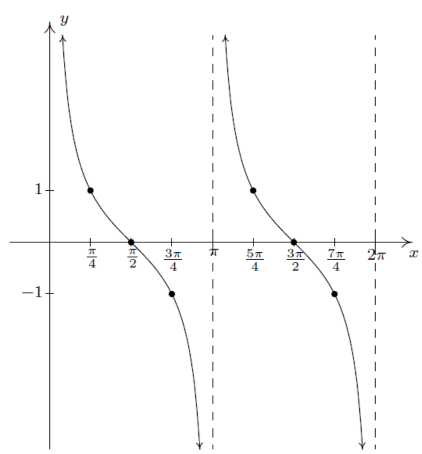
From the graph, it appears that the domain of the tangent function, $y = \tan(x)$, includes all real numbers x except for $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \cdots$. These are the x-values where $\cos(x) = 0$ and are the only numbers for which $y = \tan(x)$ is undefined. Thus, the domain of $y = \tan(x)$ is all real numbers x, excluding $x = \pi/2 + \pi k$ for any integer k. The range of $y = \tan(x)$, as observed from the graph, includes all real numbers.

Graph of the Cotangent Function

It should be no surprise that the graph of the cotangent function behaves similarly to the graph of the tangent function. Plotting $y = \cot(x)$ over the interval $[0, 2\pi]$ results in the graph below.

x	$\cot(x)$	$(x, \cot(x))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	

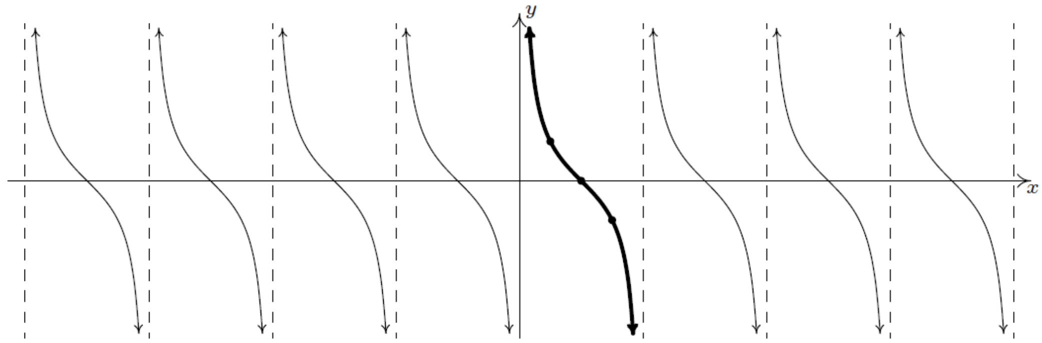
x	$\cot(x)$	$(x, \cot(x))$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	



The graph of $y = \cot(x)$ over $[0, 2\pi]$.

It clearly appears that the period of $\cot(x)$ is π, which is indeed the case and will be revisited in **Section 4.2**. The vertical asymptotes in the interval $[0, 2\pi]$ are $x = 0$, $x = \pi$ and $x = 2\pi$.

We take as one fundamental cycle the interval $(0, \pi)$. A more complete graph of $y = \cot(x)$ is below, with the fundamental cycle highlighted.



The graph of $y = \cot(x)$.

We see, from the graph, the apparent domain of the cotangent function is all real numbers *x* except for $x = 0, \pm\pi, \pm 2\pi, \cdots$. These are the *x*-values where $\sin(x) = 0$ and are the only numbers for which $y = \cot(x)$ is undefined. Thus, the domain of $y = \cot(x)$ is all real numbers *x*, excluding $x = \pi k$, for any integer *k*. The range of $y = \cot(x)$ includes all real numbers.

Note that on the intervals between their vertical asymptotes, both $y = \tan(x)$ and $y = \cot(x)$ are continuous and smooth. In other words, they are continuous and smooth *on their domains*. Both functions are odd, as you can see by the symmetry of the graphs about the origin, and as we will verify algebraically in **Section 4.1**. The following theorem summarizes the properties of the tangent and cotangent functions.

Theorem 3.4. Properties of the Tangent and Cotangent Functions:

- The function $J(x) = \tan(x)$
  - has domain $\left\{x : x \neq \frac{\pi}{2} + \pi k,\ k \text{ is any integer}\right\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(x) = \cot(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is any integer}\}$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π

Graphs of Transformations of the Tangent and Cotangent Functions

Graphing transformations of the tangent and cotangent functions is similar to graphing transformations of the sine and cosine. **Theorem 3.2**, in which $g(x) = A \cdot f(\omega x + \phi) + B$, can be used for both, but there are a few differences to be aware of.

- The period of both tangent and cotangent is π, and so the period of transformations will be $\frac{\pi}{\omega}$.
- The vertical scaling is $|A|$, but tangent and cotangent do not have amplitude since they do not possess the wavelike characteristics of the sine and cosine functions.
- The vertical shift is B, but a baseline is not defined for transformations of the tangent and cotangent functions.
- Since we do not have a baseline, vertical scaling should be completed before the vertical shift.
- The horizontal shift, $-\frac{\phi}{\omega}$, is unchanged and will be our last step in graphing transformations of the tangent and cotangent.

Example 3.3.1. Graph one cycle of the following functions. Find the period.

1. $f(x) = 3 - \tan\left(\frac{x}{2}\right)$

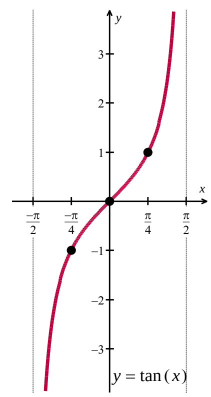
2. $g(x) = 3\cot\left(\frac{\pi}{4}x + \frac{\pi}{2}\right) + 1$

Solution.

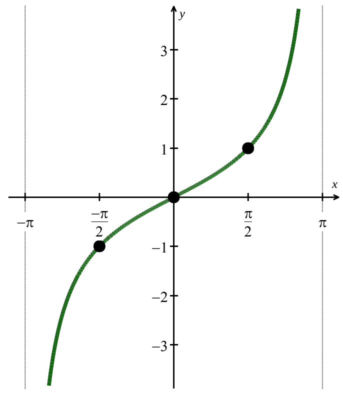
1. We will graph the function $f(x) = 3 - \tan\left(\frac{x}{2}\right)$ through a series of transformations to the fundamental cycle of the graph of $y = \tan(x)$. Before proceeding, we rewrite the function $f(x)$ in a format that will simplify this process:

$$f(x) = -\tan\left(\frac{1}{2}x + 0\right) + 3$$

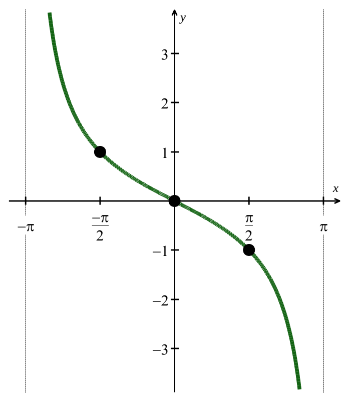
The fundamental cycle of $y = \tan(x)$ is restricted to the interval $(-\pi/2, \pi/2)$. In tracking transformations of the graph of $y = \tan(x)$, we will start with the quarter marks $x = -\pi/2$, $-\pi/4$, 0, $\pi/4$ and $\pi/2$. The first and last of these quarter marks are place markers for vertical asymptotes, but it will be important to track vertical asymptotes as well as points in the transformation process. Hence, we start with the two vertical asymptotes $x = -\pi/2$ and $x = \pi/2$, and the three points $(-\pi/4, -1)$, $(0,0)$ and $(\pi/4, 1)$.



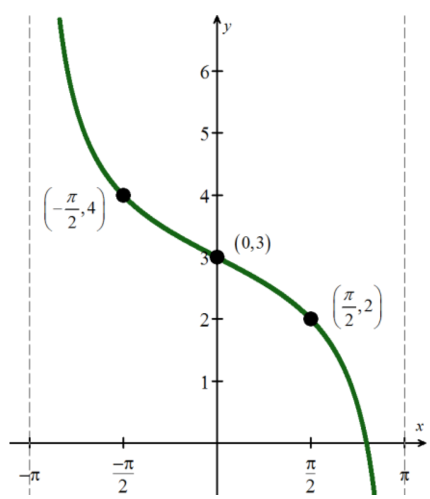
- We begin by determining the period of $f(x) = -\tan\left(\frac{1}{2}x + 0\right) + 3$. Since the coefficient of x is $\frac{1}{2}$, the period is $\frac{\pi}{(1/2)} = 2\pi$. Stretching the period π of the tangent function to the period 2π of the transformed function will result in quarter marks of $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$ and π.



- The vertical scaling is $|-1| = 1$, so the graph will not be stretched in the vertical direction. However, $-1 < 0$ tells us the graph will be reflected about the *x*-axis, as follows.

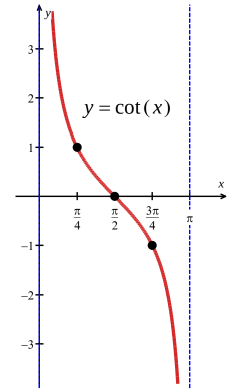


- The vertical shift is 3, indicating a shift up by 3 units.
- Finally, the horizontal shift is 0, so the graph will not be shifted left or right.



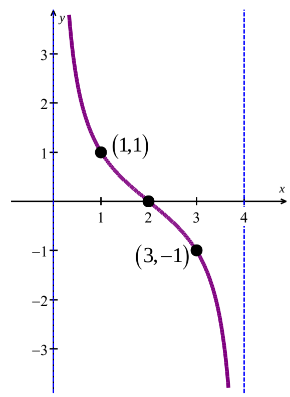
One cycle of $f(x) = 3 - \tan\left(\frac{x}{2}\right)$.

2. We graph the function $g(x)=3\cot\left(\frac{\pi}{4}x+\frac{\pi}{2}\right)+1$ using transformations of $y=\cot(x)$. The fundamental cycle of the cotangent function is on the domain $(0,\pi)$. We use the quarter marks 0, $\pi/4$, $\pi/2$, $3\pi/4$ and π. Before proceeding, the function $g(x)$ can be written as follows.

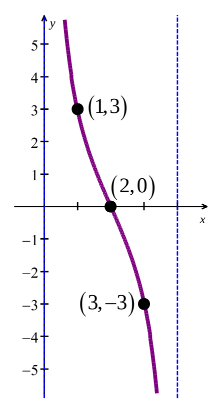


$$g(x)=3\cot\left[\frac{\pi}{4}(x+2)\right]+1$$

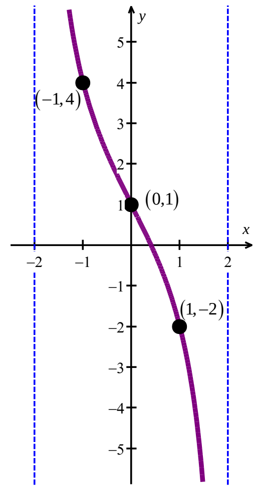
- The period of $g(x)$ is $\frac{\pi}{(\pi/4)}=4$, resulting in quarter marks 0, 1, 2, 3 and 4, shown below.



- The vertical scaling is $|3|=3$. This will stretch the graph vertically by a factor of 3. We multiply the *y*-coordinates of the points at the second and fourth quarter marks, i.e. $(1,1)$ and $(3,-1)$, by 3 to obtain the transformed points $(1,3)$ and $(3,-3)$. The middle point, $(2,0)$, does not move. Note these transformations in the graph to the right.



- Next, the graph is shifted up by 1 unit.
- Lastly, a horizontal shift of –2 moves the graph two units to the left.



One cycle of $g(x) = 3\cot\left(\frac{\pi}{4}x + \frac{\pi}{2}\right) + 1$.

□

Since the number of classical applications involving sinusoids far outnumber those involving tangent and cotangent functions, we omit the more extensive coverage here that was given to sinusoidal functions. The ambitious reader is invited to explore further results from this section. We next move on to graphs of secant and cosecant functions.

3.3 Exercises

In Exercises 1 – 6, graph one cycle of the given function. State the period of the function.

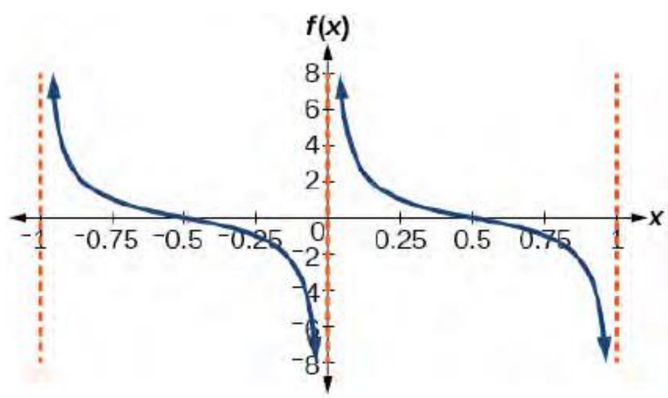
1. $y = \tan\left(x - \frac{\pi}{3}\right)$
2. $y = 2\tan\left(\frac{1}{4}x\right) - 3$
3. $y = \frac{1}{3}\tan(-2x - \pi) + 1$
4. $y = \cot\left(x + \frac{\pi}{6}\right)$
5. $y = -11\cot\left(\frac{1}{5}x\right)$
6. $y = \frac{1}{3}\cot\left(2x + \frac{3\pi}{2}\right) + 1$

In Exercises 7 – 15, graph two full periods of each function. State the period and asymptotes.

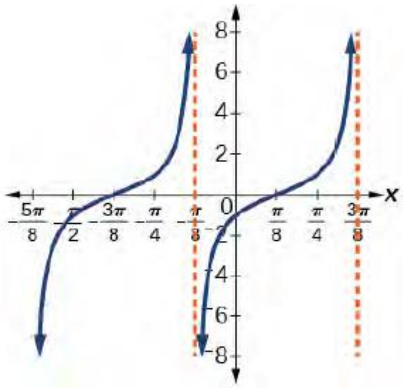
7. $f(x) = \tan(x)$
8. $f(x) = \cot(x)$
9. $f(x) = 2\tan(4x - 32)$
10. $f(x) = \tan\left(\frac{\pi}{2}x\right)$
11. $f(x) = \tan\left(x - \frac{\pi}{2}\right)$
12. $f(x) = 4\tan(x)$
13. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$
14. $f(x) = \pi\tan(\pi x - \pi) - \pi$
15. $f(x) = -3\cot(2x)$

In Exercises 16 – 17, find an equation for the graph of each function.

16.



17.



18. Verify the identity $\tan(x+\pi)=\tan(x)$ by using technology to graph the right and left hand sides.

19. Graph the function $f(x)=x-\tan(x)$ with the help of technology. Graph $y=x$ on the same set of axes and describe the behavior of *f*.

20. The function $f(x)=20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time *x*, in seconds, and distance $f(x)$, in feet.

   a. Graph the function $f(x)$ on the interval $[0,5]$.

   b. Find and interpret the vertical stretching factor, the period and any asymptotes.

   c. Evaluate $f(1)$ and $f(2.5)$ and discuss the function's values at those inputs.

3.4 Graphs of the Secant and Cosecant Functions

Learning Objectives

In this section you will:

- **Graph the secant and cosecant functions and their transformations. Identify the period and vertical asymptotes.**
- **Learn the properties of the secant and cosecant functions, including domain and range; determine whether a function is even or odd.**

Finally, we turn our attention to graphing the secant and cosecant functions.

Graph of the Secant Function

To get started, we graph the secant function using our table of values for the fundamental cycle of $y = \cos(x)$. We can take reciprocals of these cosine values since $\sec(x) = \dfrac{1}{\cos(x)}$.

x	$\cos(x)$	$\sec(x) = \dfrac{1}{\cos(x)}$	$(x, \sec(x))$
0	1	1	$(0,1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$

The domain of the secant function excludes all odd multiples of $\pi/2$ since these are the values of *x* for which $\cos(x)=0$. In our table based on the fundamental cycle of $y=\cos(x)$, the secant is undefined at $x=\pi/2$ and $x=3\pi/2$. These are both *x*-values at which vertical asymptotes occur. To determine the behavior of the graph of $y=\sec(x)$ when *x* is close to $\pi/2$ or $3\pi/2$, we will look at some values for $\sec(x)$ on both sides of $x=\pi/2$ and on both sides of $x=3\pi/2$.

- The following chart shows some values for $\sec(x)$ when *x* is less than, but close to $\pi/2$. We note that $\pi/2\approx1.571$ and include approximate values of the secant for the indicated radian measures of *x*.

x	1.5	1.55	1.56	1.57	$\cdots$	$\frac{\pi}{2}\approx1.571$
$\sec(x)=\dfrac{1}{\cos(x)}$	14	48	93	1256	$\cdots$	undefined

We note that values of $\sec(x)$ are positive, getting larger and larger, as *x* approaches $\pi/2$ from the left:

$$\text{As } x\to\frac{\pi}{2}^-,\ \sec(x)\to\infty.$$

- We next look at values of $\sec(x)$ when *x* is greater than, but close to $\pi/2$.

x	$\frac{\pi}{2}\approx1.571$	$\cdots$	1.58	1.59	1.6	1.7
$\sec(x)=\dfrac{1}{\cos(x)}$	undefined	$\cdots$	–109	–52	–34	–8

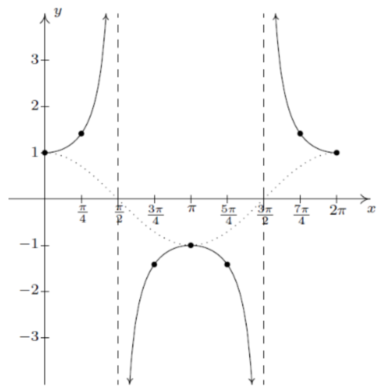
As *x* approaches $\pi/2$ from the right, the values of $\sec(x)$ get smaller and smaller, approaching negative infinity:

$$\text{As } x\to\frac{\pi}{2}^+,\ \sec(x)\to-\infty.$$

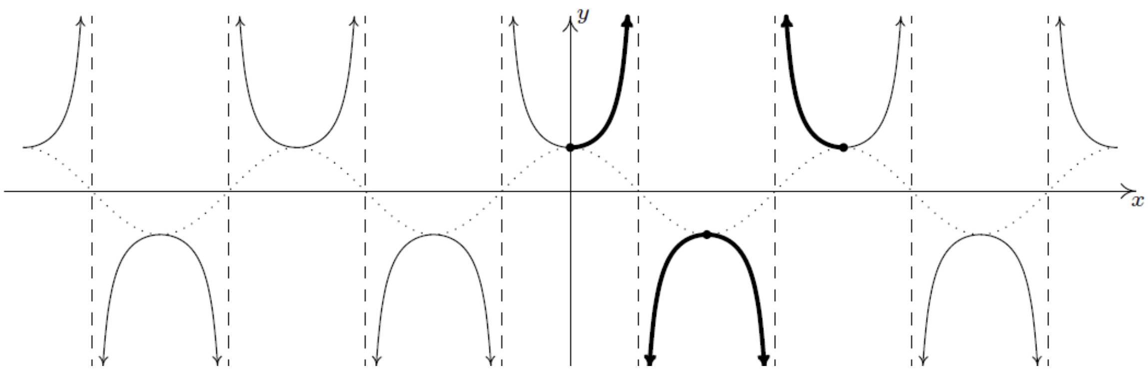
- Using a similar analysis, which we leave to the reader,

$$\text{as } x\to\frac{3\pi}{2}^-,\ \sec(x)\to-\infty, \text{ and}$$
$$\text{as } x\to\frac{3\pi}{2}^+,\ \sec(x)\to\infty.$$

Plotting points and asymptotes, with graph behavior echoing the above results, we have the following two graphs. The second graph is an extension of the first.



The graph of $y = \sec(x)$ over $[0, 2\pi]$.



The graph of $y = \sec(x)$, with fundamental cycle highlighted.

In the above illustration, the dotted graph of $y = \cos(x)$ is included for reference. It is helpful to graph the secant function by starting with a graph of the cosine function and sketching vertical asymptotes at each *x*-value for which $\cos(x) = 0$. The points where $\cos(x) = \pm 1$ are also points where $\sec(x) = \pm 1$. After drawing the asymptotes and marking the points where $\sec(x) = \pm 1$, a rough graph of $y = \sec(x)$ can quickly be completed by sketching the 'U' shapes of the secant function.

Since $\cos(x)$ is periodic with period 2π, it follows that $\sec(x)$ is also periodic with period 2π.[1] Due to the close relationship between the cosine and secant, the fundamental cycle of the secant function is the same as that of the cosine function. We previously noted that the domain of the secant function excludes all odd multiples of $\pi/2$. The range of $y = \sec(x)$, as observed graphically, includes all real numbers *y*

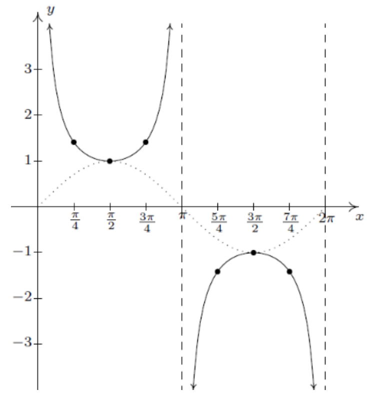
[1] Provided sec(α) and sec(β) are defined, sec(α) = sec(β) if and only if cos(α) = cos(β). Hence, sec(x) inherits its period from cos(x).

such that $y \le -1$ or $y \ge 1$, or equivalently $|y| \ge 1$. By thinking of the secant function as being the reciprocal of the cosine function, a similar result can be obtained algebraically.

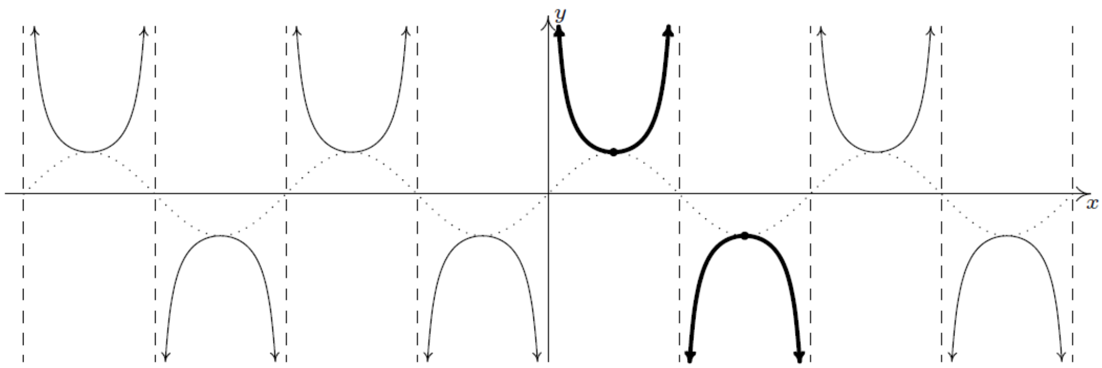
Graph of the Cosecant Function

As one would expect, to graph $y = \csc(x)$, we begin with $y = \sin(x)$ and take reciprocals of the corresponding *y*-values. Here, we encounter issues at $x = 0$, $x = \pi$ and $x = 2\pi$. These are locations of vertical asymptotes. Proceeding with an analysis of graph behavior near these asymptotes, we graph the fundamental cycle of $y = \csc(x)$ followed by an extended graph of $y = \csc(x)$. A dotted graph of $y = \sin(x)$ is included for reference.

x	$\sin(x)$	$\csc(x) = \dfrac{1}{\sin(x)}$	$(x, \csc(x))$
0	0	undefined	
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\dfrac{\pi}{4}, \sqrt{2}\right)$
$\dfrac{\pi}{2}$	1	1	$\left(\dfrac{\pi}{2}, 1\right)$
$\dfrac{3\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\dfrac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\dfrac{5\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$
$\dfrac{3\pi}{2}$	-1	-1	$\left(\dfrac{3\pi}{2}, -1\right)$
$\dfrac{7\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\dfrac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	



The fundamental cycle of $y = \csc(x)$.



The graph of $y = \csc(x)$.

Since $y = \sin(x)$ and $y = \cos(x)$ are merely phase shifts of each other, so too are $y = \csc(x)$ and $y = \sec(x)$. As with the tangent and cotangent functions, both $y = \sec(x)$ and $y = \csc(x)$ are continuous and smooth on their domains. The following theorem summarizes the properties of the secant and cosecant functions. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

Theorem 3.5. Properties of the Secant and Cosecant Functions:

- The function $F(x)=\sec(x)$
  - has domain $\left\{x : x \neq \frac{\pi}{2}+\pi k,\ k \text{ is any integer}\right\}$
  - has range $\{y : |y| \geq 1\} = (-\infty,-1] \cup [1,\infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(x)=\csc(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is any integer}\}$
  - has range $\{y : |y| \geq 1\} = (-\infty,-1] \cup [1,\infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π

Graphs of Transformations of the Secant and Cosecant Functions

In the next example, we discuss graphing more general secant and cosecant functions

Example 3.4.1. Graph one cycle of the following functions. State the period of each.

1. $f(x)=1-2\sec(2x)$
2. $g(x)=\dfrac{\csc(\pi-\pi x)-5}{3}$

Solution.

1. Before graphing $f(x)=1-2\sec(2x)$ we will graph $y=1-2\cos(2x)$ to use as a guide. Using the technique from **Section 3.1**, we will start with a graph of the cosine function and apply appropriate transformations. First, however, it helps to rewrite $y=1-2\cos(2x)$ in a format suggested by **Theorem 3.2**.

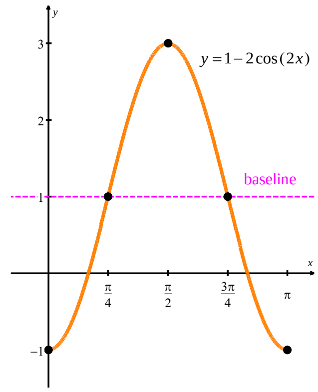
$$y=-2\cos(2x+0)+1$$

- The period is $\frac{2\pi}{2}=\pi$.

- The vertical shift is 1, for a baseline of $y=1$.

• The amplitude is $|-2| = 2$ and, since $-2 < 0$, the graph will be reflected about the baseline.

• There is no horizontal shift.

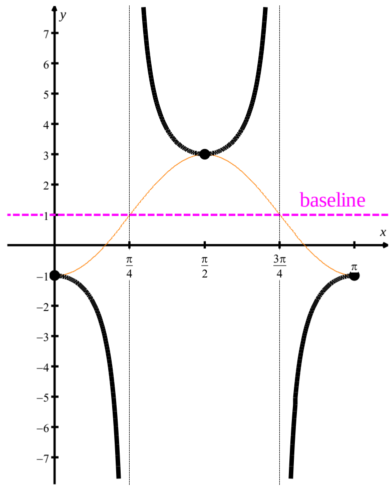


One cycle of $y = 1 - 2\cos(2x)$.

Next, to graph $f(x) = 1 - 2\sec(2x)$, we observe the following.

- Points where the graph of the cosine function crosses the baseline are vertical asymptotes of the secant function.
- Maximum values of the cosine function occur at the lowest points on the secant curve.
- Minimum values of the cosine function occur at points that are maximum values for the secant.
- Transformations of the secant function retain the familiar "U" shape between vertical asymptotes.

Below, we use $y = 1 - 2\cos(2x)$ as a guide in graphing one cycle of $f(x)$.



One cycle of $f(x)=1-2\sec(2x)$.

Since one cycle is graphed on the interval $[0,\pi]$, the period is $\pi-0=\pi$.

2. We graph the function $g(x)=\dfrac{\csc(\pi-\pi x)-5}{3}$ by first graphing the corresponding sine function, $y=\dfrac{\sin(\pi-\pi x)-5}{3}$, and then using the sine function as a guide. The first step is to rewrite the sine function in the form suggested by **Theorem 3.2**.

$$\begin{aligned} y &= \frac{\sin(\pi-\pi x)-5}{3} \\ &= \frac{1}{3}\sin(-\pi x+\pi)-\frac{5}{3} \\ &= \frac{1}{3}\sin\left[-(\pi x-\pi)\right]-\frac{5}{3} \\ &= -\frac{1}{3}\sin(\pi x-\pi)-\frac{5}{3} \end{aligned}$$ **from odd property of sine**

We proceed to graph $y=-\frac{1}{3}\sin(\pi x-\pi)-\frac{5}{3}$ as a transformation of $y=\sin(x)$, referring to **Theorem 3.2** as necessary.

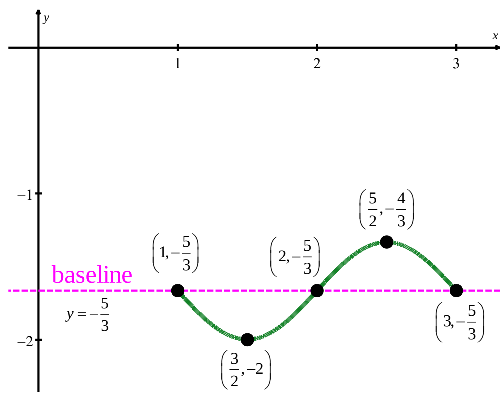
- The period is $\dfrac{2\pi}{\pi}=2$.

- The vertical shift is $-\dfrac{5}{3}$ for a baseline of $y=-\dfrac{5}{3}$.

- The amplitude is $\left|-\frac{1}{3}\right| = \frac{1}{3}$ and, since $-\frac{1}{3} < 0$, the graph will be reflected about the baseline.

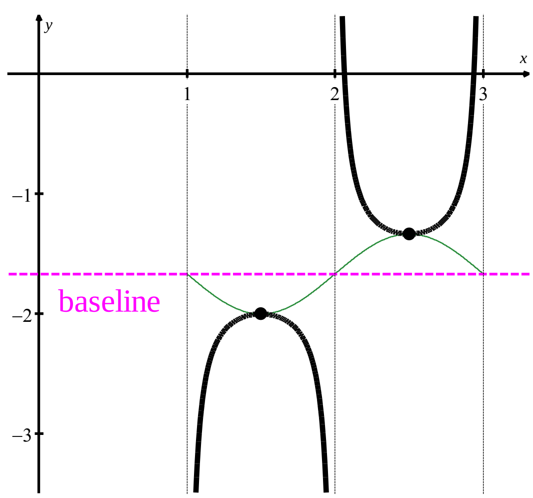
- Since $\pi x - \pi = \pi(x-1)$, the graph will be shifted to the right by 1 unit.

Locating quarter marks and corresponding points results in the following graph of $y = \frac{\sin(\pi - \pi x) - 5}{3}$.



One cycle of $y = \frac{\sin(\pi - \pi x) - 5}{3}$.

We use the transformed sine graph as a guide in sketching one cycle of $g(x) = \frac{\csc(\pi - \pi x) - 5}{3}$.



One cycle of $g(x) = \frac{\csc(\pi - \pi x) - 5}{3}$.

We find the period to be $3-1=2$.

□

While real world applications of secants and cosecants are limited, at least in comparison to the large number of available sinusoidal applications, a couple of examples are included in the Exercises. We conclude Chapter 3 with the expectation of putting to good use the properties and graphs of the trigonometric functions that have been introduced in this chapter.

3.4 Exercises

In Exercises 1 – 6, graph one cycle of the given function. State the period of the function.

1. $y = \sec\left(x - \frac{\pi}{2}\right)$

2. $y = -\csc\left(x + \frac{\pi}{3}\right)$

3. $y = -\frac{1}{3}\sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

4. $y = \csc(2x - \pi)$

5. $y = \sec(3x - 2\pi) + 4$

6. $y = \csc\left(-x - \frac{\pi}{4}\right) - 2$

In Exercises 7 – 18, graph two full periods of each function. State the period and asymptotes.

7. $f(x) = \sec(x)$

8. $f(x) = \csc(x)$

9. $f(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

10. $f(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

11. $f(x) = 2\csc(x)$

12. $f(x) = -\frac{1}{4}\csc(x)$

13. $f(x) = 4\sec(3x)$

14. $f(x) = 7\sec(5x)$

15. $f(x) = \frac{9}{10}\csc(\pi x)$

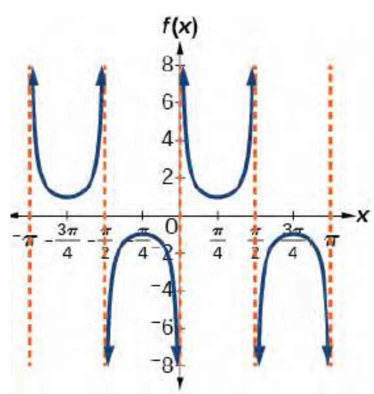
16. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

17. $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

18. $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

In Exercises 19 – 22, find an equation for the graph of each function.

19.



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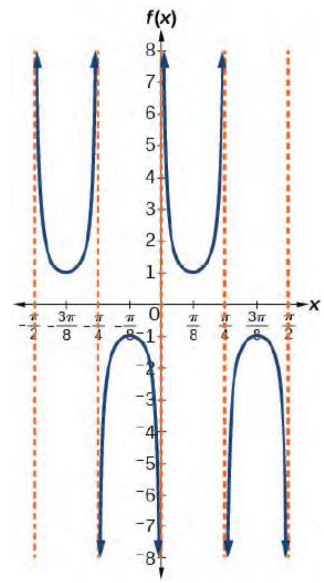

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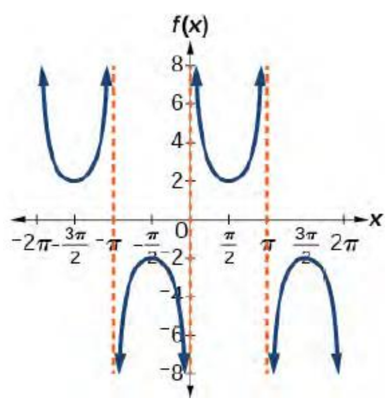
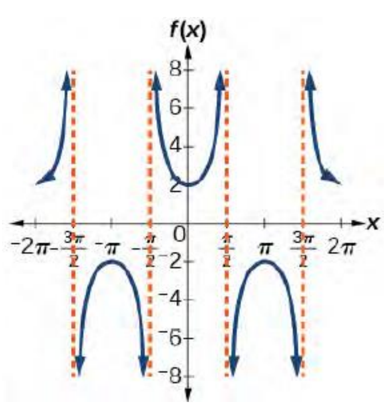


22.



23. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let *x*, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and *x* is measured negative to the left and positive to the right. The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.

a. What is a reasonable domain for $d(x)$?

b. Graph $d(x)$ on the domain.

c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.

d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the nearest hundredth.

e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the nearest hundredth.

f. What is the minimum distance between the fisherman and the boat? When does this occur?

24. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after *x* days, for *x* in the interval 0 to 30 days, is given by $g(x) = 250{,}000\csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0, 30]$.

b. Evaluate $g(5)$ and interpret the information.

c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?

d. Find and discuss the meaning of any vertical asymptotes.

CHAPTER 4
TRIGONOMETRIC IDENTITIES AND FORMULAS

Chapter Outline

4.1 The Even/Odd Identities
4.2 Sum and Difference Identities
4.3 Double Angle Identities
4.4 Power Reduction and Half Angle Formulas
4.5 Product to Sum and Sum to Product Formulas
4.6 Using Sum Identities in Determining Sinusoidal Formulas

Introduction

In Chapter 4, we begin exploring new trigonometric identities and formulas which provide us with varying ways to represent the same trigonometric expression.

Section 4.1 advances our study of even/odd identities and includes algebraic proofs of these identities. In Section 4.2, we discover the essential sum and difference identities upon which many of the remaining identities and formulas depend. Section 4.3 introduces the double angles identities, from which the power reduction and half angle formulas are derived in Section 4.4. Along with the sum and difference identities, the half angle formulas will help us obtain exact values for trigonometric functions of some 'non-common' angles. The product to sum and sum to product formulas introduced in Section 4.5 will allow further manipulation of trigonometric expressions. Finally, Section 4.6 revisits the formulas of sinusoids first introduced in Section 3.2. With the assistance of the sum and difference identities, we may now write sinusoidal formulas in a standard format that will simplify graphing.

Throughout Chapter 4, attention will be paid to finding exact values of trigonometric functions, writing trigonometric expressions in varying formats, and verifying trigonometric identities. These skills will be important in the future study of calculus.

4.1 The Even/Odd Identities

Learning Objectives

In this section you will:

- **Learn the even/odd identities.**
- **Use the even/odd identities in simplifying trigonometric expressions.**
- **Use the even/odd identities in verifying trigonometric identities.**

In **Section 2.4**, we saw the utility of the Pythagorean identities, **Theorem 2.5**, along with the reciprocal and quotient identities from **Theorem 2.3**. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we formally introduce the even/odd identities[1], while recalling their graphical significance from Chapter 3.

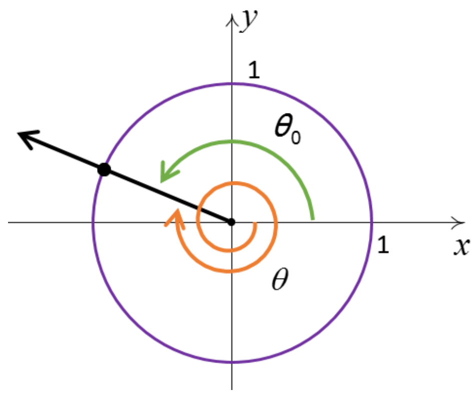
The Even/Odd Identities

Theorem 4.1. The Even/Odd Identities: For all applicable angles θ,

- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

In light of the reciprocal and quotient identities, **Theorem 2.3**, it suffices to show $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. The remaining four circular functions can be expressed in terms of $\cos(\theta)$ and $\sin(\theta)$ so the proofs of their even/odd identities are left as exercises.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \le \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive *x*-axis to the terminal side of θ as pictured to the right.) Since θ and θ_0 are coterminal, $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.



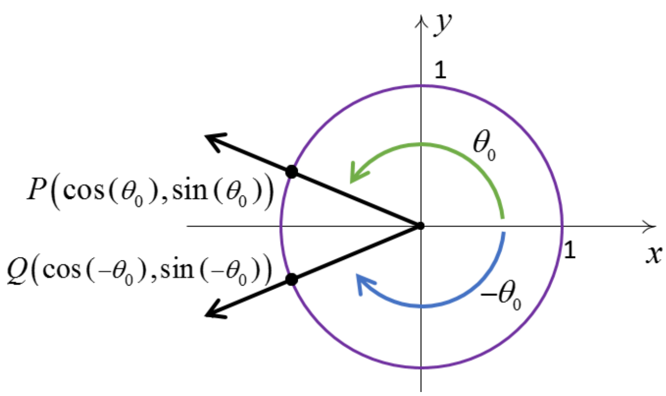
[1] As mentioned at the end of **Section 2.3**, properties of the circular functions, when thought of as functions of angles in radian measure, hold equally well if we view these functions as functions of real numbers. Not surprisingly, the even/odd properties of the circular functions are so named because they identify cosine and secant as even functions, while the remaining four circular functions are odd.

We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k so that $\theta = \theta_0 + 2\pi k$. Therefore,

$$\begin{aligned} -\theta &= -\theta_0 - 2\pi k \\ &= -\theta_0 + 2\pi(-k). \end{aligned}$$

Since k is an integer, so is $-k$, which means $-\theta$ is coterminal with $-\theta_0$. Hence, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$.

Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.



Since θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the x-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$. Since the cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required.

Simplifying Expressions

The even/odd identities are readily demonstrated using any of the common angles noted in **Section 2.2**. Their true utility, however, lies not in computation, but in simplifying expressions involving the circular functions.

Example 4.1.1. Use identities to fully simplify the expression: $(1+\sin(x))(1+\sin(-x))$.

Solution. We begin with the odd identity of the sine function.

$$\begin{aligned}(1+\sin(x))(1+\sin(-x)) &= (1+\sin(x))(1-\sin(x)) && \textbf{since } \sin(-x)=-\sin(x)\\ &= 1-\sin^2(x) && \textbf{difference of squares}\\ &= \cos^2(x) && \cos^2(x)=1-\sin^2(x)\end{aligned}$$

□

Verifying Identities

This section ends with the proof of a trigonometric identity. Looking back at **Section 2.4**, where we began verifying identities, we can now add the even/odd identities to the Pythagorean, reciprocal and quotient identities as tools in proving that identities are true.

Example 4.1.2. Verify the identity: $\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\cos(\theta)-\sin(\theta)$.

Solution. We begin with the left, more complicated, side.

$$\begin{aligned}\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{(\sin(-\theta))^2-(\cos(-\theta))^2}{\sin(-\theta)-\cos(-\theta)}\\ &= \frac{(-\sin(\theta))^2-(\cos(\theta))^2}{-\sin(\theta)-\cos(\theta)} && \textbf{even / odd identities}\\ &= \frac{(\sin(\theta))^2-(\cos(\theta))^2}{-\sin(\theta)-\cos(\theta)}\\ &= \frac{(\sin(\theta)-\cos(\theta))(\sin(\theta)+\cos(\theta))}{-(\sin(\theta)+\cos(\theta))} && \textbf{difference of squares}\\ &= \frac{\sin(\theta)-\cos(\theta)}{-1}\\ &= \cos(\theta)-\sin(\theta)\end{aligned}$$

□

4.1 Exercises

1. We know $g\left(x\right)=\cos(x)$ is an even function, and $f\left(x\right)=\sin(x)$ and $h\left(x\right)=\tan(x)$ are odd functions. What about $G\left(x\right)=\cos^2(x)$, $F\left(x\right)=\sin^2(x)$ and $H\left(x\right)=\tan^2(x)$? Are they even, odd or neither? Why?

2. Examine the graph of $f\left(x\right)=\sec(x)$ on the interval $\left[-\pi,\pi\right]$. How can we tell whether the function is even or odd by only observing the graph of $f\left(x\right)=\sec(x)$?

In Exercises 3 – 8, use identities to fully simplify the expression.

3. $\sin(-x)\cos(-x)\csc(-x)$

4. $\csc(x)+\cos(x)\cot(-x)$

5. $\dfrac{\cot(t)+\tan(t)}{\sec(-t)}$

6. $3\sin^3(t)\csc(t)+\cos^2(t)+2\cos(-t)\cos(t)$

7. $-\tan(-x)\cot(-x)$

8. $\dfrac{-\sin(-x)\cos(x)\sec(x)\csc(x)\tan(x)}{\cot(x)}$

In Exercises 9 – 14, use the even/odd identities to verify the identity. Assume all quantities are defined.

9. $\sin(3\pi-2\theta)=-\sin(2\theta-3\pi)$

10. $\cos\left(-\dfrac{\pi}{4}-5t\right)=\cos\left(5t+\dfrac{\pi}{4}\right)$

11. $\tan\left(-t^2+1\right)=-\tan\left(t^2-1\right)$

12. $\csc(-\theta-5)=-\csc(\theta+5)$

13. $\sec(-6t)=\sec(6t)$

14. $\cot(9-7\theta)=-\cot(7\theta-9)$

In Exercises 15 – 18, prove or disprove the identity.

15. $\dfrac{1}{1+\cos(x)}-\dfrac{1}{1-\cos(-x)}=-2\cot(x)\csc(x)$

16. $\dfrac{\tan(x)}{\sec(x)}\sin(-x)=\cos^2(x)$

17. $\dfrac{\sec(-x)}{\tan(x)+\cot(x)}=-\sin(-x)$

18. $\dfrac{1+\sin(x)}{\cos(x)}=\dfrac{\cos(x)}{1+\sin(-x)}$

19. Verify the even/odd identities for tangent, cosecant, secant and cotangent.

4.2 The Sum and Difference Identities

Learning Objectives

In this section you will:

- **Learn the sum and difference identities for cosine, sine and tangent.**
- **Use the sum and difference identities to find values of trigonometric functions.**
- **Use the sum and difference identities in verifying trigonometric identities.**
- **Learn and apply the cofunction identities.**

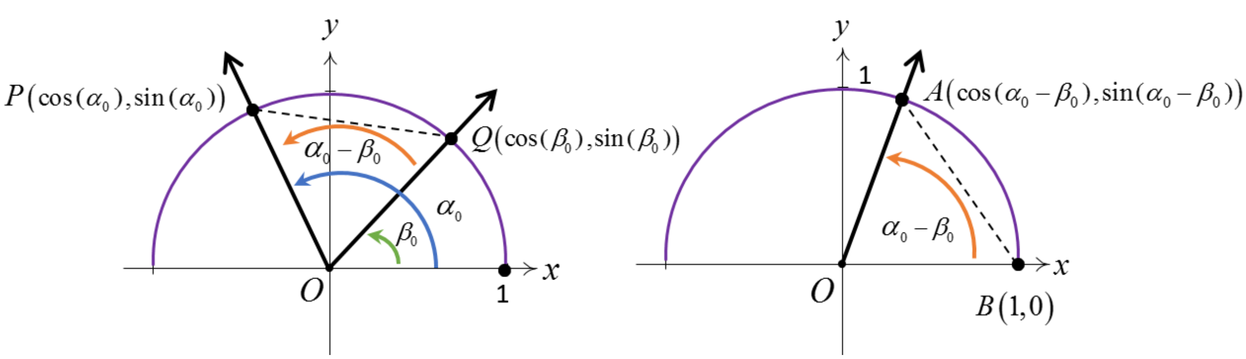
We begin with a theorem introducing the sum and difference identities for the cosine function, followed by a proof of the theorem.

The Sum and Difference Identities for the Cosine Function

Theorem 4.2: Sum and Difference Identities for Cosine.

- $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the even/odd identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $\alpha-\beta$ is coterminal with $\alpha_0-\beta_0$. Consider the following case where $\alpha_0 \geq \beta_0$.



Since the angles *POQ* and *AOB* are congruent, the distance between *P* and *Q* is equal to the distance between *A* and *B*.[1] Using the distance formula to determine distances *QP* and *BA*, we have

$$\sqrt{\left[\cos(\alpha_0)-\cos(\beta_0)\right]^2+\left[\sin(\alpha_0)-\sin(\beta_0)\right]^2}=\sqrt{\left[\cos(\alpha_0-\beta_0)-1\right]^2+\sin\left[(\alpha_0-\beta_0)-0\right]^2}$$

or, after squaring both sides:

$$\left[\cos(\alpha_0)-\cos(\beta_0)\right]^2+\left[\sin(\alpha_0)-\sin(\beta_0)\right]^2=\left[\cos(\alpha_0-\beta_0)-1\right]^2+\sin\left[(\alpha_0-\beta_0)-0\right]^2.$$

Expanding the **left hand side**, then using the Pythagorean identities $\cos^2(\alpha_0)+\sin^2(\alpha_0)=1$ and $\cos^2(\beta_0)+\sin^2(\beta_0)=1$, we get the following:

$$\begin{aligned}\left[\cos(\alpha_0)-\cos(\beta_0)\right]^2+\left[\sin(\alpha_0)-\sin(\beta_0)\right]^2 &= \cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)+\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\\ &= \cos^2(\alpha_0)+\sin^2(\alpha_0)+\cos^2(\beta_0)+\sin^2(\beta_0)-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\ &= 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0).\end{aligned}$$

Turning our attention to the **right hand side**, we will use $\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)=1$:

$$\begin{aligned}\left[\cos(\alpha_0-\beta_0)-1\right]^2+\sin\left[(\alpha_0-\beta_0)-0\right]^2 &= \cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1+\sin^2(\alpha_0-\beta_0)\\ &= 1+\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)\\ &= 2-2\cos(\alpha_0-\beta_0).\end{aligned}$$

Putting in all together, we get

$$2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)=2-2\cos(\alpha_0-\beta_0)$$

which simplifies as follows:

$$\begin{aligned}2-2\cos(\alpha_0-\beta_0) &= 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) && \textbf{after swapping sides}\\ -2\cos(\alpha_0-\beta_0) &= -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) && \textbf{after subtracting 2 from each side}.\\ \cos(\alpha_0-\beta_0) &= \cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0) && \textbf{after dividing through by - 2}\end{aligned}$$

Since α and α_0, β and β_0, $\alpha-\beta$ and $\alpha_0-\beta_0$, are all coterminal pairs of angles, we have

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[1] In the picture, the triangles *POQ* and *AOB* are congruent. However, $\alpha_0-\beta_0$ could be 0 or it could be π, neither of which makes a triangle. Or, $\alpha_0-\beta_0$ could be larger than π, which makes a triangle, just not the one we've drawn. You should think about these three cases.

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).$$

For the case where $\alpha_0 \le \beta_0$, we can apply the above argument to the angle $\beta_0-\alpha_0$ to obtain the identity

$$\cos(\beta_0-\alpha_0)=\cos(\beta_0)\cos(\alpha_0)+\sin(\beta_0)\sin(\alpha_0).$$

Applying the even identity of cosine, we get

$$\begin{aligned}\cos(\beta_0-\alpha_0)&=\cos(-(\beta_0-\alpha_0))\\&=\cos(\alpha_0-\beta_0),\end{aligned}$$

from which it follows that $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$.

To verify the sum identity for cosine, we use the difference identity along with the even/odd identities:

$$\begin{aligned}\cos(\alpha+\beta)&=\cos(\alpha-(-\beta))\\&=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta) && \textbf{difference identity for cosine}\\&=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(-\beta) && \textbf{even identity of cosine}\\&=\cos(\alpha)\cos(\beta)+\sin(\alpha)(-\sin(\beta)) && \textbf{odd identity of sine}\\&=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).\end{aligned}$$

We put these newfound identities to good use in the following example.

Example 4.2.1.

1. Find the exact value of $\cos(15^\circ)$.

2. Verify the identity $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$.

Solution.

1. In order to use **Theorem 4.2** to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ=45^\circ-30^\circ$.

$$\begin{aligned}\cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\ &= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \quad \textbf{difference identity for cosine} \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

2. This is a straightforward application of **Theorem 4.2**.

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) \\ &= (0)\cos(\theta) + (1)\sin(\theta) \\ &= \sin(\theta)\end{aligned}$$

□

The identity verified in **Example 4.2.1**, namely $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, is the first of the celebrated cofunction identities. These identities were first hinted at in the **2.1 Exercises**, problem 41.

The Cofunction Identities

From $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we get

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right) \\ &= \cos(\theta)\end{aligned}$$

which says, in words, that the ‘co’sine of an angle is the sine of its ‘co’mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 4.3. Cofunction Identities: For all applicable angles θ,

- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$
- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$

With the cofunction identities in place, we are now in the position to derive the sum and difference identities for sine.

The Sum and Difference Identities for the Sine Function

We begin with the sum identity.

$$\begin{aligned}\sin(\alpha+\beta)&=\cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) && \textbf{from } \sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)\\ &=\cos\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right)\\ &=\cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) && \textbf{difference identity for cosine}\end{aligned}$$

We can derive the difference identity for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum identity and the even/odd identities. Again, we leave the details to the reader.

Theorem 4.4. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

Example 4.2.2.

1. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$.
2. If α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$ and β is a Quadrant III angle with $\tan(\beta)=2$, find $\sin(\alpha-\beta)$.
3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution.

1. As in **Example 4.2.1**, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles.

   The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is

$$\begin{aligned}\frac{19\pi}{12} &= \frac{16\pi}{12}+\frac{3\pi}{12}\\ &= \frac{4\pi}{3}+\frac{\pi}{4}\end{aligned}$$

   from which we have

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \quad \textbf{sum identity for sine}\\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

2. Using the difference identity for sine, $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$. We know that $\sin(\alpha)=\frac{5}{13}$, but need to find $\cos(\alpha)$, $\cos(\beta)$ and $\sin(\beta)$.

   To find $\cos(\alpha)$, we use $\sin(\alpha)=\frac{5}{13}$ along with a Pythagorean identity.

$$\begin{aligned}\cos^2(\alpha)+\left(\frac{5}{13}\right)^2 &= 1 \qquad \textbf{from } \cos^2(\alpha)+\sin^2(\alpha)=1\\ \cos^2(\alpha) &= \frac{144}{169}\\ \cos(\alpha) &= \pm\frac{12}{13}\\ \cos(\alpha) &= -\frac{12}{13} \quad \textbf{since } \alpha \textbf{ is a Quadrant II angle}\end{aligned}$$

   We next use a different Pythagorean identity, along with $\tan(\beta)=2$, to find $\cos(\beta)$.

$$1+2^2=\sec^2(\beta) \quad \textbf{from } 1+\tan^2(\beta)=\sec^2(\beta)$$

$$\sec^2(\beta)=\pm\sqrt{5}$$

$$\sec(\beta)=-\sqrt{5} \quad \textbf{since } \beta \textbf{ is a Quadrant III angle}$$

$$\frac{1}{\cos(\beta)}=-\sqrt{5} \quad \textbf{reciprocal identity for secant}$$

$$\cos(\beta)=-\frac{1}{\sqrt{5}}$$

We need to determine $\sin(\beta)$, knowing that $\cos(\beta)=-\frac{1}{\sqrt{5}}$ and $\tan(\beta)=2$.

$$\sin(\beta)=\tan(\beta)\cos(\beta) \quad \textbf{from the quotient identity } \tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$$

$$\sin(\beta)=(2)\left(-\frac{1}{\sqrt{5}}\right)$$

$$\sin(\beta)=-\frac{2}{\sqrt{5}}$$

We now have all the pieces needed to find $\sin(\alpha-\beta)$.

$$\begin{aligned}\sin(\alpha-\beta)&=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\&=\left(\frac{5}{13}\right)\left(-\frac{1}{\sqrt{5}}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2}{\sqrt{5}}\right)\\&=-\frac{29}{13\sqrt{5}}\\&=-\frac{29\sqrt{5}}{65}\end{aligned}$$

3. We can start expanding $\tan(\alpha+\beta)$.

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$ **quotient identity for tangent**

$$=\frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}$$ **sum identities for cosine and sine**

$$=\frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\cdot\frac{\dfrac{1}{\cos(\alpha)\cos(\beta)}}{\dfrac{1}{\cos(\alpha)\cos(\beta)}}$$ **goal :** $\dfrac{\sin(\alpha)}{\cos(\alpha)}$ **&** $\dfrac{\sin(\beta)}{\cos(\beta)}$

$$=\frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}$$

$$=\frac{\dfrac{\sin(\alpha)}{\cos(\alpha)}+\dfrac{\sin(\beta)}{\cos(\beta)}}{1-\dfrac{\sin(\alpha)}{\cos(\alpha)}\cdot\dfrac{\sin(\beta)}{\cos(\beta)}}$$

The last step is to replace $\dfrac{\sin(\alpha)}{\cos(\alpha)}$ with $\tan(\alpha)$ and $\dfrac{\sin(\beta)}{\cos(\beta)}$ with $\tan(\beta)$.

$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

Naturally, this result is limited to those cases where all of the tangents are defined.

□

The Sum and Difference Identities for the Tangent Function

The formula developed in **Example 4.2.2** for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan\left(\alpha+(-\beta)\right)$. The reader is encouraged to fill in the details.

Below we summarize all of the sum and difference formulas for sine, cosine and tangent.

Theorem 4.5. Sum and Difference Identities: For all applicable angles α and β,

- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

In the statement of **Theorem 4.5**, we have combined the cases for the sum '+' and difference '–' of angles into one formula. The convention is that if you want the formula for the sum '+' of two angles, use the top sign in the formula; for the difference '–' use the bottom sign. For example,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}.$$

We finish this section by, as promised in **Section 3.3**, proving algebraically that the period of the tangent function is π. Recall that a function f is periodic if there is a real number p so that $f(t+p) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p, if it exists, is called the period of f.

To prove that the period of $J(x) = \tan(x)$ is π, we appeal to the sum identity for tangents.

$$\begin{aligned} J(x+\pi) &= \tan(x+\pi) \\ &= \frac{\tan(x) + \tan(\pi)}{1 - \tan(x)\tan(\pi)} \\ &= \frac{\tan(x) + 0}{1 - [\tan(x)](0)} \\ &= \tan(x) \\ &= J(x) \end{aligned}$$

This tells us that the tangent is a periodic function and that the period of $\tan(x)$ is at most π. To show that it is exactly π, suppose p is a positive real number so that $\tan(x+p) = \tan(x)$ for all real numbers x. For $x = 0$, we have

$$\begin{aligned} \tan(p) &= \tan(0+p) \\ &= \tan(0) \qquad \textbf{from } \tan(x+p) = \tan(x) \\ &= 0 \end{aligned}$$

which means p is a multiple of π. The smallest positive multiple of π is π itself, so we have established that the period of the tangent function is π.

We leave it to the reader to prove algebraically that the period of the cotangent function is also π.[2]

[2] Certainly, mimicking the proof for the period of tan(x) is an option. For another approach, consider transforming tan(x) to cot(x) using identities.

4.2 Exercises

In Exercises 1 – 15, use the sum and difference identities to find the exact value. You may have need of the quotient, reciprocal or even/odd identities as well.

1. $\cos(75^\circ)$
2. $\sec(165^\circ)$
3. $\sin(105^\circ)$
4. $\csc(195^\circ)$
5. $\cot(255^\circ)$
6. $\tan(375^\circ)$
7. $\cos\left(\frac{13\pi}{12}\right)$
8. $\sin\left(\frac{11\pi}{12}\right)$
9. $\tan\left(\frac{13\pi}{12}\right)$
10. $\cos\left(\frac{7\pi}{12}\right)$
11. $\tan\left(\frac{17\pi}{12}\right)$
12. $\sin\left(\frac{\pi}{12}\right)$
13. $\cot\left(\frac{11\pi}{12}\right)$
14. $\csc\left(\frac{5\pi}{12}\right)$
15. $\sec\left(-\frac{\pi}{12}\right)$

16. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find

(a) $\cos(\alpha+\beta)$
(b) $\sin(\alpha+\beta)$
(c) $\tan(\alpha+\beta)$
(d) $\cos(\alpha-\beta)$
(e) $\sin(\alpha-\beta)$
(f) $\tan(\alpha-\beta)$

17. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find

(a) $\cos(\alpha+\beta)$
(b) $\sin(\alpha+\beta)$
(c) $\tan(\alpha+\beta)$
(d) $\cos(\alpha-\beta)$
(e) $\sin(\alpha-\beta)$
(f) $\tan(\alpha-\beta)$

18. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$, where $\frac{3\pi}{2} < \beta < 2\pi$, find

(a) $\sin(\alpha+\beta)$
(b) $\cos(\alpha-\beta)$
(c) $\tan(\alpha-\beta)$

19. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find

(a) $\csc(\alpha-\beta)$
(b) $\sec(\alpha+\beta)$
(c) $\cot(\alpha+\beta)$

In Exercises 20 – 32, verify the identity.

20. $\cos(\theta-\pi)=-\cos(\theta)$

21. $\sin(\pi-\theta)=\sin(\theta)$

22. $\tan\left(\theta+\frac{\pi}{2}\right)=-\cot(\theta)$

23. $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)$

24. $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos(\alpha)\sin(\beta)$

25. $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos(\alpha)\cos(\beta)$

26. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\dfrac{1+\cot(\alpha)\tan(\beta)}{1-\cot(\alpha)\tan(\beta)}$

27. $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{1-\tan(\alpha)\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

28. $\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin(\alpha)\sin(\beta)$

29. $\dfrac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)}=\dfrac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

30. $\dfrac{\sin(t+h)-\sin(t)}{h}=\cos(t)\left(\dfrac{\sin(h)}{h}\right)+\sin(t)\left(\dfrac{\cos(h)-1}{h}\right)$

31. $\dfrac{\cos(t+h)-\cos(t)}{h}=\cos(t)\left(\dfrac{\cos(h)-1}{h}\right)-\sin(t)\left(\dfrac{\sin(h)}{h}\right)$

32. $\dfrac{\tan(t+h)-\tan(t)}{h}=\left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

33. Verify the cofunction identities for tangent, secant, cosecant and cotangent.

34. Verify the difference identities for sine and tangent.

4.3 Double Angle Identities

Learning Objectives

In this section you will:

- **Learn the double angle identities for sine, cosine and tangent.**
- **Find trigonometric values of double angles.**
- **Verify identities involving double angles.**

In **Section 4.2**, the sum identities for the trigonometric functions were introduced:

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$
$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Double Angle Identities

Using the sum identities, in the case where $\alpha=\beta$, we let $\theta=\alpha=\beta$ to attain the double angle identities in the following theorem.

Theorem 4.6. Double Angle Identities: For all applicable angles θ,

- $\sin(2\theta)=2\sin(\theta)\cos(\theta)$
- $\cos(2\theta)=\begin{cases}\cos^2(\theta)-\sin^2(\theta)\\ 2\cos^2(\theta)-1\\ 1-2\sin^2(\theta)\end{cases}$
- $\tan(2\theta)=\dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to exchange squares of cosine and sine via a Pythagorean identity. We verify that $\cos(2\theta)=1-2\sin^2(\theta)$:

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta) && \textbf{sum identity for cosine}\\ &= \cos^2(\theta)-\sin^2(\theta) && \textbf{verifies first form}: \cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)\\ &= \left[1-\sin^2(\theta)\right]-\sin^2(\theta) && \textbf{from } \cos^2(\theta)+\sin^2(\theta)=1\\ &= 1-2\sin^2(\theta).\end{aligned}$$

Trigonometric Values of Double Angles

Now that we have established the double angle identities, we put them to good use in determining trigonometric values of double angles.

Example 4.3.1.

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Find $\cos(2\theta)$ and $\sin(2\theta)$. Determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

2. If $\sin(\theta)=x$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of *x*.

Solution.

1. Using $x^2+y^2=r^2$, from **Theorem 2.6** in **Section 2.5**, with $x=-3$ and $y=4$, we find $r=\sqrt{x^2+y^2}=5$. Hence, $\cos(\theta)=\frac{x}{r}=-\frac{3}{5}$ and $\sin(\theta)=\frac{y}{r}=\frac{4}{5}$. It follows that

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) && \textbf{from double angle identity}\\ &= \left(-\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2\\ &= -\frac{7}{25}\end{aligned}$$

and

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) && \textbf{from double angle identity}\\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{24}{25}.\end{aligned}$$

Since both the cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III.

2. If your first reaction to $\sin(\theta) = x$ is that x should be the cosine of θ, then you have indeed learned something. However, context is everything. Here, x is just a variable. It does not necessarily represent the x-coordinate of a point on the Unit Circle. Here, x represents the quantity $\sin(\theta)$, and what we wish to know is how to express $\sin(2\theta)$ in terms of x. We will see more of this kind of thing in Chapter 5 and, as usual, this is something we need for calculus.

   We start with the double angle identity for sine:

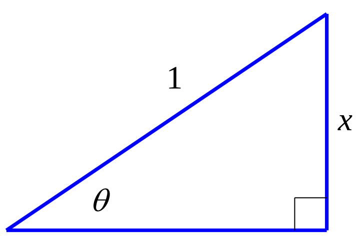
$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ &= 2x\cos(\theta) \qquad \textbf{from the problem statement that } \sin(\theta) = x\end{aligned}$$

   We need to write $\cos(\theta)$ in terms of x to finish the problem. There are two different methods that come readily to mind, both of which are good to know. The first is purely algebraic, using the Pythagorean identity:

$$\begin{aligned}\cos^2(\theta) + \sin^2(\theta) &= 1 \qquad \textbf{Pythagorean identity} \\ \cos^2(\theta) + x^2 &= 1 \qquad \textbf{from the problem statement} \\ \cos(\theta) &= \pm\sqrt{1 - x^2} \\ \cos(\theta) &= \sqrt{1 - x^2} \qquad \cos(\theta) \ge 0 \textbf{ since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\end{aligned}$$

The second method, preferred by many, provides a visual approach for determining $\cos(\theta)$. We sketch a right triangle with acute angle θ and, noting that

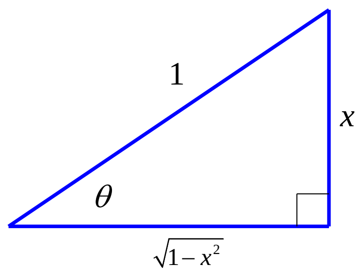
$$\begin{aligned}\sin(\theta) &= x \\ &= \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}},\end{aligned}$$



we label the hypotenuse with length 1 and the side opposite θ with length x. We then use the Pythagorean Theorem to determine the length of the side adjacent to θ.

$$\begin{aligned}\left(\text{adjacent length}\right)^2 + x^2 &= 1^2 \\ \left(\text{adjacent length}\right)^2 &= 1 - x^2 \\ \text{adjacent length} &= \sqrt{1 - x^2}\end{aligned}$$

This results in the following triangle.



From the triangle we see that $\cos(\theta)=\frac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$.

Then, back to solving for $\sin(2\theta)$, we have a final answer is $\sin(2\theta)=2x\sqrt{1-x^2}$.

□

Verifying Identities that Include Double Angles

Establishing trigonometric identities using the double angle identities is our next task. As before, starting with the more complicated side of an equation is usually a good strategy.

Example 4.3.2.

1. Verify the identity $\sin(2\theta)=\frac{2\tan(\theta)}{1+\tan^2(\theta)}$.

2. Verify the identity $\tan(2\theta)=\frac{2}{\cot(\theta)-\tan(\theta)}$.

Solution.

1. We start with the right hand side of the identity.

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} && \textbf{from Pythagorean identity} \\
&= \frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\left(\dfrac{1}{\cos^2(\theta)}\right)} && \textbf{from reciprocal \& quotient identities} \\
&= 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\sin(\theta)\cos(\theta) \\
&= \sin(2\theta) && \textbf{from double angle identity for sine}
\end{aligned}$$

2. In this case, we begin with the left side of the equation.

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \textbf{double angle identity} \\
&= \frac{2\tan(\theta)\left(\dfrac{1}{\tan(\theta)}\right)}{\left(1-\tan^2(\theta)\right)\left(\dfrac{1}{\tan(\theta)}\right)} && \textbf{goal : numerator of 2} \\
&= \frac{2}{\dfrac{1}{\tan(\theta)}-\tan(\theta)} \\
&= \frac{2}{\cot(\theta)-\tan(\theta)} && \textbf{reciprocal identity for cotangent}
\end{aligned}$$

□

Part 2 of the previous example is a case where the more complicated side of the initial equation appeared on the right, but we chose to start with the left side. Beginning with the right side would have required some thinking ahead, possibly working backwards. Try it! When using identities to simplify a trigonometric expression, solve a trigonometric equation or verify a trigonometric identity, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated, although generally one of the paths will result in a simpler solution. In verifying identities, the strategies established in **Section 2.4** will help, but there is no substitute for practice.

One last note before we move on to **Section 4.4**. While double angle identities could be established for secant, cosecant and cotangent, the identities already established in this section may be used in their place. Recall that secant, cosecant and cotangent are reciprocal identities of cosine, sine and tangent, respectively. Thus, for example, $\sec(2\theta) = \dfrac{1}{\cos(2\theta)}$ and so any of the three double angle identities for cosine may be used in determining $\sec(2\theta)$.

4.3 Exercises

1. Use the double angle identity $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$ to verify the double angle identity $\cos(2\theta)=2\cos^2(\theta)-1$.

2. Use the double angle identities for $\cos(2\theta)$ and $\sin(2\theta)$ to verify the double angle identity for $\tan(2\theta)$.

In Exercises 3 – 12, use the given information about θ to find the exact values of

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$ (c) $\tan(2\theta)$

3. $\sin(\theta)=-\dfrac{7}{25}$ where $\dfrac{3\pi}{2}<\theta<2\pi$

4. $\cos(\theta)=\dfrac{28}{53}$ where $0<\theta<\dfrac{\pi}{2}$

5. $\tan(\theta)=\dfrac{12}{5}$ where $\pi<\theta<\dfrac{3\pi}{2}$

6. $\csc(\theta)=4$ where $\dfrac{\pi}{2}<\theta<\pi$

7. $\cos(\theta)=\dfrac{3}{5}$ where $0<\theta<\dfrac{\pi}{2}$

8. $\sin(\theta)=-\dfrac{4}{5}$ where $\pi<\theta<\dfrac{3\pi}{2}$

9. $\cos(\theta)=\dfrac{12}{13}$ where $\dfrac{3\pi}{2}<\theta<2\pi$

10. $\sin(\theta)=\dfrac{5}{13}$ where $\dfrac{\pi}{2}<\theta<\pi$

11. $\sec(\theta)=\sqrt{5}$ where $\dfrac{3\pi}{2}<\theta<2\pi$

12. $\tan(\theta)=-2$ where $\dfrac{\pi}{2}<\theta<\pi$

In Exercises 13 – 22, verify the identity. Assume all quantities are defined.

13. $\sin(2\theta)=\dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$

14. $\cos(2\theta)=\dfrac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$

15. $\tan(2\theta)=\dfrac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)-1}$

16. $\left[\cos(\theta)+\sin(\theta)\right]^2=1+\sin(2\theta)$

17. $\left[\cos(\theta)-\sin(\theta)\right]^2=1-\sin(2\theta)$

18. $\tan(2\theta)=\dfrac{1}{1-\tan(\theta)}-\dfrac{1}{1+\tan(\theta)}$

19. $\csc(2\theta)=\dfrac{\cot(\theta)+\tan(\theta)}{2}$

20. $\sec(2\theta)=\dfrac{\cos(\theta)}{\cos(\theta)+\sin(\theta)}+\dfrac{\sin(\theta)}{\cos(\theta)-\sin(\theta)}$

21. $\dfrac{1}{\cos(\theta)-\sin(\theta)}+\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\cos(\theta)}{\cos(2\theta)}$

22. $\dfrac{1}{\cos(\theta)-\sin(\theta)}-\dfrac{1}{\cos(\theta)+\sin(\theta)}=\dfrac{2\sin(\theta)}{\cos(2\theta)}$

23. Suppose θ is a Quadrant I angle with $\sin(\theta)=x$. Verify the following formulas.

(a) $\cos(\theta)=\sqrt{1-x^2}$ (b) $\sin(2\theta)=2x\sqrt{1-x^2}$ (c) $\cos(2\theta)=1-2x^2$

24. Discuss with your classmates how each of the formulas, if any, in Exercise 23 change if we assume θ is a Quadrant II, III or IV angle.

25. Suppose θ is a Quadrant I angle with $\tan(\theta)=x$. Verify the following formulas.

(a) $\cos(\theta)=\dfrac{1}{\sqrt{x^2+1}}$ (b) $\sin(\theta)=\dfrac{x}{\sqrt{x^2+1}}$

(c) $\sin(2\theta)=\dfrac{2x}{x^2+1}$ (d) $\cos(2\theta)=\dfrac{1-x^2}{x^2+1}$

26. Discuss with your classmates how each of the formulas, if any, in Exercise 25 change if we assume θ is a Quadrant II, III or IV angle.

27. If $\sin(\theta)=\dfrac{x}{2}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of *x*.

28. If $\tan(\theta)=\dfrac{x}{7}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of *x*.

4.4 Power Reduction and Half Angle Formulas

Learning Objectives

In this section you will:

- **Learn and apply the power reduction formulas for sine and cosine.**
- **Learn and apply the half angle formulas for sine, cosine and tangent.**

In **Section 4.3**, the double angle identities allowed us to write $\cos(2\theta)$ as powers of sine and/or cosine. In calculus, we have occasion to do the reverse; that is, reduce the power of sine and cosine.

Power Reduction Formulas

Solving the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ and the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ result in the aptly-named 'power reduction' formulas below.

Theorem 4.7. Power Reduction Formulas: For all angles θ,

- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$
- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$

Example 4.4.1. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and/or difference of cosines to the first power.

Solution. We begin with a straightforward application of **Theorem 4.7**.

$$\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right) && \textbf{from power reduction formulas}\\
&= \frac{1}{4}\left(1-\cos^2(2\theta)\right)\\
&= \frac{1}{4}-\frac{1}{4}\cos^2(2\theta)\\
&= \frac{1}{4}-\frac{1}{4}\left(\frac{1+\cos\left(2(2\theta)\right)}{2}\right) && \textbf{replacing } \theta \textbf{ with } 2\theta \textbf{ in power reduction formula}\\
&= \frac{1}{4}-\frac{1}{8}-\frac{1}{8}\cos(4\theta)\\
&= \frac{1}{8}-\frac{1}{8}\cos(4\theta)
\end{aligned}$$

□

Another application of the power reduction formulas is the half angle formulas.

Half Angle Formulas

To start, we apply the power reduction formula to $\cos^2\left(\frac{\theta}{2}\right)$:

$$\begin{aligned}
\cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\\
&= \frac{1+\cos(\theta)}{2}.
\end{aligned}$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine. By using a quotient identity, we obtain a half angle formula for tangent.

These formulas are summarized below.

Theorem 4.8. Half Angle Formulas. For all applicable angles θ,

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of ± depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 4.4.2.

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi < \theta < 0$ with $\cos(\theta) = -\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.
3. Use the identity $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$, verified in **Example 4.3.2**, to derive the identity

   $\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$.

Solution.

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$.

$$\cos(15^\circ) = \cos\left(\frac{30^\circ}{2}\right)$$

$$= \pm\sqrt{\frac{1+\cos(30^\circ)}{2}} \quad \textbf{from half angle formula for cosine}$$

$$= +\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \quad \textbf{positive since } 15^\circ \textbf{ is in Quadrant I}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

Back in **Example 4.2.1**, we found $\cos(15^\circ)$ by using the difference identity for cosine. In that case, we determined $\cos(15^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$. The reader is encouraged to prove these two expressions are equal.

2. If $-\pi < \theta < 0$, then $-\dfrac{\pi}{2} < \dfrac{\theta}{2} < 0$, which means $\sin\left(\dfrac{\theta}{2}\right) < 0$.

$$\sin\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos(\theta)}{2}} \quad \textbf{half angle formula for sine}$$

$$= -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$$

$$= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}}$$

$$= -\sqrt{\frac{8}{10}}$$

$$= -\frac{2\sqrt{5}}{5}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$,

and we will manipulate it into the identity $\tan\left(\frac{\theta}{2}\right)=\frac{\sin(\theta)}{1+\cos(\theta)}$. If we are to use $\sin(2\theta)=\frac{2\tan(\theta)}{1+\tan^2(\theta)}$ to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$.

$$\sin(2\theta)=\frac{2\tan(\theta)}{1+\tan^2(\theta)}$$

$$\sin\left(2\left(\frac{\theta}{2}\right)\right)=\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \qquad \textbf{replacing } \theta \textbf{ with } \frac{\theta}{2}$$

$$\sin(\theta)=\frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \qquad \textbf{from a Pythagorean identity}$$

$$\sin(\theta)=2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \qquad \textbf{from reciprocal identity for secant}$$

$$\sin(\theta)=2\tan\left(\frac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \qquad \textbf{from power reduction formula for cosine}$$

$$\sin(\theta)=\tan\left(\frac{\theta}{2}\right)\left(1+\cos(\theta)\right)$$

$$\tan\left(\frac{\theta}{2}\right)=\frac{\sin(\theta)}{1+\cos(\theta)}$$

□

4.4 Exercises

In Exercises 1 – 15, use half angle formulas to find the exact value. You may have need of the quotient, reciprocal or even/odd identities as well.

1. $\cos(75^\circ)$
2. $\sin(105^\circ)$
3. $\cos(67.5^\circ)$
4. $\sin(157.5^\circ)$
5. $\tan(112.5^\circ)$
6. $\cos\left(\frac{7\pi}{12}\right)$
7. $\sin\left(\frac{\pi}{12}\right)$
8. $\cos\left(\frac{\pi}{8}\right)$
9. $\sin\left(\frac{5\pi}{8}\right)$
10. $\tan\left(\frac{7\pi}{8}\right)$
11. $\cos\left(-\frac{11\pi}{12}\right)$
12. $\sin\left(\frac{11\pi}{12}\right)$
13. $\tan\left(\frac{5\pi}{12}\right)$
14. $\tan\left(-\frac{3\pi}{12}\right)$
15. $\tan\left(-\frac{3\pi}{8}\right)$

In Exercises 16 – 29, use the given information about θ to find the exact values of

(a) $\sin\left(\frac{\theta}{2}\right)$ (b) $\cos\left(\frac{\theta}{2}\right)$ (c) $\tan\left(\frac{\theta}{2}\right)$

16. $\sin(\theta)=-\frac{7}{25}$ where $\frac{3\pi}{2}<\theta<2\pi$
17. $\cos(\theta)=\frac{28}{53}$ where $0<\theta<\frac{\pi}{2}$
18. $\tan(\theta)=\frac{12}{5}$ where $\pi<\theta<\frac{3\pi}{2}$
19. $\csc(\theta)=4$ where $\frac{\pi}{2}<\theta<\pi$
20. $\cos(\theta)=\frac{3}{5}$ where $0<\theta<\frac{\pi}{2}$
21. $\sin(\theta)=-\frac{4}{5}$ where $\pi<\theta<\frac{3\pi}{2}$
22. $\cos(\theta)=\frac{12}{13}$ where $\frac{3\pi}{2}<\theta<2\pi$
23. $\sin(\theta)=\frac{5}{13}$ where $\frac{\pi}{2}<\theta<\pi$
24. $\sec(\theta)=\sqrt{5}$ where $\frac{3\pi}{2}<\theta<2\pi$
25. $\tan(\theta)=-2$ where $\frac{\pi}{2}<\theta<\pi$
26. $\tan(\theta)=-\frac{4}{3}$, where θ is in Quadrant IV
27. $\sin(\theta)=-\frac{12}{13}$, where θ is in Quadrant III
28. $\csc(\theta)=7$, where θ is in Quadrant II
29. $\sec(\theta)=-4$, where θ is in Quadrant II

30. Without using your calculator, show that $\dfrac{\sqrt{2+\sqrt{3}}}{2}=\dfrac{\sqrt{6}+\sqrt{2}}{4}$.

31. Let θ be a Quadrant III angle with $\cos(\theta)=-\dfrac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\dfrac{\theta}{2}\right)$ by first assuming $3\pi<\theta<\dfrac{7\pi}{2}$ and then assuming $\pi<\theta<\dfrac{3\pi}{2}$. Compute $\sin\left(\dfrac{\theta}{2}\right)$ in both cases.

4.5 Product to Sum and Sum to Product Formulas

Learning Objectives

In this section you will:

- **Learn and apply the Product to Sum Formulas.**
- **Learn and apply the Sum to Product Formulas.**

This section begins with an example that uses identities from **Sections 4.2** and **4.3** to write a trigonometric expression as the sum of trigonometric expressions.

Example 4.5.1. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

Solution. The double angle identity $\cos(2\theta)=2\cos^2(\theta)-1$ expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta) && \textbf{from sum identity for cosine}\\ &= \left(2\cos^2(\theta)-1\right)\cos(\theta)-\left(2\sin(\theta)\cos(\theta)\right)\sin(\theta) && \textbf{from double angle identities}\\ &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\end{aligned}$$

Finally, we exchange $\sin^2(\theta)$ for $1-\cos^2(\theta)$, courtesy of a Pythagorean identity, and get

$$\begin{aligned}\cos(3\theta) &= 2\cos^3(\theta)-\cos(\theta)-2\sin^2(\theta)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\left(1-\cos^2(\theta)\right)\cos(\theta)\\ &= 2\cos^3(\theta)-\cos(\theta)-2\cos(\theta)+2\cos^3(\theta)\\ &= 4\cos^3(\theta)-3\cos(\theta).\end{aligned}$$

Thus, $\cos(3\theta)$ can be expressed as the polynomial $4\cos^3(\theta)-3\cos(\theta)$.

□

Having just shown how we could rewrite $\cos(3\theta)$ as the sum of powers of $\cos(\theta)$, it might occur to you that similar operations could be applied to $\cos(4\theta)$ or $\cos(5\theta)$ to rewrite the expressions as sums of powers of $\cos(\theta)$. This will be of use in calculus, as will the formulas yet to be presented in this section.

Product to Sum Formulas

Our next batch of identities, the Product to Sum Formulas[1], are easily verified by expanding each of the right hand sides in accordance with the Sum and Difference Identities. The details are left as exercises.

Theorem 4.9. Product to Sum Formulas. For all angles α and β,

- $\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\cos(\alpha)\cos(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta)=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Example 4.5.1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.

Solution. Identifying $\alpha=2\theta$ and $\beta=6\theta$, we use the Product to Sum Formula for $\cos(\alpha)\cos(\beta)$.

$$\begin{aligned}\cos(2\theta)\cos(6\theta)&=\frac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right]\\&=\frac{1}{2}\cos(-4\theta)+\frac{1}{2}\cos(8\theta)\\&=\frac{1}{2}\cos(4\theta)+\frac{1}{2}\cos(8\theta) &&\textbf{even property of cosine}\end{aligned}$$

□

Sum to Product Formulas

Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Chapter 6. These are easily verified using the Product to Sum Formulas and, as such, their proofs are left as exercises.

[1] These are also known as the Prosthaphaeresis Formulas and have a rich history. Conduct some research on them as your schedule allows.

Theorem 4.10. Sum to Product Formulas. For all angles α and β,

- $\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right)$
- $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$
- $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

Example 4.5.2. Write $\sin(\theta) - \sin(3\theta)$ as a product.

Solution. Using the Sum to Product Formula for $\sin(\alpha) - \sin(\beta)$, with $\alpha = \theta$ and $\beta = 3\theta$, yields the following.

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta) \end{aligned}$$

Where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

□

The reader is reminded that all of the identities presented in this chapter which regard the circular functions as functions of angles in radian measure apply equally well to the circular (trigonometric) functions regarded as functions of real numbers.

4.5 Exercises

In Exercises 1 – 6, write the given product as a sum. You may need to use an even/odd identity.

1. $\cos(3\theta)\cos(5\theta)$
2. $\sin(2\theta)\sin(7\theta)$
3. $\sin(9\theta)\cos(\theta)$
4. $\cos(2\theta)\cos(6\theta)$
5. $\sin(3\theta)\sin(2\theta)$
6. $\cos(\theta)\sin(3\theta)$

In Exercises 7 – 12, write the given sum as a product. You may need to use an even/odd or cofunction identity.

7. $\cos(3\theta)+\cos(5\theta)$
8. $\sin(2\theta)-\sin(7\theta)$
9. $\sin(5\theta)-\cos(6\theta)$
10. $\sin(9\theta)-\sin(-\theta)$
11. $\sin(\theta)+\cos(\theta)$
12. $\cos(\theta)-\sin(\theta)$

In Exercises 13 – 20, verify the identity. Assume all quantities are defined.

13. $8\sin^4(\theta)=\cos(4\theta)-4\cos(2\theta)+3$
14. $8\cos^4(\theta)=\cos(4\theta)+4\cos(2\theta)+3$
15. $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$
16. $\cos(4\theta)=8\cos^4(\theta)-8\cos^2(\theta)+1$
17. $\sin(4\theta)=4\sin(\theta)\cos^3(\theta)-4\sin^3(\theta)\cos(\theta)$
18. $32\sin^2(\theta)\cos^4(\theta)=2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)$
19. $32\sin^4(\theta)\cos^2(\theta)=2-\cos(2\theta)-2\cos(4\theta)+\cos(6\theta)$
20. $\cos(8\theta)=128\cos^8(\theta)-256\cos^6(\theta)+160\cos^4(\theta)-32\cos^2(\theta)+1$ (HINT: Use result for 16.)

21. In **Example 4.5.1**, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 16, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number *n*?

22. In Exercise 15, we had you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

23. Verify the Product to Sum Identities.

24. Verify the Sum to Product Identities.

4.6 Using Sum Identities in Determining Sinusoidal Formulas

Learning Objectives

In this section you will:

- **Write a trigonometric function of sines and cosines in general sinusoidal format:** $C(x) = A\cos(\omega x + \phi) + B$ **or** $S(x) = A\sin(\omega x + \phi) + B$**.**

The motivation for this section lies in the occasions when a trigonometric function is defined in terms of both sines and cosines. By rewriting the function as a sine function or as a cosine function, properties of the function, such as amplitude and period, will become more apparent. This will be a tool used in future courses such as differential equations.

To get started, if we use the sum identity for cosine, we can expand $C(x) = A\cos(\omega x + \phi) + B$ to yield

$$C(x) = A\cos(\omega x)\cos(\phi) - A\sin(\omega x)\sin(\phi) + B.$$

Similarly, using the sum identity for sine, $S(x) = A\sin(\omega x + \phi) + B$ is equivalent to

$$S(x) = A\sin(\omega x)\cos(\phi) + A\cos(\omega x)\sin(\phi) + B.$$

Making these observations allows us to recognize (and graph) functions as sinusoids which, at first glance, don't appear to fit the forms of either $C(x)$ or $S(x)$.

Example 4.6.1. Consider the function $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$.

1. Find a formula for $f(x)$ in the form $C(x) = A\cos(\omega x + \phi) + B$ for $\omega > 0$.
2. Find a formula for $f(x)$ in the form $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$.

Check your answers analytically using identities.

Solution.

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. Equating $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $C(x) = A\cos(\omega x + \phi) + B$, we get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\cos(\omega x)\cos(\phi) - A\sin(\omega x)\sin(\phi) + B$$

or

$$\cos(2x)-\sqrt{3}\sin(2x)+0 = A\cos(\omega x)\cos(\phi)-A\sin(\omega x)\sin(\phi)+B.$$

By matching up corresponding coefficients and constants, we get $\omega = 2$ and $B = 0$. To determine A and ϕ, a bit more work is involved. Rewriting the equation will help.

$$1\cdot\cos(2x)-\sqrt{3}\sin(2x) = A\cos(\phi)\cos(2x)-A\sin(\phi)\sin(2x)$$

On the left hand side, the coefficient of $\cos(2x)$ is 1, while on the right hand side it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have that $A\cos(\phi)=1$. Similarly, we find by equating the coefficients of $\sin(2x)$ that $A\sin(\phi)=\sqrt{3}$. We now have a system of nonlinear equations that will allow us to determine values for A and ϕ.

$$\begin{aligned}
\cos^2(\phi)+\sin^2(\phi) &= 1 && \textbf{Pythagorean Identity}\\
\left(\frac{1}{A}\right)^2+\left(\frac{\sqrt{3}}{A}\right)^2 &= 1 && \textbf{from } A\cos(\phi)=1 \textbf{ and } A\sin(\phi)=\sqrt{3}\\
A^2\left(\frac{1}{A}\right)^2+A^2\left(\frac{\sqrt{3}}{A}\right)^2 &= A^2 && \textbf{multiplying through by } A^2\\
1+3 &= A^2\\
A &= \pm 2
\end{aligned}$$

Choosing $A=2$, we have $2\cos(\phi)=1$ and $2\sin(\phi)=\sqrt{3}$ or, after some rearrangement, $\cos(\phi)=\frac{1}{2}$ and $\sin(\phi)=\frac{\sqrt{3}}{2}$. One such angle which satisfies this criteria is $\phi=\frac{\pi}{3}$. Hence, one way to write $f(x)$ as a sinusoid is $f(x)=2\cos\left(2x+\frac{\pi}{3}\right)$.

We can easily check our answer using the sum formula for cosine.

$$\begin{aligned}
f(x) &= 2\cos\left(2x+\frac{\pi}{3}\right)\\
&= 2\left[\cos(2x)\cos\left(\frac{\pi}{3}\right)-\sin(2x)\sin\left(\frac{\pi}{3}\right)\right]\\
&= 2\left[\cos(2x)\left(\frac{1}{2}\right)-\sin(2x)\left(\frac{\sqrt{3}}{2}\right)\right]\\
&= \cos(2x)-\sqrt{3}\sin(2x)
\end{aligned}$$

This verifies that $f(x)=2\cos\left(2x+\frac{\pi}{3}\right)$ is equivalent to $f(x)=\cos(2x)-\sqrt{3}\sin(2x)$.

2. Proceeding as before, we equate $f(x)=\cos(2x)-\sqrt{3}\sin(2x)$ with the expanded form of $S(x)=A\sin(\omega x+\phi)+B$ to get

$$\cos(2x)-\sqrt{3}\sin(2x)=A\sin(\omega x)\cos(\phi)+A\cos(\omega x)\sin(\phi)+B$$

or

$$\cos(\mathbf{2x})-\sqrt{3}\sin(\mathbf{2x})+\mathbf{0}=A\cos(\boldsymbol{\omega x})\sin(\phi)+A\sin(\boldsymbol{\omega x})\cos(\phi)+\boldsymbol{B}$$

Once again, we may take $\omega=2$ and $B=0$. To determine A and ϕ, we begin by rewriting the equation.

$$\mathbf{1}\cdot\cos(2x)+\left(-\sqrt{3}\right)\sin(2x)=\boldsymbol{A\sin(\phi)}\cos(2x)+\boldsymbol{A\cos(\phi)}\sin(2x)$$

We equate the coefficients of $\cos(2x)$, then $\sin(2x)$, on either side and get $A\sin(\phi)=1$ and $A\cos(\phi)=-\sqrt{3}$. Using the Pythagorean identity $\cos^2(\phi)+\sin^2(\phi)=1$ as before, we get $A=\pm 2$. Then $2\sin(\phi)=1$, or $\sin(\phi)=\frac{1}{2}$, and $2\cos(\phi)=-\sqrt{3}$, which means $\cos(\phi)=-\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi=\frac{5\pi}{6}$. Hence, we have $f(x)=2\sin\left(2x+\frac{5\pi}{6}\right)$.

We check our work analytically, using the sum formula for sine.

$$\begin{aligned}
f(x)&=2\sin\left(2x+\frac{5\pi}{6}\right)\\
&=2\left[\sin(2x)\cos\left(\frac{5\pi}{6}\right)+\cos(2x)\sin\left(\frac{5\pi}{6}\right)\right]\\
&=2\left[\sin(2x)\left(-\frac{\sqrt{3}}{2}\right)+\cos(2x)\left(\frac{1}{2}\right)\right]\\
&=-\sqrt{3}\sin(2x)+\cos(2x)\\
&=\cos(2x)-\sqrt{3}\sin(2x)
\end{aligned}$$

Thus, $f(x)=2\sin\left(2x+\frac{5\pi}{6}\right)$ is equivalent to $f(x)=\cos(2x)-\sqrt{3}\sin(2x)$.

Graphing the three formulas for $f(x)$ on a graphing calculator of computer graphing program will result in three identical graphs, verifying our analytical work.

□

It is worth mentioning that, had we chosen $A=-2$ instead of $A=2$ as we worked through the preceding example, our final answers would have looked different. The reader is encouraged to rework the example using $A=-2$ and to then use identities to show that the formulas are all equivalent.

It is important to note that in order for the technique presented in the example to fit a function into the form of one of the general equations, $C(x)=A\cos(\omega x+\phi)+B$ or $S(x)=A\sin(\omega x+\phi)+B$, the arguments of the cosine and sine function must match. That is, while $f(x)=\cos(2x)-\sqrt{3}\sin(2x)$ is a sinusoid, $g(x)=\cos(2x)-\sqrt{3}\sin(3x)$ is not.[1] The general equations of sinusoids will be explored further in the exercises.

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[1] This graph does, however, exhibit sinusoid-like characteristics. Check it out!

4.6 Exercises

In Exercises 1 – 10, use **Example 4.6.1** as a guide to show that the function is a sinusoid by rewriting it in the forms $C(x) = A\cos(\omega x + \phi) + B$ and $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$ and $0 \le \phi < 2\pi$.

1. $f(x) = \sqrt{2}\sin(x) + \sqrt{2}\cos(x) + 1$
2. $f(x) = 3\sqrt{3}\sin(3x) - 3\cos(3x)$
3. $f(x) = -\sin(x) + \cos(x) - 2$
4. $f(x) = -\frac{1}{2}\sin(2x) - \frac{\sqrt{3}}{2}\cos(2x)$
5. $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x)$
6. $f(x) = \frac{3}{2}\cos(2x) - \frac{3\sqrt{3}}{2}\sin(2x) + 6$
7. $f(x) = -\frac{1}{2}\cos(5x) - \frac{\sqrt{3}}{2}\sin(5x)$
8. $f(x) = -6\sqrt{3}\cos(3x) - 6\sin(3x) - 3$
9. $f(x) = \frac{5\sqrt{2}}{2}\sin(x) - \frac{5\sqrt{2}}{2}\cos(x)$
10. $f(x) = 3\sin\left(\frac{x}{6}\right) - 3\sqrt{3}\cos\left(\frac{x}{6}\right)$

11. In Exercises 1 – 10, you should have noticed a relationship between the phases ϕ for $C(x)$ and $S(x)$. Show that if $f(x) = A\sin(\omega x + \alpha) + B$, then $f(x) = A\cos(\omega x + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.

12. Let ϕ be an angle measured in radians and let $P(a,b)$ be a point on the terminal side of ϕ when it is drawn in standard position. Use **Theorem 2.6** and the sum identity for sine to show that $f(x) = a\sin(\omega x) + b\cos(\omega x) + B$ (with $\omega > 0$) can be written as $f(x) = \sqrt{a^2 + b^2}\sin(\omega x + \phi) + B$.

CHAPTER 5
THE INVERSE TRIGONOMETRIC FUNCTIONS

Chapter Outline

5.1 Properties of the Inverse Cosine and Sine Functions
5.2 Properties of the Inverse Tangent and Cotangent Functions
5.3 Properties of the Inverse Secant and Cosecant Functions
5.4 Calculators and the Inverse Circular Functions

Introduction

Chapter 5 introduces the valuable inverse (circular) trigonometric functions. The first three sections are devoted to defining these inverse functions and identifying properties of each. Emphasis is on determining function values and rewriting expressions containing inverse trigonometric functions. Section 5.1 focuses on the inverse cosine and sine functions. In Section 5.2, the inverse tangent and cotangent functions are added, followed by the inverse secant and cosecant in Section 5.3. Throughout the first three sections, relationships between the inverse functions are developed.

Because of the necessity for using inverse trigonometric functions in solving real-world applications, the calculation of degree or radian measure is often desired. Section 5.4 includes techniques for determining approximate values of inverse trigonometric functions through technology. In addition to real-world applications of inverse trigonometric functions, Section 5.4 includes finding domains and ranges, and verifying domains and ranges through graphing technology.

Chapter 5 is essential to the understanding of trigonometric functions and their uses. The inverse trigonometric functions will be needed in solving trigonometric equations and solving triangles, the focus of the next two chapters.

5.1 Properties of the Inverse Cosine and Sine Functions

Learning Objectives

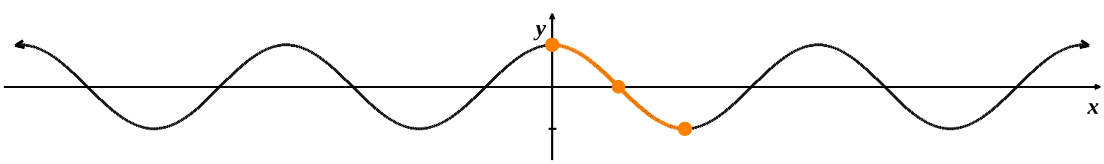
In this section you will:

- **Learn and be able to apply properties of the inverse cosine and sine functions, including domain and range.**
- **Find exact values of inverse cosine and sine functions, and of their composition with other trigonometric functions.**
- **Convert compositions of trigonometric and inverse cosine or sine functions to algebraic expressions.**

In this chapter, we concern ourselves with finding inverses of the (circular) trigonometric functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domain of each circular function to obtain a one-to-one function.

The Inverse Cosine Function

We first consider $f(x)=\cos(x)$. Choosing the interval $[0,\pi]$ allows us to keep the range as $[-1,1]$ along with the properties of being smooth and continuous.



Restricting the domain of $f(x)=\cos(x)$ to $[0,\pi]$.

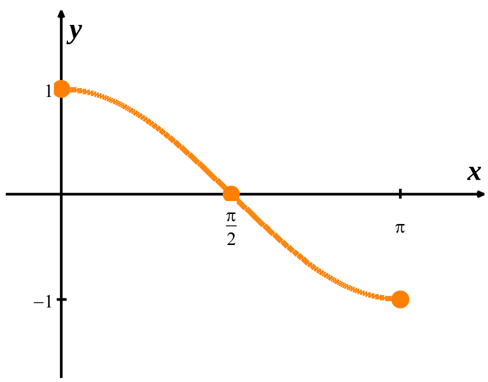
Recall that the inverse of a function f is typically denoted f^{-1}. The notation for the inverse of $f(x)=\cos(x)$ is denoted as either $f^{-1}(x)=\cos^{-1}(x)$ or $f^{-1}(x)=\arccos(x)$, read 'arc-cosine of x'.[1]

To understand the 'arc' in arccosine, recall that an inverse function, by definition, reverses the process of the original function. The function $f(t)=\cos(t)$ takes a real number input t, associates it with the angle $\theta=t$ radians, and returns the value $\cos(\theta)$. Digging deeper, we have that $\cos(\theta)=\cos(t)$ is

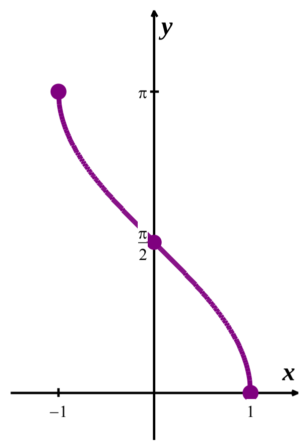
[1] The obvious pitfall here is our convention of writing $(\cos(x))^2$ as $\cos^2(x)$, $(\cos(x))^3$ as $\cos^3(x)$ and so on. It is easy to confuse $\cos^{-1}(x)$ as $(\cos(x))^{-1}$, which is equivalent to $\sec(x)$, not the inverse of $\cos(x)$. Always be careful to check the context of $\cos^{-1}(x)$!

the x-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1,0)$. Hence, we may view the inputs to $f(t)=\cos(t)$ as oriented arcs and the outputs as x-coordinates on the Unit Circle. The function f^{-1}, then, would take x-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arccosine.

Below are the graphs of $f(x)=\cos(x)$ and $f^{-1}(x)=\arccos(x)$, where we obtain the latter from the former by reflecting it across the line $y=x$. This is achieved by switching the x and y coordinates.



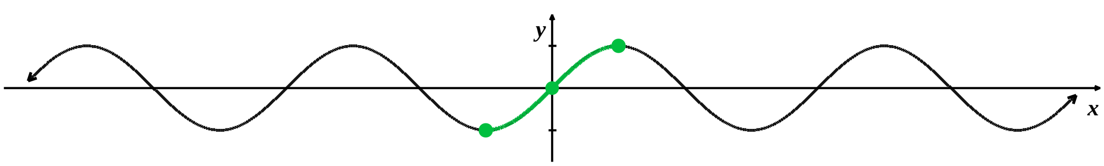
$f(x)=\cos(x),\ 0\le x\le\pi$



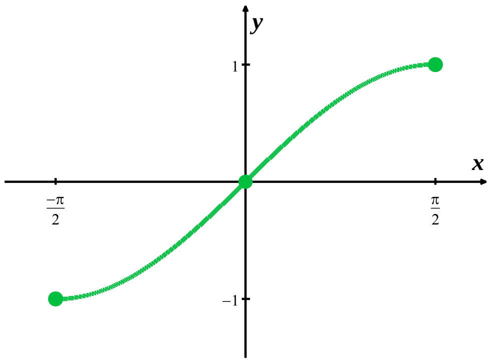
$f^{-1}(x)=\arccos(x)$

The Inverse Sine Function

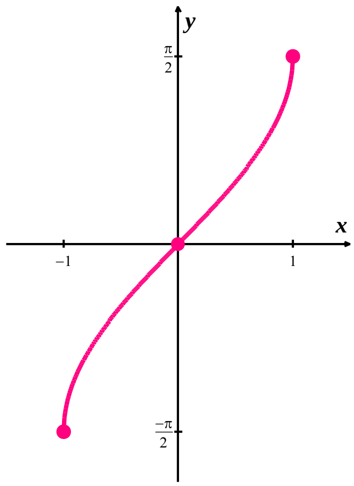
We restrict $g(x)=\sin(x)$ in a similar manner, although the interval of choice is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



The inverse of $g(x)=\sin(x)$, denoted $g^{-1}(x)=\arcsin(x)$, is read 'arcsine of x'.



$g(x) = \sin(x), \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$g^{-1}(x) = \arcsin(x)$

We list some important facts about the arccosine and arcsine functions in the following theorem.

Theorem 5.1. Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arccos(x)$
  - Domain: $[-1,1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $0 \le t \le \pi$ and $\cos(t) = x$
  - $\cos(\arccos(x)) = x$ provided $-1 \le x \le 1$
  - $\arccos(\cos(x)) = x$ provided $0 \le x \le \pi$
  - arccosine is neither even nor odd
- Properties of $G(x) = \arcsin(x)$
  - Domain: $[-1,1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $\sin(t) = x$
  - $\sin(\arcsin(x)) = x$ provided $-1 \le x \le 1$
  - $\arcsin(\sin(x)) = x$ provided $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
  - arcsine is odd

Everything in **Theorem 5.1** is a direct consequence of the fact that $f(x)=\cos(x)$ for $0 \le x \le \pi$ and $F(x)=\arccos(x)$ are inverses of each other, as are $g(x)=\sin(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $G(x)=\arcsin(x)$. It's about time for an example.

Example 5.1.1. Find the exact values of the following.

1. $\arccos\left(\frac{1}{2}\right)$
2. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
3. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
4. $\sin^{-1}\left(-\frac{1}{2}\right)$
5. $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$
6. $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)$
7. $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$
8. $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

Solution.

1. To find $\arccos\left(\frac{1}{2}\right)$, we need to find the real number t (or, equivalently, an angle measuring t radians) with $\cos(t)=\frac{1}{2}$ and $0 \le t \le \pi$. We know that $t=\frac{\pi}{3}$ meets these criteria, so $\arccos\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

2. The value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ is a real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t)=\frac{\sqrt{2}}{2}$. The number we seek is $t=\frac{\pi}{4}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$.

3. We begin by observing that $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ is equivalent to $\arccos\left(-\frac{\sqrt{2}}{2}\right)$. The number $t=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ lies in the interval $[0,\pi]$ with $\cos(t)=-\frac{\sqrt{2}}{2}$. Our answer is $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$.

4. To find $\sin^{-1}\left(-\frac{1}{2}\right)$, we seek the number *t* in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ with $\sin(t)=-\frac{1}{2}$. The answer is $t=-\frac{\pi}{6}$ so that $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$.

5. Since $0\le\frac{\pi}{6}\le\pi$, we could simply invoke **Theorem 5.1** to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$. However, in order to make sure we understand why this is the case, we choose to work the example through using the definition of arccosine.

   Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)=\arccos\left(\frac{\sqrt{3}}{2}\right)$. Now, $\arccos\left(\frac{\sqrt{3}}{2}\right)$ is the real number *t* with $0\le t\le\pi$ and $\cos(t)=\frac{\sqrt{3}}{2}$. We find $t=\frac{\pi}{6}$, so that

$$\begin{aligned}\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)&=\arccos\left(\frac{\sqrt{3}}{2}\right)\\&=\frac{\pi}{6}.\end{aligned}$$

6. Since $\frac{11\pi}{6}$ does not fall between 0 and π, **Theorem 5.1** does not apply. We are forced to work through from the inside out starting with $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$. Hence,

$$\begin{aligned}\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)&=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\\&=\frac{\pi}{6}.\end{aligned}$$

7. One way to simplify $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$ is to use **Theorem 5.1** directly. Since $-\frac{3}{5}$ is between –1 and 1, we have $\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)=-\frac{3}{5}$ and we are done.

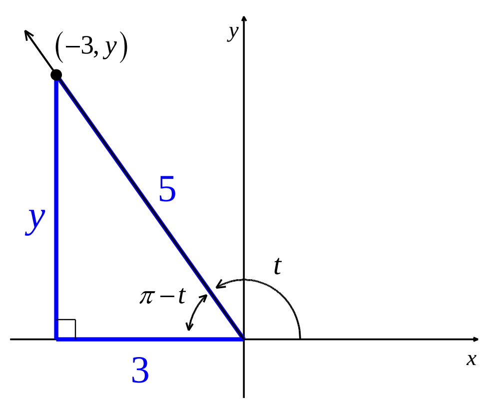
However, as before, to really understand why this cancellation occurs, we let $t=\cos^{-1}\left(-\frac{3}{5}\right)$.

Then, by definition, $\cos(t)=-\frac{3}{5}$. Hence,

$$\begin{aligned}\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)&=\cos(t)\\&=-\frac{3}{5}.\end{aligned}$$

8. To evaluate $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$, as in the previous example, we let $t=\arccos\left(-\frac{3}{5}\right)$ so that $\cos(t)=-\frac{3}{5}$ for some t where $0\le t\le\pi$. Since $\cos(t)<0$, we can narrow this down a bit and conclude that $\frac{\pi}{2}<t<\pi$, so that t corresponds to an angle in Quadrant II. We move on to finding $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)=\sin(t)$.

A geometric approach is to sketch the angle t, along with its corresponding reference angle $\pi-t$. We then introduce a 'reference triangle' in Quadrant II. Since $\cos(t)=-\frac{3}{5}$, the reference triangle will have $\cos(\pi-t)=\frac{3}{5}$. We label the adjacent side with length 3 and the hypotenuse with length 5.



The Pythagorean Theorem can be used to find the length y of the opposite side.

$$y^2 + 3^2 = 5^2$$
$$y^2 = 16$$
$$y = 4 \quad \text{As a length, y is positive.}$$

Then $\sin(\pi - t) = \frac{y}{5} = \frac{4}{5}$. And, since sine is positive in Quadrant II, $\sin(t) = \frac{4}{5}$. Finally,

$$\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$$
$$= \frac{4}{5}.$$

□

Most of the common errors encountered in dealing with the inverse circular functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$ as opposed to $\frac{11\pi}{6}$, as observed in part 6 of the previous example.

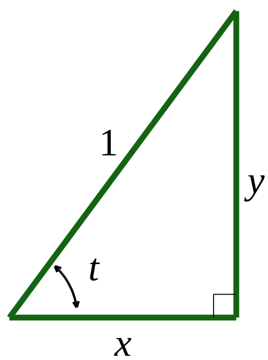
Example 5.1.2. Rewrite the following as algebraic expressions of *x* and state the domain on which the equivalence is valid.

1. $\tan\left(\arccos(x)\right)$
2. $\cos\left(2\arcsin(x)\right)$

Solution.

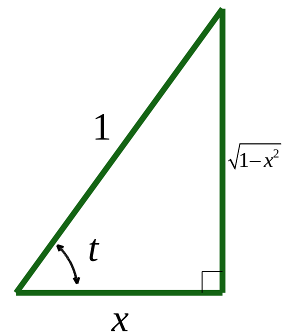
1. We begin this solution by letting $t = \arccos(x)$, so that $\cos(t) = x$.

   We sketch a right triangle representing $\cos(t) = \frac{x}{1} = \frac{\text{adjacent}}{\text{hypotenuse}}$. To find the length of the opposite side, *y*, in terms of *x*, the Pythagorean Theorem yields



$$x^2 + y^2 = 1^2$$
$$y^2 = 1 - x^2$$
$$y = \sqrt{1 - x^2}$$

This results in



$$\tan(\arccos(x)) = \tan(t)$$
$$= \frac{\sqrt{1-x^2}}{x} \quad \textbf{since tangent is } \frac{\textbf{opposite}}{\textbf{adjacent}}$$

To determine the values of *x* for which this equivalence is valid, we consider our substitution $t = \arccos(x)$. The domain of $\arccos(x)$ is $[-1,1]$, or $-1 \le x \le 1$. Additionally, since the tangent is not defined when the cosine is 0, we need to discard $x = 0$. Hence, $\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is valid for *x* in $[-1,0) \cup (0,1]$.

Note that for *x* in $[-1,0)$, we have $\frac{\pi}{2} < \arccos(x) \le \pi$, resulting in values of the tangent being less than or equal to zero. This corresponds correctly in sign with the values obtained for $\frac{\sqrt{1-x^2}}{x}$ since $-1 \le x < 0$.

2. We proceed as in the previous problem by writing $t = \arcsin(x)$ so that so that *t* lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of *x*. We have three choices for rewriting $\cos(2t)$: $\cos^2(t) - \sin^2(t)$, $2\cos^2(t) - 1$ or $1 - 2\sin^2(t)$. Since we know $x = \sin(t)$, it is easiest to use the last form.[2]

$$\cos(2\arcsin(x)) = \cos(2t)$$
$$= 1 - 2\sin^2(t)$$
$$= 1 - 2x^2$$

[2] To use the first or second form, we could use a right triangle, like we did in part 1, to evaluate sin(*t*) in terms of *x*.

To find the restrictions on *x*, we revisit our substitution $t = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \le x \le 1$, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid only on $[-1,1]$.

□

Even though the expression we arrived at in part 2 of the last example, namely $1-2x^2$, is defined for all real numbers, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid for only $-1 \le x \le 1$. This is similar to the fact that while the expression *x* is defined for all real numbers, the equivalence $\left(\sqrt{x}\right)^2 = x$ is valid only for $x \ge 0$. For this reason, it pays to be careful when we determine the intervals where such equivalences are valid.

5.1 Exercises

In Exercises 1 – 18, find the exact value.

1. $\arcsin(-1)$
2. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin(0)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin(1)$
10. $\cos^{-1}(-1)$
11. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\cos^{-1}\left(-\frac{1}{2}\right)$
14. $\arccos(0)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
17. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos(1)$

In Exercises 19 – 44, find the exact value or state that it is undefined.

19. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
20. $\sin\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$
21. $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$
22. $\sin(\arcsin(-0.42))$
23. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$
24. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
25. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$
26. $\cos\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$
27. $\cos(\arccos(-0.998))$
28. $\cos(\arccos(\pi))$
29. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$
30. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$
31. $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$
32. $\sin^{-1}\left(\sin\left(\frac{11\pi}{6}\right)\right)$
33. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$
34. $\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$
35. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$
36. $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$
37. $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$
38. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$
39. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

40. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

41. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

42. $\sin\left(\sin^{-1}\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

43. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

44. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

In Exercises 45 – 54, rewrite the quantities as algebraic expressions of *x* and state the domain on which the equivalence is valid.

45. $\sin\left(\arccos\left(x\right)\right)$

46. $\tan\left(\sin^{-1}(x)\right)$

47. $\sin\left(2\cos^{-1}(x)\right)$

48. $\sin\left(\arccos\left(2x\right)\right)$

49. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

50. $\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$

51. $\sin\left(2\arcsin\left(7x\right)\right)$

52. $\sin\left(2\sin^{-1}\left(\frac{x\sqrt{3}}{3}\right)\right)$

53. $\cos\left(2\arcsin\left(4x\right)\right)$

54. $\sin\left(\sin^{-1}(x)+\cos^{-1}(x)\right)$

55. If $\sin(\theta)=\frac{x}{2}$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, find an expression for $\theta+\sin(2\theta)$ in terms of *x*.

56. Show that $\arcsin\left(x\right)+\arccos\left(x\right)=\frac{\pi}{2}$ for $-1\le x\le 1$.

57. Discuss with your classmates why $\sin^{-1}\left(\frac{1}{2}\right)\neq 30^{\circ}$.

58. Why do the functions $f\left(x\right)=\sin^{-1}(x)$ and $g\left(x\right)=\cos^{-1}(x)$ have different ranges?

59. Since the functions $y=\cos(x)$ and $y=\cos^{-1}(x)$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

5.2 Properties of the Inverse Tangent and Cotangent Functions

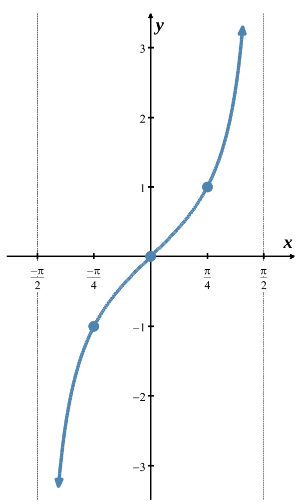
Learning Objectives

In this section you will:

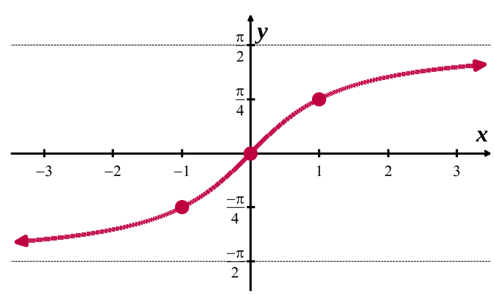
- **Learn and apply properties of the inverse tangent and cotangent functions, including domain and range.**
- **Find exact values of inverse tangent and cotangent functions, and of their composition with other trigonometric functions.**
- **Convert compositions of trigonometric and inverse tangent or cotangent functions to algebraic expressions.**

The Inverse Tangent Function

We restrict $f(x)=\tan(x)$ to its fundamental cycle on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ to obtain the inverse of tangent. The inverse, $f^{-1}(x)=\arctan(x)$, named arctangent, is also denoted $\tan^{-1}(x)$. In the following graphs, we reflect $f(x)=\tan(x)$ about the line $y=x$ to obtain $f^{-1}(x)=\arctan(x)$. Note that the vertical asymptotes $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$ from the graph of $f(x)=\tan(x)$ become the horizontal asymptotes $y=-\frac{\pi}{2}$ and $y=\frac{\pi}{2}$ on the graph of $f^{-1}(x)=\arctan(x)$.



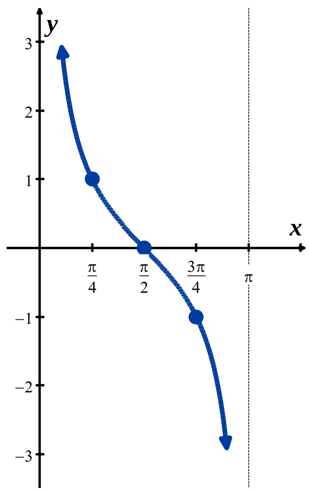
$f(x)=\tan(x)$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$



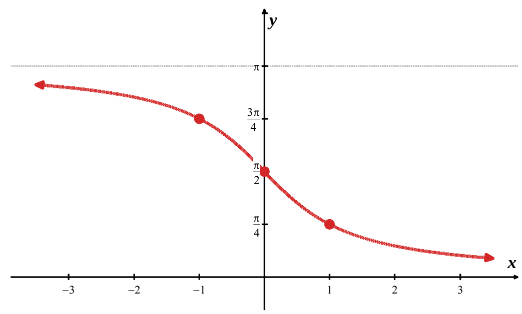
$f^{-1}(x)=\arctan(x)$

The Inverse Cotangent Function

Next, we restrict $g(x) = \cot(x)$ to its fundamental cycle on $(0, \pi)$ to obtain the arccotangent: $g^{-1}(x) = \operatorname{arccot}(x)$ or $g^{-1}(x) = \cot^{-1}(x)$. Once again, the vertical asymptotes $x = 0$ and $x = \pi$ of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$ when the graph of the cotangent is reflected about the line $y = x$ to obtain the graph of the arccotangent.



$g(x) = \cot(x)$, $0 < x < \pi$



$g^{-1}(x) = \operatorname{arccot}(x)$

Theorem 5.2. Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\dfrac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \dfrac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$ and $\tan(t) = x$
  - $\arctan(x) = \operatorname{arccot}\left(\dfrac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(x)) = x$ provided $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
  - arctangent is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range $(0, \pi)$
  - as $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $0 < t < \pi$ and $\cot(t) = x$
  - $\operatorname{arccot}(x) = \arctan\left(\dfrac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$
  - arccotangent is neither even nor odd

Example 5.2.1. Find the exact values of the following.

1. $\arctan\left(\sqrt{3}\right)$
2. $\operatorname{arccot}\left(-\sqrt{3}\right)$
3. $\cot(\operatorname{arccot}(-5))$
4. $\sin\left(\tan^{-1}\left(-\dfrac{3}{4}\right)\right)$

Solution.

1. We know $\arctan\left(\sqrt{3}\right)$ is the real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t)=\sqrt{3}$. We find $t=\frac{\pi}{3}$, so $\arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$.

2. The real number $t=\operatorname{arccot}\left(-\sqrt{3}\right)$ lies in the interval $(0,\pi)$ with $\cot(t)=-\sqrt{3}$. We get $\operatorname{arccot}\left(-\sqrt{3}\right)=\frac{5\pi}{6}$.

3. We can apply **Theorem 5.2** directly and obtain $\cot\left(\operatorname{arccot}(-5)\right)=-5$. However, working it through provides us with yet another opportunity to understand why this is the case.

   If we let $t=\operatorname{arccot}(-5)$, then $\cot(t)=-5$ for some t, $0<t<\pi$. Hence,

$$\begin{aligned}\cot\left(\operatorname{arccot}(-5)\right)&=\cot(t)\\&=-5.\end{aligned}$$

4. We start simplifying $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)$ by letting $t=\tan^{-1}\left(-\frac{3}{4}\right)$. Then $\tan(t)=-\frac{3}{4}$ for some t, $-\frac{\pi}{2}<t<\frac{\pi}{2}$. Since $\tan(t)<0$, we know, in fact, $-\frac{\pi}{2}<t<0$.

   One way to proceed is to use the Pythagorean identity $1+\cot^2(t)=\csc^2(t)$, since this relates the reciprocals of $\tan(t)$ and $\sin(t)$ and is valid for all t under consideration.[1] Along with this identity, we use $\cot(t)=-\frac{4}{3}$, from $\tan(t)=-\frac{3}{4}$, to solve for $\csc(t)$.

---

[1] It's always a good idea to make sure the identities used in these situations are valid for all values t under consideration. Check our work back in **Example 5.1.1**. Were the identities we used there valid for all t under consideration?

$$\begin{aligned} 1+\cot^2(t) &= \csc^2(t) \\ 1+\left(-\frac{4}{3}\right)^2 &= \csc^2(t) \\ \frac{25}{9} &= \csc^2(t) \\ \csc(t) &= \pm\frac{5}{3} \end{aligned}$$

With $-\frac{\pi}{2} < t < 0$, we choose $\csc(t) = -\frac{5}{3}$ so that $\sin(t) = -\frac{3}{5}$. Hence,

$$\begin{aligned} \sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right) &= \sin(t) \\ &= -\frac{3}{5}. \end{aligned}$$

□

Example 5.2.2. Rewrite the following as algebraic expressions of *x* and state the domain on which the equivalence is valid.

1. $\tan(2\arctan(x))$
2. $\cos\left(\cot^{-1}(2x)\right)$

Solution.

1. If we let $t = \arctan(x)$, then $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$. We look for a way to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of *x*.

   Before we get started using identities, we note that $\tan(2t)$ is undefined, for any integer *k*, when

   $$\begin{aligned} 2t &= \frac{\pi}{2} + \pi k, \text{ or} \\ t &= \frac{\pi}{4} + \frac{\pi}{2}k. \end{aligned}$$

   The only members of this family which lie in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are $t = \pm\frac{\pi}{4}$, which means the values of *t* under consideration for $\tan(2t)$ are $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Returning to $\arctan(2t)$, we note the double angle identity $\tan(2t) = \dfrac{2\tan(t)}{1-\tan^2(t)}$ is valid for all of the values of *t* under consideration. Hence, we get

$$\begin{aligned}\tan\left(2\arctan(x)\right) &= \tan(2t) \\ &= \frac{2\tan(t)}{1-\tan^2(t)} \\ &= \frac{2x}{1-x^2}.\end{aligned}$$

To find where this equivalence is valid, we check back with our substitution $t = \arctan(x)$.

Since the domain of $\arctan(x)$ is all real numbers, the only exclusions come from $t = \pm\dfrac{\pi}{4}$, the values of *t* we discarded earlier. We exclude corresponding values of *x*:

$$\begin{aligned}x &= \tan(t) \\ &= \tan\left(\pm\frac{\pi}{4}\right) \\ &= \pm 1.\end{aligned}$$

Hence, the equivalence $\tan\left(2\arctan(x)\right) = \dfrac{2x}{1-x^2}$ holds for all *x* in $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

2. To get started, we let $t = \cot^{-1}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. Then $\cos\left(\cot^{-1}(2x)\right) = \cos(t)$ and, since $\cos(t)$ is always defined, there are no additional restrictions on *t*. Our goal is to then express $\cos(t)$ in terms of *x*.

   Using the identity $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$, which is valid for *t* in $(0,\pi)$, we can write $\cos(t) = \cot(t)\sin(t)$. Thus

$$\begin{aligned}\cos\left(\cot^{-1}(2x)\right) &= \cos(t) \\ &= \cot(t)\sin(t) \\ &= 2x\sin(t).\end{aligned}$$

   With $\cot(t) = 2x$ and $\csc(t) = \dfrac{1}{\sin(t)}$, the Pythagorean identity $1+\cot^2(t) = \csc^2(t)$ provides a path for expressing $\sin(t)$ in terms of *x*.

$$1+\cot^2(t)=\csc^2(t)$$
$$1+(2x)^2=\csc^2(t)$$
$$\csc(t)=\pm\sqrt{4x^2+1}$$

Since *t* is between 0 and *π*, $\csc(t)>0$. Thus, $\csc(t)=\sqrt{4x^2+1}$ and $\sin(t)=\dfrac{1}{\sqrt{4x^2+1}}$.

Finally,

$$\begin{aligned}\cos\left(\cot^{-1}(2x)\right)&=2x\sin(t)\\&=\frac{2x}{\sqrt{4x^2+1}}.\end{aligned}$$

This is true for all real numbers *x* since $\cot^{-1}(2x)$ is defined for all real numbers *x* and we encountered no additional restrictions on *t*.

□

5.2 Exercises

In Exercises 1 – 14, find the exact value.

1. $\arctan\left(-\sqrt{3}\right)$
2. $\tan^{-1}(-1)$
3. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
4. $\arctan(0)$
5. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
6. $\tan^{-1}(1)$
7. $\tan^{-1}\left(\sqrt{3}\right)$
8. $\cot^{-1}\left(-\sqrt{3}\right)$
9. $\operatorname{arccot}(-1)$
10. $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
11. $\cot^{-1}(0)$
12. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right)$
13. $\operatorname{arccot}(1)$
14. $\operatorname{arccot}\left(\sqrt{3}\right)$

In Exercises 15 – 48, find the exact value or state that it is undefined.

15. $\tan\left(\tan^{-1}(-1)\right)$
16. $\tan\left(\tan^{-1}\left(\sqrt{3}\right)\right)$
17. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$
18. $\tan(\arctan(0.965))$
19. $\tan\left(\tan^{-1}(3\pi)\right)$
20. $\cot(\operatorname{arccot}(1))$
21. $\cot\left(\cot^{-1}\left(-\sqrt{3}\right)\right)$
22. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$
23. $\cot\left(\cot^{-1}(-0.001)\right)$
24. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$
25. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$
26. $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$
27. $\tan^{-1}(\tan(\pi))$
28. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
29. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$
30. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$
31. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$
32. $\operatorname{arccot}(\cot(\pi))$
33. $\cot^{-1}\left(\cot\left(\frac{\pi}{2}\right)\right)$
34. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$
35. $\sin(\arctan(-2))$
36. $\sin\left(\cot^{-1}\left(\sqrt{5}\right)\right)$
37. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$
38. $\cos\left(\cot^{-1}(3)\right)$

39. $\tan\left(\sin^{-1}\left(-\frac{2\sqrt{5}}{5}\right)\right)$

40. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

41. $\tan(\operatorname{arccot}(12))$

42. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

43. $\cot\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

44. $\cot(\tan^{-1}(0.25))$

45. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$

46. $\sin(2\tan^{-1}(2))$

47. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

48. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 49 – 55, rewrite the quantities as algebraic expressions of *x* and state the domain on which the equivalence is valid.

49. $\cos(\tan^{-1}(x))$

50. $\sin(2\tan^{-1}(x))$

51. $\cos(2\arctan(x))$

52. $\cos(\tan^{-1}(3x))$

53. $\cos(\arcsin(x)+\arctan(x))$

54. $\tan(2\sin^{-1}(x))$

55. $\sin\left(\frac{1}{2}\arctan(x)\right)$

56. If $\tan(\theta)=\frac{x}{7}$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, find an expression for $\frac{1}{2}\theta-\frac{1}{2}\sin(2\theta)$ in terms of *x*.

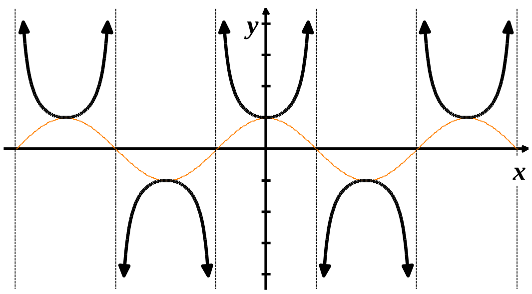
5.3 Properties of the Inverse Secant and Cosecant Functions

Learning Objectives

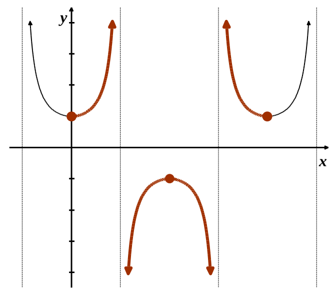
In this section you will:

- **Learn and apply properties of the inverse secant and cosecant functions, including domain and range.**
- **Find exact values of inverse secant and cosecant functions, and of their composition with other trigonometric functions.**
- **Convert compositions of trigonometric and inverse secant or cosecant functions to algebraic expressions.**

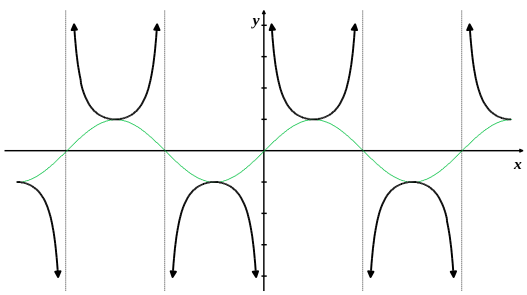
The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in **Section 3.4**, are given below with the fundamental cycles highlighted.



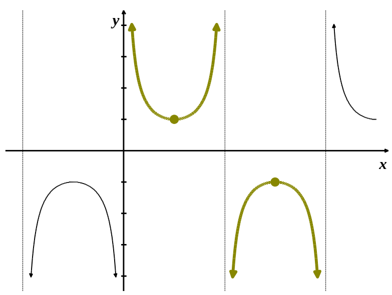
The graph of $y = \sec(x)$



Fundamental cycle of $y = \sec(x)$



The graph of $y = \csc(x)$

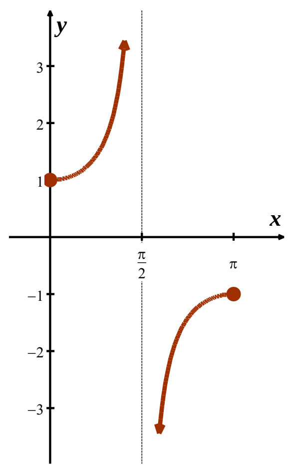


Fundamental cycle of $y = \csc(x)$

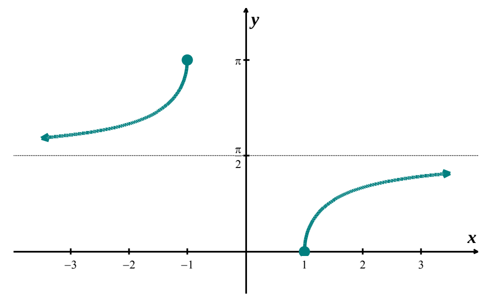
It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty,-1]\cup[1,\infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus, in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1,\infty)$, and another piece to cover the bottom, namely $(-\infty,-1]$.

The Inverse Secant Function

For $f(x)=\sec(x)$, we restrict the domain to $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, corresponding to that of cosine, and reflect $f(x)=\sec(x)$ about the line $y=x$ to obtain the graph of $f^{-1}(x)=\operatorname{arcsec}(x)$.



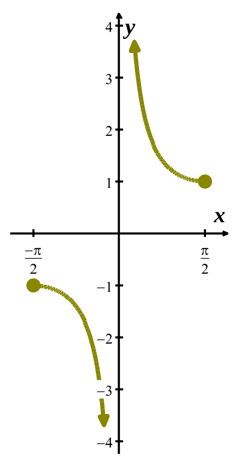
$f(x)=\sec(x)$ on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$



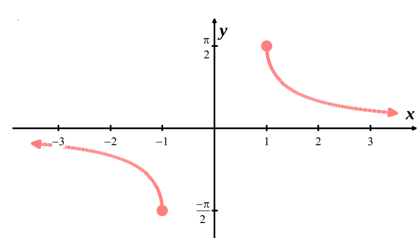
$f^{-1}(x)=\operatorname{arcsec}(x)$

The Inverse Cosecant Function

We restrict $g(x)=\csc(x)$ to $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, in correspondence with the sine, and reflect about the line $y=x$ to obtain $g^{1}(x)=\operatorname{arccsc}(x)$.



$g(x)=\csc(x)$ on $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$



$g^{-1}(x)=\operatorname{arccsc}(x)$

Note that $\operatorname{arcsec}(x)$ and $\operatorname{arccsc}(x)$ may also be written as $\sec^{-1}(x)$ and $\csc^{-1}(x)$, respectively. For both arcsecant and arccosecant, the domain is $(-\infty,-1]\cup[1,\infty)$, which can be written as $\{x:|x|\geq 1\}$.

Theorem 5.3. Properties of the Arcsecant and Arccosecant Functions

- Properties of $F(x)=\operatorname{arcsec}(x)$
  - Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
  - Range: $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
  - as $x\to-\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{+}$; as $x\to\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x)=t$ if and only if $0\leq t<\frac{\pi}{2}$ or $\frac{\pi}{2}<t\leq\pi$ and $\sec(t)=x$
  - $\operatorname{arcsec}(x)=\arccos\left(\frac{1}{x}\right)$ provided $|x|\geq 1$
  - $\sec(\operatorname{arcsec}(x))=x$ provided $|x|\geq 1$
  - $\operatorname{arcsec}(\sec(x))=x$ provided $0\leq x<\frac{\pi}{2}$ or $\frac{\pi}{2}<x\leq\pi$
  - arcsecant is neither even nor odd
- Properties of $G(x)=\operatorname{arccsc}(x)$
  - Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
  - Range: $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
  - as $x\to-\infty$, $\operatorname{arccsc}(x)\to 0^{-}$; as $x\to\infty$, $\operatorname{arccsc}(x)\to 0^{+}$
  - $\operatorname{arccsc}(x)=t$ if and only if $-\frac{\pi}{2}\leq t<0$ or $0<t\leq\frac{\pi}{2}$ and $\csc(t)=x$
  - $\operatorname{arccsc}(x)=\arcsin\left(\frac{1}{x}\right)$ provided $|x|\geq 1$
  - $\csc(\operatorname{arccsc}(x))=x$ provided $|x|\geq 1$
  - $\operatorname{arccsc}(\csc(x))=x$ provided $-\frac{\pi}{2}\leq x<0$ or $0<x\leq\frac{\pi}{2}$
  - arccosecant is odd

Example 5.3.1. Find the exact values of the following.

1. $\operatorname{arcsec}(2)$
2. $\csc^{-1}(-2)$
3. $\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
4. $\cot(\operatorname{arccsc}(-3))$

Solution.

1. Since $|2| \geq 1$, we can use **Theorem 5.3** and have

$$\begin{aligned}\operatorname{arcsec}(2) &= \arccos\left(\frac{1}{2}\right) \\ &= \frac{\pi}{3}.\end{aligned}$$

2. Once again, with $|-2| \geq 1$, **Theorem 5.3** comes to our aid giving

$$\begin{aligned}\csc^{-1}(-2) &= \sin^{-1}\left(-\frac{1}{2}\right) \\ &= -\frac{\pi}{6}.\end{aligned}$$

3. Since $\dfrac{5\pi}{4}$ doesn't fall between 0 and $\dfrac{\pi}{2}$ or between $\dfrac{\pi}{2}$ and π, we cannot use the inverse property stated in **Theorem 5.3**. We can, nevertheless, begin by working inside out which yields

$$\begin{aligned}\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right) &= \sec^{-1}\left(-\sqrt{2}\right) & \textbf{Note: } \left|-\sqrt{2}\right| \geq 1 \\ &= \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) & \textbf{from Theorem 5.3} \\ &= \frac{3\pi}{4}.\end{aligned}$$

4. One way to begin to simplify $\cot(\operatorname{arccsc}(-3))$ is to let $t = \operatorname{arccsc}(-3)$. Then $\csc(t) = -3$ and, since this is negative, we have that t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$. We are after $\cot(t)$, knowing $t = \operatorname{arccsc}(-3)$, so we use the Pythagorean identity $1 + \cot^2(t) = \csc^2(t)$.

$$\begin{aligned}1 + \cot^2(t) &= \csc^2(t) \\ 1 + \cot^2(t) &= (-3)^2 \\ \cot(t) &= \pm\sqrt{8} \\ \cot(t) &= \pm 2\sqrt{2}\end{aligned}$$

Since $-\frac{\pi}{2} \leq t < 0$, $\cot(t) < 0$, so we get $\cot(\text{arccsc}(-3)) = -2\sqrt{2}$.

□

Example 5.3.2. Rewrite the following as algebraic expressions of *x* and state the domain on which the equivalence is valid.

1. $\tan(\text{arcsec}(x))$
2. $\cos(\csc^{-1}(4x))$

Solution.

1. We begin simplifying $\tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for *t* in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all *t* values under consideration, there are no additional restrictions on *t*.

   To relate $\sec(t)$ to $\tan(t)$, we use the identity $1 + \tan^2(t) = \sec^2(t)$, which is valid for all *t* under consideration.

$$\begin{aligned} 1 + \tan^2(t) &= \sec^2(t) \\ 1 + \tan^2(t) &= x^2 \\ \tan(t) &= \pm\sqrt{x^2 - 1} \end{aligned}$$

   If *t* belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$; if, on the other hand, *t* belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a piecewise defined function for $\tan(t)$.

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, \text{ if } 0 \leq \text{t} < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, \text{ if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

   Now we need to determine what these conditions on *t* mean for $x = \sec(t)$. When $0 \leq t < \frac{\pi}{2}$, it follows that $x \geq 1$, and when $\frac{\pi}{2} < t \leq \pi$, we have $x \leq -1$. With no further restrictions on *t*, we can express $\tan(\text{arcsec}(x))$ as an algebraic expression of *x*.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2-1}, \text{ if } x \ge 1 \\ -\sqrt{x^2-1}, \text{ if } x \le -1 \end{cases}$$

2. To simplify $\cos\left(\csc^{-1}(4x)\right)$, we start by letting $t = \csc^{-1}(4x)$. Then $\csc(t) = 4x$ for *t* in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, and we now set about finding an expression for $\cos\left(\csc^{-1}(4x)\right) = \cos(t)$. Since $\cos(t)$ is defined for all *t*, there are no additional restrictions on *t*. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$. To find $\cos(t)$, we can make use of the identity $\cos^2(t) + \sin^2(t) = 1$.

$$\cos^2(t) + \sin^2(t) = 1$$
$$\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$$
$$\cos(t) = \pm\sqrt{\frac{16x^2-1}{16x^2}}$$
$$\cos(t) = \pm\frac{\sqrt{16x^2-1}}{4|x|}$$

Since *t* belongs to $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$, we know $\cos(t) \ge 0$, so we choose $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|}$. (The absolute value here is necessary since *x* could be negative.)

To find the values of *x* for which this equivalence is valid, we look back at our original substitution $t = \csc^{-1}(4x)$. The domain of $\csc^{-1}(x)$ requires its argument *x* to satisfy $|x| \ge 1$, and so the domain of $\csc^{-1}(4x)$ will require $|4x| \ge 1$.

$$|4x| \ge 1$$
$$4x \le -1 \text{ or } 4x \ge 1$$
$$x \le -\frac{1}{4} \text{ or } x \ge \frac{1}{4}$$

With no additional restrictions on *t*, the equivalence $\cos\left(\csc^{-1}(4x)\right) = \frac{\sqrt{16x^2-1}}{4|x|}$ holds for all *x* in $\left(-\infty,-\frac{1}{4}\right]\cup\left[\frac{1}{4},\infty\right)$.

□

5.3 Exercises

In Exercises 1 – 16, find the exact value.

1. $\operatorname{arcsec}(2)$
2. $\csc^{-1}(2)$
3. $\sec^{-1}(\sqrt{2})$
4. $\csc^{-1}(\sqrt{2})$
5. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
6. $\operatorname{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
7. $\sec^{-1}(1)$
8. $\operatorname{arccsc}(1)$
9. $\operatorname{arcsec}(-2)$
10. $\sec^{-1}(-\sqrt{2})$
11. $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$
12. $\sec^{-1}(-1)$
13. $\csc^{-1}(-2)$
14. $\operatorname{arccsc}(-\sqrt{2})$
15. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
16. $\operatorname{arccsc}(-1)$

In Exercises 17 – 52, find the exact value or state that it is undefined.

17. $\sec(\sec^{-1}(2))$
18. $\sec(\operatorname{arcsec}(-1))$
19. $\sec\left(\sec^{-1}\left(\frac{1}{2}\right)\right)$
20. $\sec(\sec^{-1}(0.75))$
21. $\sec(\operatorname{arcsec}(117\pi))$
22. $\csc(\csc^{-1}(\sqrt{2}))$
23. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$
24. $\csc\left(\csc^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
25. $\csc(\operatorname{arccsc}(1.0001))$
26. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$
27. $\sec^{-1}\left(\sec\left(\frac{\pi}{4}\right)\right)$
28. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$
29. $\sec^{-1}\left(\sec\left(\frac{5\pi}{6}\right)\right)$
30. $\sec^{-1}\left(\sec\left(-\frac{\pi}{2}\right)\right)$
31. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$
32. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$
33. $\csc^{-1}\left(\csc\left(\frac{5\pi}{4}\right)\right)$
34. $\csc^{-1}\left(\csc\left(\frac{2\pi}{3}\right)\right)$
35. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$
36. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$
37. $\sec^{-1}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

38. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

39. $\sin\left(\csc^{-1}(-3)\right)$

40. $\cos\left(\sec^{-1}(5)\right)$

41. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

42. $\cot\left(\csc^{-1}\left(\sqrt{5}\right)\right)$

43. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

44. $\sec\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$

45. $\sec\left(\arctan(10)\right)$

46. $\sec\left(\cot^{-1}\left(-\frac{\sqrt{10}}{10}\right)\right)$

47. $\csc\left(\operatorname{arccot}(9)\right)$

48. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

49. $\csc\left(\tan^{-1}\left(-\frac{2}{3}\right)\right)$

50. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

51. $\sin\left(2\csc^{-1}\left(\frac{13}{5}\right)\right)$

52. $\cos\left(2\sec^{-1}\left(\frac{25}{7}\right)\right)$

In Exercises 53 – 55, rewrite the quantities as algebraic expressions of *x* and state the domain on which the equivalence is valid.

53. $\sec\left(\arctan(x)\right)$

54. $\csc\left(\arccos(x)\right)$

55. $\sec\left(\arctan(2x)\right)\tan\left(\arctan(2x)\right)$

56. If $\sec(\theta)=\frac{x}{4}$ for $0<\theta<\frac{\pi}{2}$, find an expression for $4\tan(\theta)-4\theta$ in terms of *x*.

57. Show that $\operatorname{arcsec}(x)=\arccos\left(\frac{1}{x}\right)$ for $|x|\geq 1$ as long as we use $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ as the range of $f(x)=\operatorname{arcsec}(x)$.

58. Show that $\csc^{-1}(x)=\sin^{-1}\left(\frac{1}{x}\right)$ for $|x|\geq 1$ as long as we use $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ as the range of $f(x)=\csc^{-1}(x)$.

5.4 Calculators and the Inverse Circular Functions

Learning Objectives

In this section you will:

- **Use technology to evaluate approximate values of the inverse trigonometric functions.**
- **Find domains and ranges of inverse trigonometric functions.**
- **Use inverse trigonometric functions to solve real-world applications.**

In the sections to come, we will have need to approximate values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labeled as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked for an arccotangent, arcsecant or arccosecant, we often need to employ some ingenuity, as the next example illustrates.

Using a Calculator to Find Values

Example 5.4.1. Use a calculator to approximate the following values to four decimal places.

1. $\operatorname{arccot}(2)$ 2. $\sec^{-1}(5)$ 3. $\cot^{-1}(-2)$ 4. $\operatorname{arccsc}\left(-\frac{3}{2}\right)$

Solution.

1. Since $2>0$, we can use a property from **Theorem 5.2** to rewrite $\operatorname{arccot}(2)$ as $\arctan\left(\frac{1}{2}\right)$.

   After verifying that our calculator or other graphing tool is in radian mode, we find

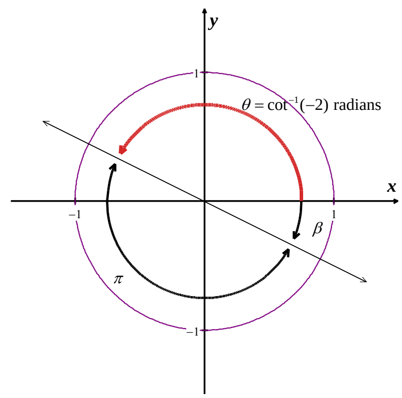
$$\begin{aligned}\operatorname{arccot}(2) &= \arctan\left(\frac{1}{2}\right)\\ &\approx 0.4636 \text{ radians.}\end{aligned}$$

2. Noting that $5\geq 1$, we can use a property from **Theorem 5.3** to write

$$\begin{aligned}\sec^{-1}(5) &= \cos^{-1}\left(\frac{1}{5}\right)\\ &\approx 1.3694 \text{ radians.}\end{aligned}$$

3. Since the argument –2 is negative, we cannot directly apply **Theorem 5.2** to help us find $\cot^{-1}(-2)$. We will, however, be able to use $\tan^{-1}\left(\frac{1}{-2}\right)$ by first establishing a relationship between $\cot^{-1}(-2)$ and $\tan^{-1}\left(-\frac{1}{2}\right)$.

   - By definition, the real number $t = \tan^{-1}\left(-\frac{1}{2}\right)$ satisfies $\tan(t) = -\frac{1}{2}$ with $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know more specifically that $-\frac{\pi}{2} < t < 0$, so t corresponds to an angle β in Quadrant IV with $\tan(\beta) = -\frac{1}{2}$.



   - We next visualize the angle $\theta = \cot^{-1}(-2)$ radians and note that θ is a Quadrant II angle with $\tan(\theta) = -\frac{1}{2}$.

   This means θ is exactly π units away from β, and we get

$$\begin{aligned}\theta &= \pi + \beta \\ &= \pi + \tan^{-1}\left(-\frac{1}{2}\right) \\ &\approx 2.6779.\end{aligned}$$

   Hence, $\cot^{-1}(-2) \approx 2.6779$ radians.

4. To approximate $\operatorname{arccsc}\left(-\frac{3}{2}\right)$, noting that $\left|-\frac{3}{2}\right| \geq 1$, we can use **Theorem 5.3**:

$$\begin{aligned}\operatorname{arccsc}\left(-\frac{3}{2}\right) &= \arcsin\left(-\frac{2}{3}\right) \\ &\approx -0.7297 \text{ radians.}\end{aligned}$$

□

Domain and Range of Inverse Trigonometric Functions

Example 5.4.2. Find the domain and range of the following functions. Check your answers using graphing technology.

1. $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$
2. $f(x) = 3\tan^{-1}(4x)$
3. $g(x) = \operatorname{arccot}\left(\frac{x}{2}\right) + \pi$

Solution.

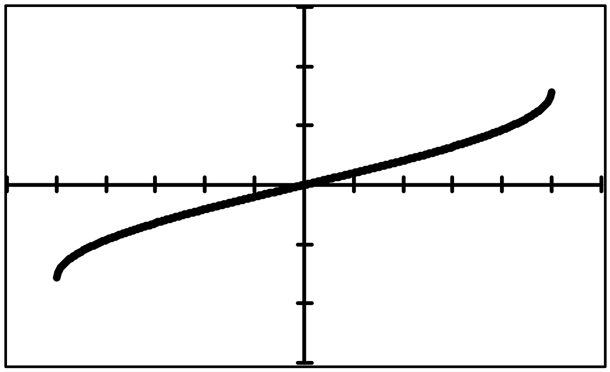
1. Since the domain of $F(x) = \arccos(x)$ is $-1 \le x \le 1$, we can find the domain of $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$ by setting the argument of the arccosine, in this case $\frac{x}{5}$, between –1 and 1.

$$-1 \le \frac{x}{5} \le 1$$
$$(-1)(5) \le \left(\frac{x}{5}\right)(5) \le (1)(5)$$
$$-5 \le x \le 5$$

So the domain of $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$ is $[-5, 5]$.

To determine the range of *f*, we select three key points on the graph of $F(x) = \arccos(x)$: $(-1, \pi)$, $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$. We use transformations to track these points to $\left(-5, -\frac{\pi}{2}\right)$, $(0, 0)$ and $\left(5, \frac{\pi}{2}\right)$ on the graph of $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$. Plotting these values tells us that the range of *f* is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The following graphing calculator screen confirms a domain of $[-5, 5]$ and range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

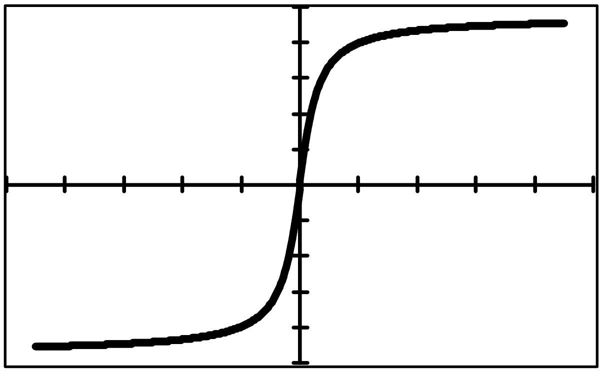


$$y = f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$$

2. To find the domain of $f(x) = 3\tan^{-1}(4x)$, we note the domain of $F(x) = \tan^{-1}(x)$ is all real numbers. The only restrictions, if any, on the domain of $f(x) = 3\tan^{-1}(4x)$ come from the argument, 4*x*, and since 4*x* is defined for all real numbers, we have established that the domain of *f* is all real numbers.

   To determine the range of *f*, we can choose the key point $(0,0)$ along with horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ from the graph of $y = F(x) = \tan^{-1}(x)$. We find that the graph of $y = f(x) = 3\tan^{-1}(4x)$ differs by a horizontal compression with a factor of 4 and a vertical stretch with a factor of 3. It is the latter which affects the range, producing a range of $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

   We confirm the domain of $(-\infty, \infty)$ and range of $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ using graphing technology.



$y = f(x) = 3\tan^{-1}(4x)$

3. To find the domain of $g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$, we proceed as above. Since the domain of $G(x) = \text{arccot}(x)$ is $(-\infty, \infty)$, and $\frac{x}{2}$ is defined for all *x*, we get the domain of *g* is $(-\infty, \infty)$ as well.

As for the range, we note that the range of $G(x) = \text{arccot}(x)$, like that of $F(x) = \tan^{-1}(x)$, is limited by a pair of horizontal asymptotes, in this case $y = 0$ and $y = \pi$. We graph $y = g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$ starting with $y = G(x) = \text{arccot}(x)$ and first performing a horizontal expansion by a factor of 2, followed by a vertical shift upwards by π. This latter transformation is the one which affects the range, making it $(\pi, 2\pi)$.

To check this graphically using technology, it may be necessary to create a piecewise defined function for $g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$ that makes use of the arctangent function.

- Using **Theorem 5.2**, we have that $\text{arccot}\left(\frac{x}{2}\right) = \arctan\left(\frac{2}{x}\right)$ when $\frac{x}{2} > 0$, or, in this case, when $x > 0$. Hence, for $x > 0$, we have $g(x) = \arctan\left(\frac{2}{x}\right) + \pi$.

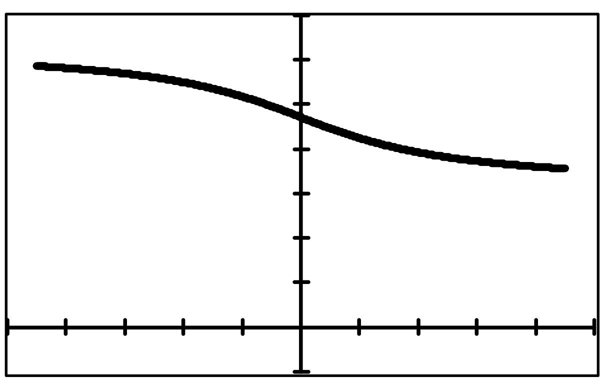
- When $\frac{x}{2} < 0$, we can use the same argument in **Example 5.4.1**, part 3, that gave us $\cot^{-1}(-2) = \pi + \tan^{-1}\left(-\frac{1}{2}\right)$ to give us $\operatorname{arccot}\left(\frac{x}{2}\right) = \pi + \arctan\left(\frac{2}{x}\right)$. Thus, for $x < 0$,

$$\begin{aligned} g(x) &= \operatorname{arccot}\left(\frac{x}{2}\right) + \pi \\ &= \pi + \arctan\left(\frac{2}{x}\right) + \pi \\ &= \arctan\left(\frac{2}{x}\right) + 2\pi. \end{aligned}$$

- What about $x = 0$? We know $g(0) = \operatorname{arccot}(0) + \pi = \pi$, and neither of the formulas for g involving arctangent will produce this result.

Finally, we have a piecewise function to use when graphing $y = g(x)$ with technology such as graphing calculators.

$$y = g(x) = \operatorname{arccot}\left(\frac{x}{2}\right) + \pi = \begin{cases} \arctan\left(\frac{2}{x}\right) + 2\pi & \text{when } x < 0 \\ \pi & \text{when } x = 0 \\ \arctan\left(\frac{2}{x}\right) + \pi & \text{when } x < 0 \end{cases}$$



$$y = g(x) = \operatorname{arccot}\left(\frac{x}{2}\right) + \pi$$

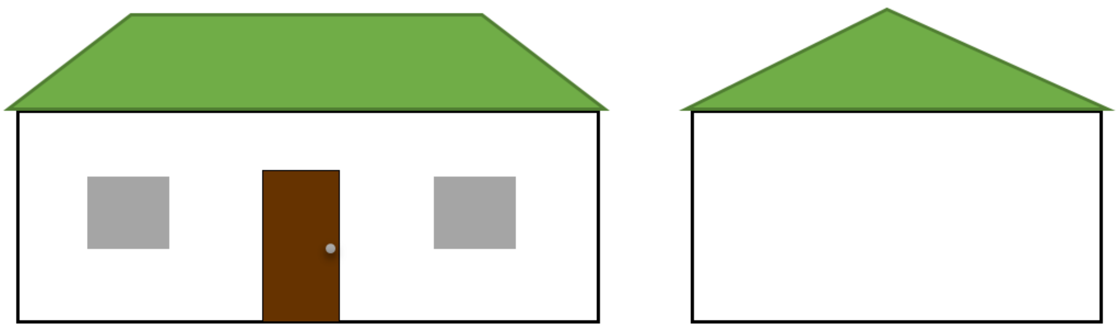
The graph confirms a domain of $(-\infty, \infty)$ and range of $(\pi, 2\pi)$.

□

Applications of Inverse Trigonometric Functions

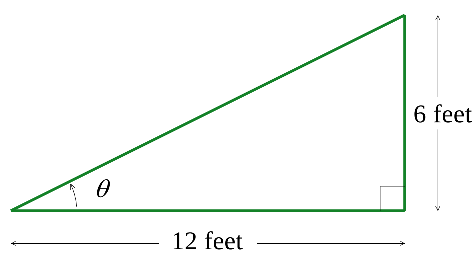
The inverse trigonometric functions are typically found in applications where the measure of an angle is required. One such scenario is presented in the following example.

Example 5.4.3.[1] The roof on the house below has a 6/12 pitch. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.



Front View   Side View

Solution. If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. Using trigonometric functions of right triangles, we find the angle of inclination, labeled θ below, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$.



Since θ is an acute angle, we can use the arctangent function and we find (using a calculator in degree mode)

[1] Thanks to Dan Stitz for this problem.

$$\theta = \arctan\left(\frac{1}{2}\right)$$
$$\approx 26.57^\circ.$$

□

5.4 Exercises

In Exercises 1 – 6, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

1. $\cos^{-1}(-0.4)$
2. $\arcsin(0.23)$
3. $\arccos\left(\frac{3}{5}\right)$
4. $\cos^{-1}(0.8)$
5. $\tan^{-1}(6)$
6. $\arctan(-6)$

In Exercises 7 – 18, find the domain of the given function. Write your answers in interval notation.

7. $f(x) = \sin^{-1}(5x)$
8. $f(x) = \cos^{-1}\left(\frac{3x-1}{2}\right)$
9. $f(x) = \arcsin\sin(2x^2)$
10. $f(x) = \arccos\left(\frac{1}{x^2-4}\right)$
11. $f(x) = \arctan(4x)$
12. $f(x) = \cot^{-1}\left(\frac{2x}{x^2-9}\right)$
13. $f(x) = \tan^{-1}(\ln(2x-1))$
14. $f(x) = \operatorname{arccot}\left(\sqrt{2x-1}\right)$
15. $f(x) = \sec^{-1}(12x)$
16. $f(x) = \operatorname{arccsc}(x+5)$
17. $f(x) = \operatorname{arcsec}\left(\frac{x^3}{8}\right)$
18. $f(x) = \csc^{-1}(e^{2x})$

19. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it touches level ground 360 feet from the base of the tower. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

20. At Cliffs of Insanity Point, the Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

21. Shelving that is being built at the college library is to be 14 inches deep. An 18-inch rod will be attached to the wall and to the underside of the shelf, at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

22. A parasailor is being pulled by a boat on Lake Powell. The cable is 300 feet long and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

23. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park is begun. From a 200 foot tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly towards the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

24. Suppose a 13-foot ladder leans against the side of a house, reaching to the bottom of a second-floor window 12 feet above the ground. What angle does the ladder make with the house? Round your answer to the nearest tenth of a degree.

CHAPTER 6
TRIGONOMETRIC EQUATIONS

Chapter Outline

6.1 Solving Equations Using the Inverse Trigonometric Functions
6.2 Solving Equations Involving a Single Trigonometric Function
6.3 Solving Equations of Multiple Trigonometric Functions/Arguments

Introduction

Chapter 6 focuses on solving trigonometric equations through the implementation of tools and formulas accumulated throughout the preceding chapters. In Section 6.1, inverse trigonometric functions are used to find acute angles within right triangles, as well as to find exact solutions of trigonometric equations. Section 6.2 applies identities and inverse functions in solving equations of a single trigonometric function. The process continues in Section 6.3 for equations that include multiple trigonometric functions and/or differing arguments.

This essential chapter is primarily used to develop algebraic skills necessary for solving equations. In Chapter 7, these skills will be put to use in the many applications of triangles made possible through the Law of Sines and the Law of Cosines.

6.1 Solving Equations Using the Inverse Trigonometric Functions

Learning Objectives

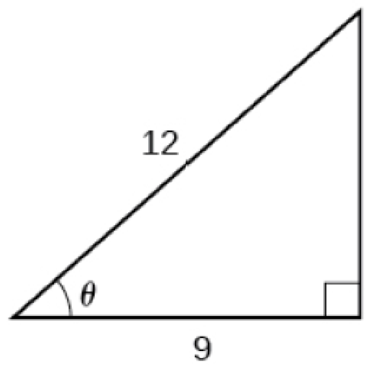
In this section you will:

- **Use inverse trigonometric functions to solve right triangles**
- **Use inverse trigonometric functions to solve for angles in trigonometric equations.**

Solving for Angles in Right Triangles

In **Section 5.4**, we used inverse trigonometric functions to solve real-world applications. Here, we apply the same technique in solving for acute angles within right triangles. Using inverse trigonometric functions, we can determine an angle, being given only the value of a trigonometric function, such as cosine in the following example. Thus, inverse trigonometric functions truly serve as inverses with the angle representing the inverse of the trigonometric function.

Example 6.1.1. Solve the following triangle for the angle θ.



Solution. Because we know the lengths of the hypotenuse and the side adjacent to the angle θ, it makes sense for us to use the cosine function.

$$\cos(\theta) = \frac{9}{12}$$

$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \qquad \text{from properties of the arccosine}$$

$$\theta \approx 0.7227 \text{ radians or } \theta \approx 41.4096^\circ$$

□

Knowing the measure of one acute angle in a right triangle, we can easily determine the measure of the second acute angle. In the example above, the measure of the angle opposite the side of length 9 would be, approximately, $180^\circ - 90^\circ - 41.4096^\circ = 48.5904^\circ$. Note that the exact measure is $\sin^{-1}\left(\frac{9}{12}\right)$.

Solving for Angles in Trigonometric Equations

In **Section 2.3**, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ and $\tan(t) = -1$. In each case, we appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of common angles. If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or to solve $\tan(t) = -2$ for real numbers *t*, we would have been hard pressed to do so. With the introduction of the inverse trigonometric functions, we are now in a position to solve these equations.

A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know there are answers but we can't express them using 'friendly' numbers.[1] To solve $x^2 = 7$, we make use of the square root function and write $x = \pm\sqrt{7}$. We can certainly approximate these answers using a calculator, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$. In the same way, we will use the arcsine function to solve $\sin(\theta) = \frac{1}{3}$, as seen in the following example.

Example 6.1.2. Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.

2. Find all real numbers *t* for which $\tan(t) = -2$.

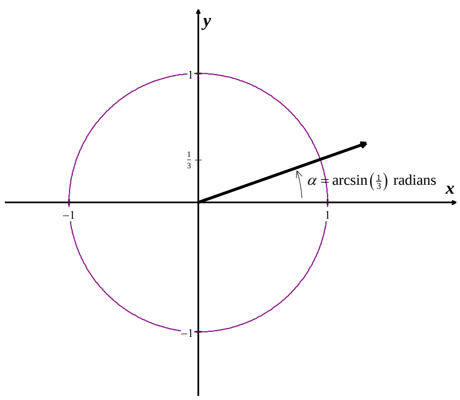
---

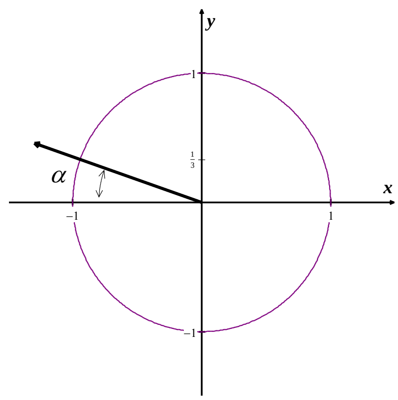
[1] This is all, of course, a matter of opinion. Many find $\pm\sqrt{7}$ just as nice as ± 2.

3. Solve $\sec(x) = -\frac{5}{3}$ for *x*.

Solution.

1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and in Quadrant II. If we let α denote the acute solution to the equation, then all of the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all solutions in Quadrant II.





Noting that $\frac{1}{3}$ is not the sine of any of the common angles, we use the arcsine function to express our answers. The real number $t = \arcsin\left(\frac{1}{3}\right)$ is defined so it satisfies $0 < t < \frac{\pi}{2}$ with $\sin(t) = \frac{1}{3}$. Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians.

Since the solutions in Quadrant I are all coterminal with α, we get part of our solution to be

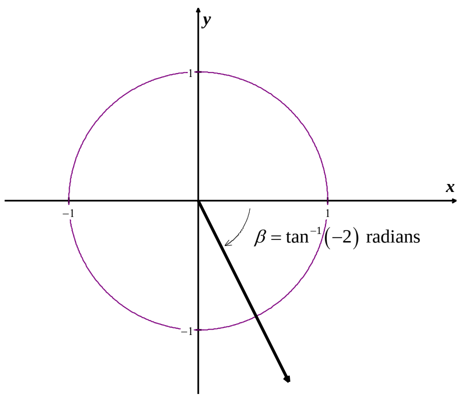
$$\begin{aligned}\theta &= \alpha + 2\pi k \\ &= \arcsin\left(\frac{1}{3}\right) + 2\pi k, \text{ for integers } k.\end{aligned}$$

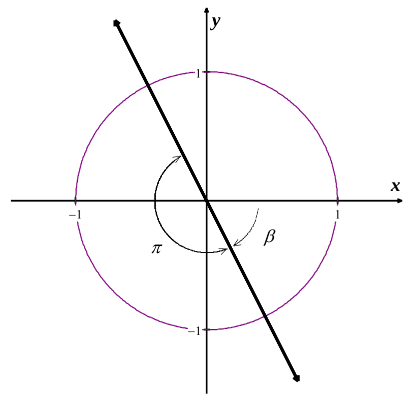
Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are

$$\theta = \pi - \alpha + 2\pi k$$
$$= \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k, \text{ for integers } k.$$

Our final answer is that the solution to $\sin(\theta) = \frac{1}{3}$ is $\theta = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for all integers *k*.

2. We may visualize the solutions to $\tan(t) = -2$ as angles θ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.





We note that none of the common angles have tangent –2, so we need to use the arctangent, or inverse tangent, function to express our answers. The real number $t = \tan^{-1}(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \tan^{-1}(-2)$ radians, we see that all of the Quadrant IV solutions to $\tan(t) = -2$ are coterminal with β. Moreover, the solutions from Quadrant II differ by exactly π units from the solutions in Quadrant IV, so all the solutions to $\tan(t) = -2$ are of the form

$$\theta = \beta + \pi k$$
$$= \tan^{-1}(-2) + \pi k \text{ for some integer } k.$$

Switching back to the variable *t*, we record our final answer to $\tan(t) = -2$ as $t = \tan^{-1}(-2) + \pi k$ for integers *k*.

3. The last equation we are asked to solve, $\sec(x) = -\frac{5}{3}$, poses an immediate problem. We are not told whether or not *x* represents an angle or a real number. We assume the latter, but note that we will use angles and the Unit Circle to solve the equation regardless.

Adopting an angle approach, we consider $\sec(\theta) = -\frac{5}{3}$ and note that our solutions lie in Quadrants II and III. Since $-\frac{5}{3}$ isn't the secant of any of the common angles, we'll need to express our solutions in terms of the arcsecant function. The real number $x = \text{arcsec}\left(-\frac{5}{3}\right)$ is defined so that $\frac{\pi}{2} < x < \pi$ with $\sec(x) = -\frac{5}{3}$.

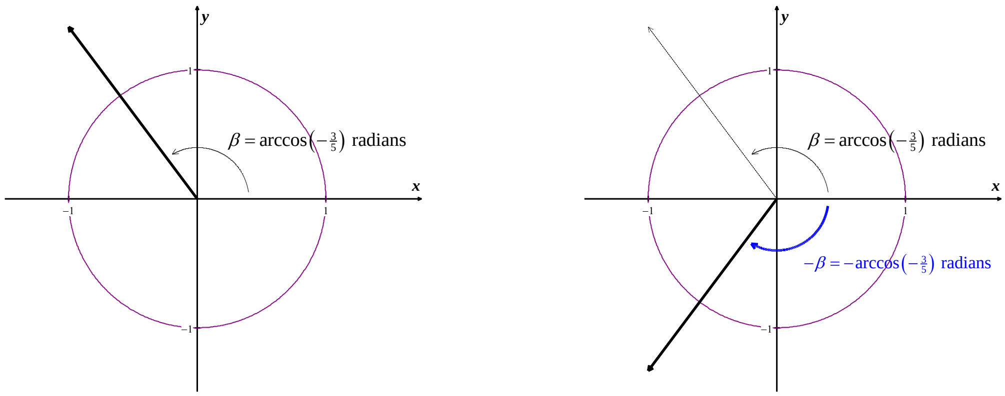
If we let $\beta = \text{arcsec}\left(-\frac{5}{3}\right)$, we see that β is a Quadrant II angle. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\text{arcsec}\left(-\frac{5}{3}\right)$. Since all angle solutions are coterminal with β or $-\beta$, we get our solutions for $\sec(\theta) = -\frac{5}{3}$ to be, for integers *k*,

$$\theta = \beta + 2\pi k \quad \text{or} \quad \theta = -\beta + 2\pi k$$

$$\theta = \text{arcsec}\left(-\frac{5}{3}\right) + 2\pi k \quad \text{or} \quad \theta = -\text{arcsec}\left(-\frac{5}{3}\right) + 2\pi k.$$

Switching back to the variable *x*, we record our final answer to $\sec(x) = -\frac{5}{3}$ as

$x = \text{arcsec}\left(-\frac{5}{3}\right) + 2\pi k$ or $x = -\text{arcsec}\left(-\frac{5}{3}\right) + 2\pi k$ for integers *k*.



Note that, with $\left|-\frac{5}{3}\right| \geq 1$, it follows from **Theorem 5.3** that $\operatorname{arcsec}\left(-\frac{5}{3}\right) = \arccos\left(-\frac{3}{5}\right)$.

Another way to write our solution to $\sec(x) = -\frac{5}{3}$ is $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers *k*.

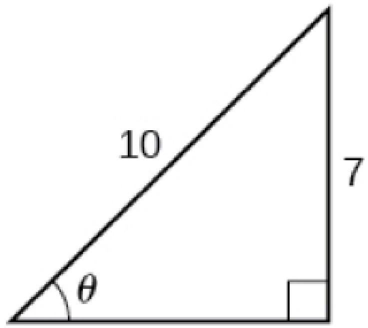
□

The reader is encouraged to check the answers found in **Example 6.1.2**, both analytically and with technology.

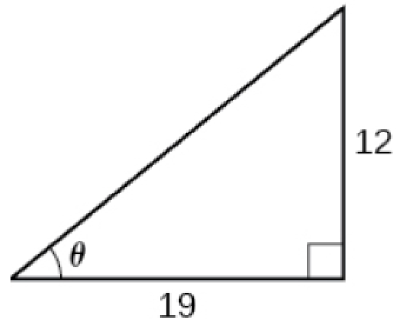
6.1 Exercises

In Exercises 1 – 2, find the angle θ in the given right triangle. Express your answers using degree measure rounded to two decimal places.

1.



2.



In Exercises 3 – 5, find the two acute angles in the right triangle whose sides have the given lengths. Express your answers using degree measure rounded to two decimal places.

3. lengths 3, 4 and 5

4. lengths 5, 12 and 13

5. lengths 336, 527 and 625

In Exercises 6 – 25, solve the equation using the techniques discussed in **Example 6.1.2**. Then approximate the solutions which lie in the interval $[0, 2\pi)$ to four decimal places.

6. $\sin(x) = \frac{7}{11}$

7. $\cos(x) = -\frac{2}{9}$

8. $\sin(x) = -0.569$

9. $\cos(x) = 0.117$

10. $\sin(x) = 0.008$

11. $\cos(x) = \frac{359}{360}$

12. $\tan(x) = 117$

13. $\cot(x) = -12$

14. $\sec(x) = \frac{3}{2}$

15. $\csc(x) = -\frac{90}{17}$

16. $\tan(x) = -\sqrt{10}$

17. $\sin(x) = \frac{3}{8}$

18. $\cos(x) = -\frac{7}{16}$

19. $\tan(x) = 0.03$

20. $\sin(x) = 0.3502$

21. $\sin(x) = -0.721$

22. $\cos(x) = 0.9824$

23. $\cos(x) = -0.5637$

24. $\cot(x) = \frac{1}{117}$

25. $\tan(x) = -0.6109$

6.2 Solving Equations Involving a Single Trigonometric Function

Learning Objectives

In this section you will:

- **Write complete solutions to equations containing a single trigonometric function.**
- **Evaluate exact solutions in the interval $[0,2\pi)$.**

In this section, we continue solving some basic equations involving trigonometric functions. Below we summarize the techniques that were first introduced in **Section 2.3**. Note that we use the letter *u* as the argument of each circular function for generality.

Strategies for Solving Basic Equations Involving Trigonometric Functions

- To solve $\cos(u)=c$ or $\sin(u)=c$ for $-1\le c\le 1$, first solve for *u* in the interval $[0,2\pi)$ and add integer multiples of the period 2π. If $c<-1$ or $c>1$, there are no real solutions.
- To solve $\sec(u)=c$ or $\csc(u)=c$ for $c\le -1$ or $c\ge 1$, convert to cosine or sine, respectively, and solve as above. If $-1<c<1$, there are no real solutions.
- To solve $\tan(u)=c$ for any real number *c*, first solve for *u* in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and add integer multiples of the period π.
- To solve $\cot(u)=c$ for $c\ne 0$, convert to tangent and solve as above. If $c=0$, the solution to $\cot(u)=0$ is $u=\frac{\pi}{2}+\pi k$ for integers *k*.

Using the above guidelines, we can comfortably solve $\sin(x)=\frac{1}{2}$ and find the solution $x=\frac{\pi}{6}+2\pi k$ or $x=\frac{5\pi}{6}+2\pi k$ for integers *k*. How do we solve something like $\sin(3x)=\frac{1}{2}$? This equation has the form $\sin(u)=\frac{1}{2}$, so we know the solutions take the form $u=\frac{\pi}{6}+2\pi k$ or $u=\frac{5\pi}{6}+2\pi k$ for integers *k*. Then, since the argument of sine is 3*x*, we have $3x=\frac{\pi}{6}+2\pi k$ or $3x=\frac{5\pi}{6}+2\pi k$ for integers *k*. To solve for x, we divide both sides[1] of the equations by 3, and obtain our solution of $x=\dfrac{\pi}{18}+\dfrac{2\pi}{3}k$ or $x=\dfrac{5\pi}{18}+\dfrac{2\pi}{3}k$ for integers k.

In the remainder of this section, we look at examples of equations which contain a single trigonometric function. The solutions provide practice with, and extensions of, the technique applied in solving $\sin(3x)=\dfrac{1}{2}$. We will add to the general solution of each equation specific solutions that fall in the interval $\left[0,2\pi\right)$.

Equations Involving Cosines or Sines

Example 6.2.1. Solve the equation $\cos(2x)=-\dfrac{\sqrt{3}}{2}$, giving the exact solutions which lie in $\left[0,2\pi\right)$.

Solution. The solutions to $\cos(u)=-\dfrac{\sqrt{3}}{2}$ are $u=\dfrac{5\pi}{6}+2\pi k$ or $u=\dfrac{7\pi}{6}+2\pi k$ for integers k. Since the argument of cosine here is $2x$, this means $2x=\dfrac{5\pi}{6}+2\pi k$ or $2x=\dfrac{7\pi}{6}+2\pi k$ for integers k. Solving for x gives $x=\dfrac{5\pi}{12}+\pi k$ or $x=\dfrac{7\pi}{12}+\pi k$ for integers k.

To determine which of our solutions lie in $\left[0,2\pi\right)$, we substitute the integer values $k=0,\pm 1,\pm 2,\cdots$ into $x=\dfrac{5\pi}{12}+\pi k$ and $x=\dfrac{7\pi}{12}+\pi k$:

k	…	–2	–1	0	1	2	…
$x=\dfrac{5\pi}{12}+\pi k$	$\cdots$	$-\dfrac{19\pi}{12}$	$-\dfrac{7\pi}{12}$	$\dfrac{5\pi}{12}$	$\dfrac{17\pi}{12}$	$\dfrac{29\pi}{12}$	$\cdots$
$x=\dfrac{7\pi}{12}+\pi k$	$\cdots$	$-\dfrac{17\pi}{12}$	$-\dfrac{5\pi}{12}$	$\dfrac{7\pi}{12}$	$\dfrac{19\pi}{12}$	$\dfrac{31\pi}{12}$	$\cdots$

[1] Don't forget to divide the $2\pi k$ by 3 as well!

The solutions in the interval $[0, 2\pi)$ correspond to $k = 0$ and $k = 1$. They are $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$ and $\frac{19\pi}{12}$.

□

In the preceding example, the solutions $x = \frac{5\pi}{12} + \pi k$ and $x = \frac{7\pi}{12} + \pi k$ can be checked analytically by substituting them into the left hand side of the original equation, $\cos(2x) = -\frac{\sqrt{3}}{2}$.

- Starting with $x = \frac{5\pi}{12} + \pi k$, we have

$$\begin{aligned} \cos\left(2\left(\frac{5\pi}{12} + \pi k\right)\right) &= \cos\left(\frac{5\pi}{6} + 2\pi k\right) \\ &= \cos\left(\frac{5\pi}{6}\right) \qquad \textbf{since the period of cosine is } 2\pi \\ &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

- Similarly, we find, for $x = \frac{7\pi}{12} + \pi k$,

$$\begin{aligned} \cos\left(2\left(\frac{7\pi}{12} + \pi k\right)\right) &= \cos\left(\frac{7\pi}{6} + 2\pi k\right) \\ &= \cos\left(\frac{7\pi}{6}\right) \\ &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

This confirms the solution of $x = \frac{5\pi}{12} + \pi k$ or $x = \frac{7\pi}{12} + \pi k$ for integers *k*.

Equations Involving Tangents or Cotangents

Next, we look at an example of a cotangent function.

Example 6.2.2. Solve $\cot(3x) = 0$, giving the exact solutions which lie in $[0, 2\pi)$.

Solution. Since $\cot(3x)=0$ has the form $\cot(u)=0$, we know $u=\frac{\pi}{2}+\pi k$. So, in this case,

$3x=\frac{\pi}{2}+\pi k$ for integers *k*. Solving for *x* yields $x=\frac{\pi}{6}+\frac{\pi}{3}k$.

We next determine which of our solutions lie in $[0,2\pi)$.

k	**…**	**–1**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**…**
$x=\frac{\pi}{6}+\frac{\pi}{3}k$	…	$-\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	…

The solutions in $[0,2\pi)$ are $x=\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{3\pi}{2}$ and $\frac{11\pi}{6}$, corresponding to $k=0$ through $k=5$.

□

To check the solution of $x=\frac{\pi}{6}+\frac{\pi}{3}k$, we start with the left side of $\cot(3x)=0$.

$$\begin{aligned}\cot\left(3\left(\frac{\pi}{6}+\frac{\pi}{3}k\right)\right)&=\cot\left(\frac{\pi}{2}+\pi k\right)\\&=\cot\left(\frac{\pi}{2}\right)\qquad\text{since the period of cotangent is }\pi\\&=0.\end{aligned}$$

This confirms our solution of $x=\frac{\pi}{6}+\frac{\pi}{3}k$ for integers *k*.

Equations Involving Secants or Cosecants

We look next at an equation involving a cosecant function which we will solve through conversion to its reciprocal, a sine function.

Example 6.2.3. Solve $\csc\left(\frac{1}{3}x-\pi\right)=\sqrt{2}$, giving the exact solutions which lie in $[0,2\pi)$.

Solution. Noting that this equation has the form $\csc(u)=\sqrt{2}$, we rewrite it as $\sin(u)=\frac{\sqrt{2}}{2}$ and find $u=\frac{\pi}{4}+2\pi k$ or $u=\frac{3\pi}{4}+2\pi k$ for integers *k*. Since the argument of cosecant here is $\left(\frac{1}{3}x-\pi\right)$, $\frac{1}{3}x-\pi=\frac{\pi}{4}+2\pi k$ or $\frac{1}{3}x-\pi=\frac{3\pi}{4}+2\pi k$.

We solve the first of these equations for *x*.

$$\frac{1}{3}x-\pi=\frac{\pi}{4}+2\pi k$$

$$\frac{1}{3}x=\frac{\pi}{4}+2\pi k+\pi \quad \text{add } \pi \text{ to both sides}$$

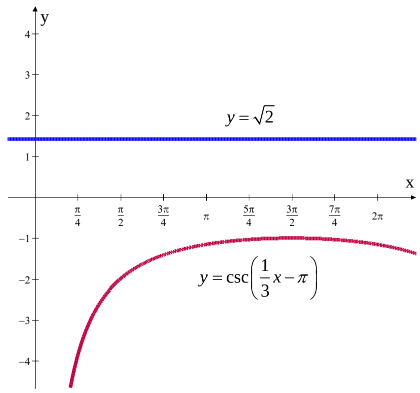
$$\frac{1}{3}x=\frac{5\pi}{4}+2\pi k \quad \text{combine the like terms } \frac{\pi}{4} \text{ and } \pi$$

$$x=3\left(\frac{5\pi}{4}+2\pi k\right) \quad \text{multiply both sides by 3}$$

$$x=\frac{15\pi}{4}+6\pi k$$

Solving the other equation, $\frac{1}{3}x-\pi=\frac{3\pi}{4}+2\pi k$, produces $x=\frac{21\pi}{4}+6\pi k$. Putting these two solutions together, we have $x=\frac{15\pi}{4}+6\pi k$ or $x=\frac{21\pi}{4}+6\pi k$ for integers *k*.

Despite the infinitely many solutions of $\csc\left(\frac{1}{3}x-\pi\right)=\sqrt{2}$, we find that *none* of them lie in $[0,2\pi)$. This can be verified graphically by plotting $y=\csc\left(\frac{1}{3}x-\pi\right)$ and $y=\sqrt{2}$, and observing that the two functions do not intersect at all over the interval $[0,2\pi)$.



□

The reader is encouraged to check the solutions of **Example 6.2.3** as we did following the first two examples in this section. We next solve an equation that at first glance does not fit the profile of equations thus far in this section.

Example 6.2.4. Solve $\sec^2(x)=4$. List the solutions which lie in the interval $[0,2\pi)$.

Solution. The complication in solving an equation like $\sec^2(x)=4$ comes not from the argument of secant, which is just *x*, but rather from the fact that secant is being squared. Thus, we begin by extracting square roots to get $\sec(x)=\pm 2$. Converting to cosines, we have $\cos(x)=\pm\frac{1}{2}$.

- For $\cos(x)=\frac{1}{2}$, we get $x=\frac{\pi}{3}+2\pi k$ or $x=\frac{5\pi}{3}+2\pi k$ for integers *k*.
- For $\cos(x)=-\frac{1}{2}$, we get $x=\frac{2\pi}{3}+2\pi k$ or $x=\frac{4\pi}{3}+2\pi k$ for integers *k*.

If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of $\frac{\pi}{3}$. As a result, these solutions can be combined and we may write our solutions as $x=\frac{\pi}{3}+\pi k$ and $x=\frac{2\pi}{3}+\pi k$ for integers *k*.

The solutions in the interval $[0,2\pi)$ come from the values $k=0$ and $k=1$ as indicated in the following table.

k	…	–1	0	1	2	…
$x=\frac{\pi}{3}+\pi k$	$\ldots$	$-\frac{2\pi}{3}$	$\frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{7\pi}{3}$	$\ldots$
$x=\frac{2\pi}{3}+\pi k$	$\ldots$	$-\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{3}$	$\frac{8\pi}{3}$	$\ldots$

The solutions in the interval $[0,2\pi)$ are $x=\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

□

Solutions that Include Inverse Trigonometric Functions

Our remaining two examples require inverse trigonometric functions in their solutions. Otherwise, the solution process is similar to that of the first four examples.

Example 6.2.5. Solve $\tan\left(\frac{x}{2}\right)=-3$ and determine solutions that lie in the interval $[0,2\pi)$.

Solution. The equation $\tan\left(\frac{x}{2}\right)=-3$ has the form $\tan(u)=-3$, whose solution is

$u=\arctan(-3)+\pi k$. Hence, $\frac{x}{2}=\arctan(-3)+\pi k$, so $x=2\arctan(-3)+2\pi k$ for integers *k*.

To determine which of our answers lie in the interval $[0,2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator,[2] we proceed analytically.

Since $-3<0$, it follows that $-\frac{\pi}{2}<\arctan(-3)<0$. Multiplying through by 2 gives $-\pi<2\arctan(-3)<0$. We are now in a position to argue which of the solutions $x=2\arctan(-3)+2\pi k$ lie in $[0,2\pi)$.

- For $k=0$, we get $x=2\arctan(-3)<0$, so we discard this answer and all answers $x=2\arctan(-3)+2\pi k$ where $k<0$.

[2] Your instructor will let you know if you should abandon the analytic route at this point and use your calculator. But seriously, what fun would that be?

- Next, we turn our attention to $k=1$ and get $x=2\arctan(-3)+2\pi$. Starting with the inequality $-\pi<2\arctan(-3)<0$, we add 2π and get $\pi<2\arctan(-3)+2\pi<2\pi$. This means $x=2\arctan(-3)+2\pi$ lies in $[0,2\pi)$.
- Advancing k to 2 produces $x=2\arctan(-3)+4\pi$. Once again, we get from $-\pi<2\arctan(-3)<0$ that $3\pi<2\arctan(-3)+4\pi<4\pi$. Since this is outside the interval $[0,2\pi)$, we discard $x=2\arctan(-3)+4\pi$ and all solutions of the form $x=2\arctan(-3)+2\pi k$ for $k>2$.

Thus, the only solution of $\tan\left(\frac{x}{2}\right)=-3$ in the interval $[0,2\pi)$ is $x=2\arctan(-3)+2\pi\approx 3.7851$.

□

A similar process determines the solutions of the following equation involving a sine function.

Example 6.2.6. Solve $\sin(2x)=0.87$ and find solutions in the interval $[0,2\pi)$.

Solution. To solve $\sin(2x)=0.87$, we first note that the equation has the form $\sin(u)=0.87$, which has the family of solutions $u=\arcsin(0.87)+2\pi k$ or $u=\pi-\arcsin(0.87)+2\pi k$ for integers k. Since the argument of sine here is $2x$, we get

$$2x=\arcsin(0.87)+2\pi k \text{ or } 2x=\pi-\arcsin(0.87)+2\pi k$$

which gives

$$x=\frac{1}{2}\arcsin(0.87)+\pi k \text{ or } x=\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)+\pi k \text{ for integers } k.$$

To determine which of these solutions lie in $[0,2\pi)$, we first need to get an idea of the value of $x=\frac{1}{2}\arcsin(0.87)$. Once again, we could use a calculator but we adopt an analytic route here.

$$0<\arcsin(0.87)<\frac{\pi}{2} \quad \textbf{by definition}$$

$$0<\frac{1}{2}\arcsin(0.87)<\frac{\pi}{4} \quad \textbf{after multiplying through by } \frac{1}{2}$$

- Starting with the family of solutions $x=\frac{1}{2}\arcsin\left(0.87\right)+\pi k$, we use the same kind of arguments as in our solution to **Example 6.2.5** and find only the solutions corresponding to $k=0$ and $k=1$ lie in $\left[0,2\pi\right)$: $x=\frac{1}{2}\arcsin(0.87)$ and $x=\frac{1}{2}\arcsin(0.87)+\pi$.

- Next, we move to the family $x=\frac{\pi}{2}-\frac{1}{2}\arcsin\left(0.87\right)+\pi k$. Here we need to get a better estimate of $\frac{\pi}{2}-\frac{1}{2}\arcsin\left(0.87\right)$.

$$
\begin{aligned}
&0<\frac{1}{2}\arcsin(0.87)<\frac{\pi}{4} && \textbf{from above}\\
&0>-\frac{1}{2}\arcsin(0.87)>-\frac{\pi}{4} && \textbf{multiply through by - 1}\\
&\frac{\pi}{2}>\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)>\frac{\pi}{4} && \textbf{add } \frac{\pi}{2}\\
&\frac{\pi}{4}<\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)<\frac{\pi}{2}
\end{aligned}
$$

Proceeding with the usual arguments, we find the only solutions which lie in $\left[0,2\pi\right)$ correspond to $k=0$ and $k=1$, namely $x=\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)$ and $x=\frac{3\pi}{2}-\frac{1}{2}\arcsin(0.87)$.

All told, we have found four solutions to $\sin(2x)=0.87$ in $\left[0,2\pi\right)$: $x=\frac{1}{2}\arcsin(0.87)$, $x=\frac{1}{2}\arcsin(0.87)+\pi$, $x=\frac{\pi}{2}-\frac{1}{2}\arcsin(0.87)$ and $x=\frac{3\pi}{2}-\frac{1}{2}\arcsin(0.87)$.

□

We will continue solving equations containing trigonometric functions in **Section 6.3**, with the added complexity of multiple trigonometric functions and/or arguments.

6.2 Exercises

In Exercises 1 –18, find the exact solutions of the equation and then list those solutions which are in the interval $\left[0, 2\pi\right)$.

1. $\sin(5x) = 0$
2. $\cos(3x) = \frac{1}{2}$
3. $\sin(-2x) = \frac{\sqrt{3}}{2}$
4. $\tan(6x) = 1$
5. $\csc(4x) = -1$
6. $\sec(3x) = \sqrt{2}$
7. $\cot(2x) = -\frac{\sqrt{3}}{3}$
8. $\cos(9x) = 9$
9. $\sin\left(\frac{x}{3}\right) = \frac{\sqrt{2}}{2}$
10. $\cos\left(x + \frac{5\pi}{6}\right) = 0$
11. $\sin\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$
12. $2\cos\left(x + \frac{7\pi}{4}\right) = -\frac{1}{2}$
13. $\csc(x) = 0$
14. $\tan(2x - \pi) = 1$
15. $\tan^2(x) = 3$
16. $\sec^2(x) = \frac{4}{3}$
17. $\cos^2(x) = \frac{1}{2}$
18. $\sin^2(x) = \frac{3}{4}$

In Exercises 19 – 26, solve the equation.

19. $\arccos(2x) = \pi$
20. $\pi - 2\arcsin(x) = 2\pi$
21. $4\arctan(3x - 1) - \pi = 0$
22. $6\operatorname{arccot}(2x) - 5\pi = 0$
23. $4\operatorname{arcsec}\left(\frac{x}{2}\right) = \pi$
24. $12\operatorname{arccsc}\left(\frac{x}{3}\right) = 2\pi$
25. $9\arcsin^2(x) - \pi^2 = 0$
26. $9\arccos^2(x) - \pi^2 = 0$

27. With the help of your classmates, determine the number of solutions to $\sin(x) = \frac{1}{2}$ in $\left[0, 2\pi\right)$. Then find the number of solutions to $\sin(2x) = \frac{1}{2}$, $\sin(3x) = \frac{1}{2}$ and $\sin(4x) = \frac{1}{2}$ in $\left[0, 2\pi\right)$. A pattern should emerge. Explain how this pattern would help you solve equations like $\sin(11x) = \frac{1}{2}$. Now consider $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ and $\sin\left(\frac{5x}{2}\right) = \frac{1}{2}$. What do you find? Replace $\frac{1}{2}$ with –1 and repeat the whole exploration.

6.3 Solving Equations of Multiple Trigonometric Functions/Arguments

Learning Objectives

In this section you will:

- **Solve equations containing multiple trigonometric functions and/or arguments.**
- **Evaluate exact solutions in the interval [0,2π).**

Each of the examples in **Section 6.2** featured one trigonometric function. If an equation involves two different trigonometric functions, or if the equation contains the same trigonometric function but with different arguments, we will need to use identities and algebra to reduce the equation to a form similar to the equations in **Section 6.2**.

Equations with Different Powers of the Same Function

Example 6.3.1. Solve the equation $3\sin^3(x)=\sin^2(x)$ and list the solutions which lie in the interval $[0,2\pi)$.

Solution. We resist the temptation to divide both sides of $3\sin^3(x)=\sin^2(x)$ by $\sin^2(x)$ (What goes wrong if you do?) and instead gather all of the terms to one side of the equation and factor.

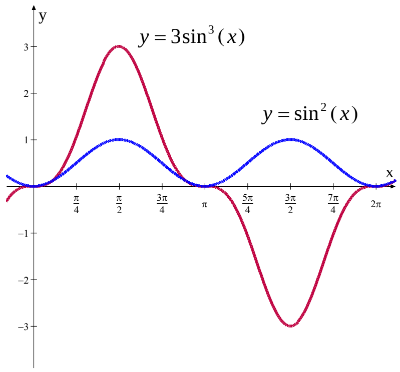
$$\begin{aligned} 3\sin^3(x) &= \sin^2(x) \\ 3\sin^3(x)-\sin^2(x) &= 0 \\ \sin^2(x)\left(3\sin(x)-1\right) &= 0 \end{aligned}$$

Factor out $\sin^2(x)$ from both terms.

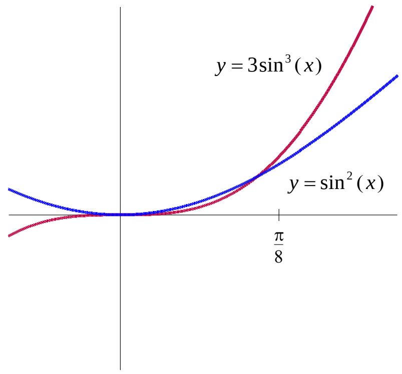
We get $\sin^2(x)=0$ or $3\sin(x)-1=0$. Solving for $\sin(x)$, we find $\sin(x)=0$ or $\sin(x)=\frac{1}{3}$.

- The solution to $\sin(x)=0$ is $x=\pi k$, with $x=0$ and $x=\pi$ being two solutions which lie in $[0,2\pi)$.
- To solve $\sin(x)=\frac{1}{3}$, we use the arcsine function to get $x=\arcsin\left(\frac{1}{3}\right)+2\pi k$ or $x=\pi-\arcsin\left(\frac{1}{3}\right)+2\pi k$ for integers *k*. We find the two solutions here which lie in $[0,2\pi)$ to be $x=\arcsin\left(\frac{1}{3}\right)$ and $x=\pi-\arcsin\left(\frac{1}{3}\right)$.

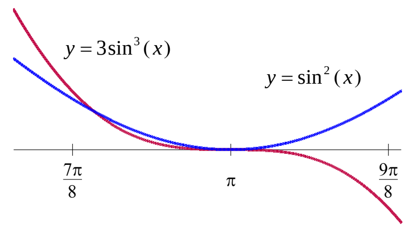
We can visualize the solutions by graphing $y = 3\sin^3(x)$ and $y = \sin^2(x)$. It may be necessary to 'zoom in' close to $x = 0$ and $x = \pi$ to verify that the graphs do in fact intersect four times.



interval $[0, 2\pi)$



close to 0



close to π

To summarize, the solutions to $3\sin^3(x) = \sin^2(x)$ in the interval $[0, 2\pi)$ are $x = 0$, $x = \arcsin\left(\frac{1}{3}\right)$, $x = \pi - \arcsin\left(\frac{1}{3}\right)$ and $x = \pi$.[1]

□

Equations Containing Multiple Trigonometric Functions

In the next example, we make use of a Pythagorean identity.

Example 6.3.2. Solve the equation $\sec^2(x) = \tan(x) + 3$ and list the solutions in $[0, 2\pi)$.

Solution. Analysis of $\sec^2(x) = \tan(x) + 3$ reveals two different trigonometric functions, so an identity is in order. Since $\sec^2(x) = 1 + \tan^2(x)$, we have

---

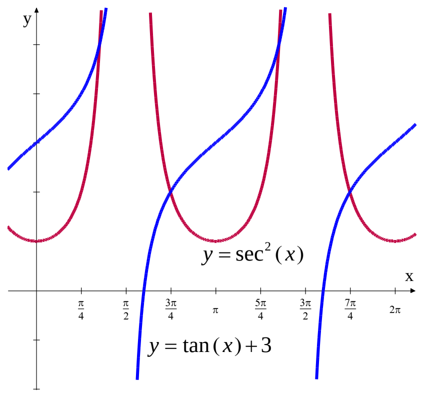
[1] Note that we are *not* counting $x = 2\pi$ as a solution since it is not in the interval $[0, 2\pi)$. In the forthcoming solutions, remember that while $x = 2\pi$ may be a solution to the equation, it isn't counted among the solutions in $[0, 2\pi)$.

$$\sec^2(x) = \tan(x) + 3$$
$$1 + \tan^2(x) = \tan(x) + 3 \quad \textbf{since } \sec^2(x) = 1 + \tan^2(x)$$
$$\tan^2(x) - \tan(x) - 2 = 0$$
$$u^2 - u - 2 = 0 \quad \textbf{for } u = \tan(x)$$
$$(u+1)(u-2) = 0$$

This gives $u = -1$ or $u = 2$. Since $u = \tan(x)$, we have $\tan(x) = -1$ or $\tan(x) = 2$.

- From $\tan(x) = -1$, we get $x = -\frac{\pi}{4} + \pi k$ for integers *k*. The solutions which lie in $[0, 2\pi)$ are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.
- To solve $\tan(x) = 2$, we employ the arctangent function and get $x = \arctan(2) + \pi k$ for integers *k*. Using the same sort of argument we saw in **Example 6.2.5**, we get $x = \arctan(2)$ and $x = \pi + \arctan(2)$ as solutions in $[0, 2\pi)$.

The points of intersection on the following graph of $y = \sec^2(x)$ and $y = \tan(x) + 3$ correspond to the four solutions of $\sec^2(x) = \tan(x) + 3$ on the interval $[0, 2\pi)$.



As discussed above, these solutions include $x = \arctan(2)$, $x = \frac{3\pi}{4}$, $x = \pi + \arctan(2)$ and $x = \frac{7\pi}{4}$.

□

Equations Containing Multiple Arguments of the Same Function

Some trigonometric equations can be solved by treating them as quadratic equations. Before proceeding in this manner with the following example, we will need to apply a double angle identity.

Example 6.3.3. Solve the equation $\cos(2x) = 3\cos(x) - 2$.

Solution. In the equation $\cos(2x) = 3\cos(x) - 2$ we have the same trigonometric function, namely cosine, on both sides, but the arguments differ. Using the identity $\cos(2x) = 2\cos^2(x) - 1$, we obtain an equation quadratic in form and can proceed as we have done in the past.

$$\begin{aligned}
\cos(2x) &= 3\cos(x) - 2 \\
2\cos^2(x) - 1 &= 3\cos(x) - 2 \quad \textbf{since } \cos(2x) = 2\cos(x) - 1 \\
2\cos^2(x) - 3\cos(x) + 1 &= 0 \\
2u^2 - 3u + 1 &= 0 \quad \textbf{for } u = \cos(x) \\
(2u - 1)(u - 1) &= 0
\end{aligned}$$

This gives $u = \frac{1}{2}$ or $u = 1$. Since $u = \cos(x)$, we have $\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$.

- Solving $\cos(x) = \frac{1}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$ for integers *k*.
- From $\cos(x) = 1$, we get $x = 2\pi k$ for integers *k*.

The answers which lie in $[0, 2\pi)$ are $x = 0$, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Try graphing $y = \cos(2x)$ and $y = 3\cos(x) - 2$ to verify that the curves intersect in three places on $[0, 2\pi)$, and that the *x*-coordinates of these points confirm our results.

□

Next, we look at an example that uses a technique similar to **Example 6.3.3**, but relies on an identity established in **Example 4.5.1**.

Example 6.3.4. Solve the equation $\cos(3x) = 13\cos(x)$.

Solution. From **Example 4.5.1**, we know that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$. This transforms our equation into a polynomial in terms of $\cos(x)$.

$$\begin{aligned}
\cos(3x) &= 13\cos(x) \\
4\cos^3(x) - 3\cos(x) &= 13\cos(x) \\
4\cos^3(x) - 16\cos(x) &= 0 \\
4u^3 - 16u &= 0 \qquad \textbf{for } u = \cos(x) \\
4u\left(u^2 - 4\right) &= 0 \\
4u\left(u-2\right)\left(u+2\right) &= 0
\end{aligned}$$

We get $u = 0$, $u = 2$ and $u = -2$. Since $u = \cos(x)$, our solutions would result from $\cos(x) = 0$, $\cos(x) = 2$ or $\cos(x) = -2$. Both 2 and –2 are outside the range of cosine. Thus, the only real solution to $\cos(3x) = 13\cos(x)$ is $\cos(x) = 0$, so that $x = \frac{\pi}{2} + \pi k$ for integers *k*. The only solutions which lie in $\left[0, 2\pi\right)$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

□

Equations Containing Different Functions and Different Arguments

The following example includes different trigonometric functions and different arguments.

Example 6.3.5. Solve $\sin(2x) = \sqrt{3}\cos(x)$ for *x*.

Solution. In examining the equation $\sin(2x) = \sqrt{3}\cos(x)$, not only do we have different functions involved, namely sine and cosine, we also have different arguments to contend with, namely 2*x* and *x*. Using the identity $\sin(2x) = 2\sin(x)\cos(x)$ makes all of the arguments the same and we proceed as we would solving any nonlinear equation, We gather all of the nonzero terms on one side of the equation and factor.

$$\begin{aligned}
\sin(2x) &= \sqrt{3}\cos(x) \\
2\sin(x)\cos(x) &= \sqrt{3}\cos(x) \\
2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 \\
\cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0
\end{aligned}$$

Then $\cos(x)=0$ or $\sin(x)=\frac{\sqrt{3}}{2}$. From $\cos(x)=0$, we obtain $x=\frac{\pi}{2}+\pi k$ for integers k. From $\sin(x)=\frac{\sqrt{3}}{2}$, we get $x=\frac{\pi}{3}+2\pi k$ or $x=\frac{2\pi}{3}+2\pi k$ for integers k.

The answers which lie in $[0,2\pi)$ are $x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

□

The last example in this section, which also includes different functions and arguments, tests our memory a bit and introduces another solution technique.

Example 6.3.6. Solve the following equation for x: $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$.

Solution. Unlike the previous problem, there seems to be no quick way to get the circular functions or their argument to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$. If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for $\sin\left(x+\frac{x}{2}\right)$.

$$\begin{aligned} \sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)&=1 \\ \sin\left(x+\frac{x}{2}\right)&=1 \\ \sin\left(\frac{3}{2}x\right)&=1 \end{aligned}$$

Hence, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right)=1$. We proceed in solving for x.

$$\begin{aligned} \sin\left(\frac{3}{2}x\right)&=1 \\ \frac{3}{2}x&=\frac{\pi}{2}+2\pi k && \textbf{since sine is equal to 1} \\ x&=\frac{\pi}{3}+\frac{4\pi}{3}k && \textbf{after multiplying both sides by } \frac{2}{3} \end{aligned}$$

The solution of $x = \frac{\pi}{3} + \frac{4\pi}{3}k$ for integers k has the following two solutions in the interval $\left[0, 2\pi\right)$:

$x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

□

When solving trigonometric equations, try something! Practice will help, but trying different solution techniques will improve your problem solving skills. After working through the examples in this section, spend some time with the problems in the Exercises. Try checking your solutions through viewing the intersection of graphs, as in **Example 6.3.1** and **Example 6.3.2**.

6.3 Exercises

In Exercises 1 –40, solve the equation, giving the exact solutions which lie in $[0, 2\pi)$.

1. $\sin(x) = \cos(x)$
2. $\sin(2x) = \sin(x)$
3. $\sin(2x) = \cos(x)$
4. $\cos(2x) = \sin(x)$
5. $\cos(2x) = \cos(x)$
6. $\cos(2x) = 2 - 5\cos(x)$
7. $3\cos(2x) + \cos(x) + 2 = 0$
8. $\cos(2x) = 5\sin(x) - 2$
9. $3\cos(2x) = \sin(x) + 2$
10. $2\sec^2(x) = 3 - \tan(x)$
11. $\tan^2(x) = 1 - \sec(x)$
12. $\cot^2(x) = 3\csc(x) - 3$
13. $\sec(x) = 2\csc(x)$
14. $\cos(x)\csc(x)\cot(x) = 6 - \cot^2(x)$
15. $\sin(2x) = \tan(x)$
16. $\cot^4(x) = 4\csc^2(x) - 7$
17. $\cos(2x) + \csc^2(x) = 0$
18. $\tan^3(x) = 3\tan(x)$
19. $\tan^2(x) = \frac{3}{2}\sec(x)$
20. $\cos^3(x) = -\cos(x)$
21. $\tan(2x) - 2\cos(x) = 0$
22. $\csc^3(x) + \csc^2(x) = 4\csc(x) + 4$
23. $2\tan(x) = 1 - \tan^2(x)$
24. $\tan(x) = \sec(x)$
25. $\sin(6x)\cos(x) = -\cos(6x)\sin(x)$
26. $\sin(3x)\cos(x) = \cos(3x)\sin(x)$
27. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$
28. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$
29. $\sin(x) + \cos(x) = 1$
30. $\sin(x) + \sqrt{3}\cos(x) = 1$
31. $\sqrt{2}\cos(x) - \sqrt{2}\sin(x) = 1$
32. $\sqrt{3}\sin(2x) + \cos(2x) = 1$
33. $\cos(2x) - \sqrt{3}\sin(2x) = \sqrt{2}$
34. $3\sqrt{3}\sin(3x) - 3\cos(3x) = 3\sqrt{3}$
35. $\cos(3x) = \cos(5x)$
36. $\cos(4x) = \cos(2x)$

37. $\sin(5x) = \sin(3x)$

38. $\cos(5x) = -\cos(2x)$

39. $\sin(6x) + \sin(x) = 0$

40. $\tan(x) = \cos(x)$

CHAPTER 7
BEYOND RIGHT TRIANGLES

Chapter Outline

7.1 Solving Triangles with the Law of Sines
7.2 Applications of the Law of Sines
7.3 The Law of Cosines

Introduction

Chapter 7 introduces some new tools that will help in solving obtuse triangles, and in solving real-life applications. In Section 7.1, triangles are defined as Angle-Side-Angle, Angle-Angle-Side and Side-Side-Angle. The Law of Sines is introduced to assist with solving triangles of these types. Section 7.2 further promotes the Law of Sines as a tool in finding the area of a triangle and in applying the Law of Sines to various applications. In Section 7.3, the Law of Cosines is used to solve triangles of the types Side-Angle-Side and Side-Side-Side. Solutions to many applications are made possible through the Law of Cosines. Heron's Formula, which is proved through the Law of Cosines, provides a simple method for finding the area of a triangle in which the lengths of all three sides are known.

This is a critical chapter in developing aptitude for solving real-life trigonometric applications. It is followed by Chapter 8, in which we delve into the more theoretical, but still very application-oriented, polar coordinates and complex numbers.

7.1 Solving Triangles with the Law of Sines

Learning Objectives

In this section you will:

- **Use the Law of Sines to solve oblique triangles.**
- **Distinguish between ASA, AAS and ASS triangles.**
- **Determine the existence of, and values for, multiple solutions of oblique triangles.**
- **Determine when given criteria will not result in a triangle.**

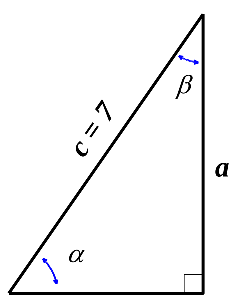
Trigonometry literally means 'measuring triangles' and with Chapters 1 – 6 under our belts, we are more than prepared to do just that. The main goal of this chapter is to develop theorems which allow us to solve triangles – that is, find the length of each side of a triangle and the measure of each of its angles.

Solving Right Triangles

We have had some experience solving right triangles. The following example reviews what we know.

Example 7.1.1. Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Express the angles in decimal degrees, rounded to the nearest hundredth of a degree.

Solution. For definitiveness, we label the triangle below.



- To find the length of the missing side a, we use the Pythagorean Theorem to get $a^2 + 4^2 = 7^2$, which then yields $a = \sqrt{33}$ units.

Now that all three sides of the triangle are known, there are several ways we can find α and β using the inverse trigonometric functions. To decrease the chances of propagating error, however, we stick to the data given to us in the problem. In this case, the lengths 4 and 7 were given.

- We want to relate the lengths 4 and 7 to *α*. Since $\cos(\alpha) = \frac{4}{7}$ and *α* is an acute angle,

  $\alpha = \arccos\left(\frac{4}{7}\right)$ radians. Converting to degrees, we find $\alpha \approx 55.15^\circ$.

- We see that $\sin(\beta) = \frac{4}{7}$, so $\beta = \arcsin\left(\frac{4}{7}\right)$ radians and we have $\beta \approx 34.85^\circ$.

Note that we could have used the measure of angle *α* to find the measure of angle *β*, using the fact that *α* and *β* are complements, from which $\alpha + \beta = 90^\circ$.

□

A few remarks about **Example 7.1.1** are in order.

1. First, we adhere to the convention that a lower case Greek letter denotes an angle[1] and the corresponding lowercase English letter represents the side[2] opposite that angle. Thus, *a* is the side opposite *α*, *b* is the side opposite *β* and *c* is the side opposite *γ*. Taken together, the pairs (α, a), (β, b) and (γ, c) are called *angle-side opposite pairs*.
2. Second, as mentioned earlier, we will strive to solve for quantities using the original data given in the problem whenever possible. While this is not always the easiest or fastest way to proceed, it minimizes the chances of propagated error.[3]
3. Third, since many of the applications which require solving triangles 'in the wild' rely on degree measure, we shall adopt this convention for the time being.[4]

The Law of Sines

The Pythagorean Theorem along with the definitions of the trigonometric functions in **Section 2.1** allow us to easily handle any given right triangle problem, but what if the triangle isn't a right triangle? Any triangle that is not a right triangle is an **oblique triangle**. In certain cases, we can use the Law of Sines to solve oblique triangles.

[1] as well as the measure of said angle
[2] as well as the length of said side
[3] Your Science teachers should thank us for this.
[4] Don't worry! Radians will be back before you know it!

Theorem 7.1. The Law of Sines: Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following ratios hold

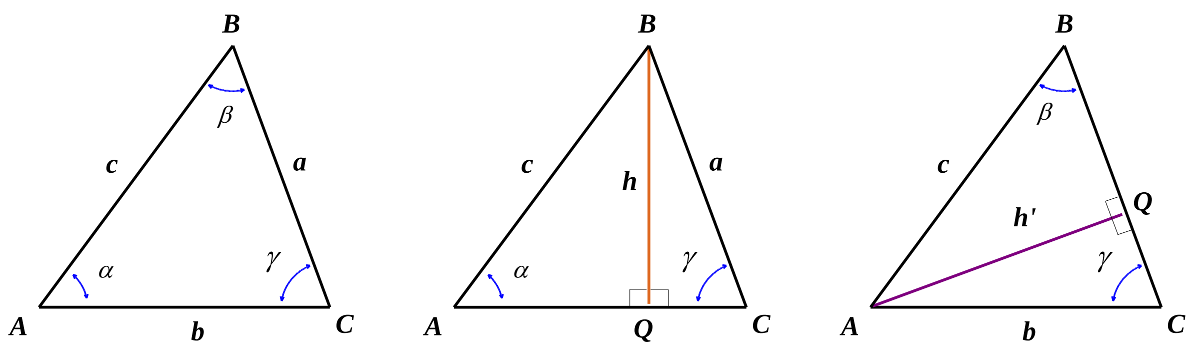
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Or, equivalently,

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

The proof of the Law of Sines can be broken into three cases.

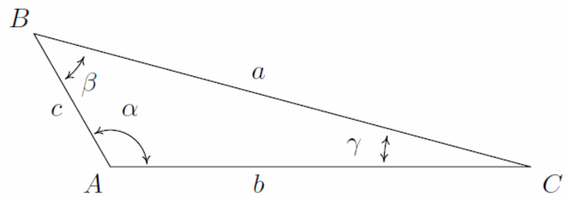
1. For our first case, consider the triangle $\triangle ABC$ below, all of whose angles are acute, with angle-side pairs (α, a), (β, b) and (γ, c).

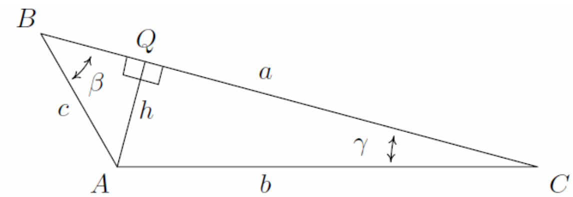


If we drop an altitude from vertex *B*, we divide the triangle into two right triangles: $\triangle ABQ$ and $\triangle BCQ$. If we call the length of the altitude *h* (for height), we get that $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$ so that $h = c\sin(\alpha) = a\sin(\gamma)$. After some rearrangement of the last equation, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

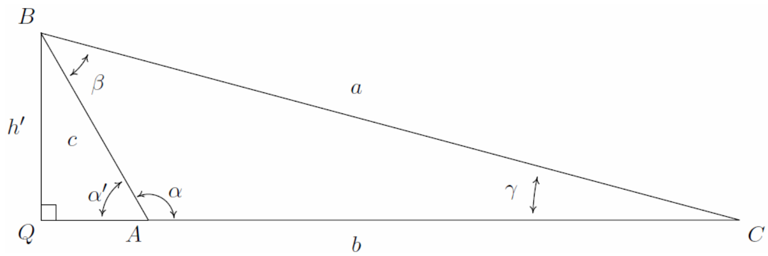
If we drop an altitude from vertex *A*, we can proceed as above using the triangles $\triangle ABQ$ and $\triangle ACQ$ to get $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$, completing the proof for this case.

2. For our next case, consider the triangle $\triangle ABC$ below with obtuse angle α. Extending an altitude from vertex A gives two right triangles, as in the previous case: $\triangle ABQ$ and $\triangle ACQ$. Proceeding as before, we get $h = b\sin(\gamma)$ and $h = c\sin(\beta)$ so that $\dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$.





Dropping an altitude from vertex B also generates two right triangles, $\triangle ABQ$ and $\triangle BCQ$.



We know that $\sin(\alpha') = \dfrac{h'}{c}$ so that $h' = c\sin(\alpha')$. Since $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so in fact we have $h' = c\sin(\alpha)$. Proceeding to $\triangle BCQ$, we get $\sin(\gamma) = \dfrac{h'}{a}$ so $h' = a\sin(\gamma)$. Putting this together with the previous equation, we get $\dfrac{\sin(\gamma)}{c} = \dfrac{\sin(\alpha)}{a}$, and we are finished with this case.

3. The remaining case is when $\triangle ABC$ is a right triangle. In this case, the definitions of trigonometric functions from **Section 2.1** can be used to verify the Law of Sines and this verification is left to the reader.

In order to use the Law of Sines to solve a triangle, we need at least three measurements of angles and/or sides, including at least one of the sides. Also, note that we need to be given, or be able to find, at least one angle-side opposite pair. We will investigate three possible oblique triangle problem situations.

AAS (Angle-Angle-Side)

Here, we know the measurements of two angles and a side that is not between the known angles.

Example 7.1.2. Solve the triangle: $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. Knowing an angle-side opposite pair, namely $\alpha = 120^\circ$ and $a = 7$, we may proceed in using the Law of Sines.

$$\frac{b}{\sin(45^\circ)} = \frac{7}{\sin(120^\circ)} \quad \textbf{since } \beta = 45^\circ$$

$$b = \frac{7\sin(45^\circ)}{\sin(120^\circ)}$$

$$b = \frac{7\left(\sqrt{2}/2\right)}{\sqrt{3}/2}$$

$$b = \frac{7\sqrt{2}}{\sqrt{3}} \approx 5.72 \text{ units}$$

Now that we have two angle-side pairs, it is time to find the third. To find γ, we use the fact that the sum of the measures of the angles in a triangle is 180°. Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$. To find c, we have no choice but to use the derived value $\gamma = 15^\circ$, yet we can minimize the propagation of error here by using the given angle-side opposite pair (α, a). The Law of Sines gives us

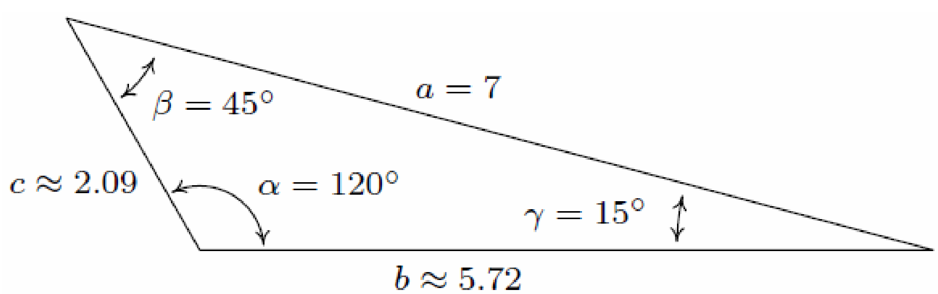
$$\frac{c}{\sin(15^\circ)} = \frac{7}{\sin(120^\circ)}$$

$$c = \frac{7\sin(15^\circ)}{\sin(120^\circ)}$$

$$c \approx 2.09 \text{ units}$$

The exact value of $\sin(15^\circ)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus 'exact' here means

$c = \dfrac{7\sin(15^\circ)}{\sin(120^\circ)}$.



□

ASA (Angle-Side-Angle)

In this case, we know the measurements of two angles and the included side.

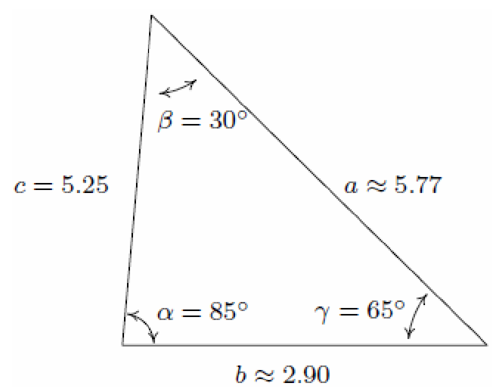
Example 7.1.3. Solve the triangle: $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. In this example, we are not immediately given an angle-side opposite pair, but as we have the measures of α and β, we can solve for γ since $\gamma = 180^\circ - 85^\circ - 30^\circ = 65^\circ$. As in the previous example, we are forced to use a derived value in our computations since the only angle-side pair available is (γ, c). The Law of Sines gives

$$\frac{a}{\sin(85^\circ)} = \frac{5.25}{\sin(65^\circ)}$$
$$a = \frac{5.25\sin(85^\circ)}{\sin(65^\circ)}$$
$$a \approx 5.77 \text{ units}$$

To find b we use the angle-side pair (γ, c) which yields

$$\frac{b}{\sin(30^\circ)} = \frac{5.25}{\sin(65^\circ)}$$
$$b = \frac{5.25\sin(30^\circ)}{\sin(65^\circ)}$$
$$b \approx 2.90 \text{ units}$$



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ASS (Angle-Side-Side)

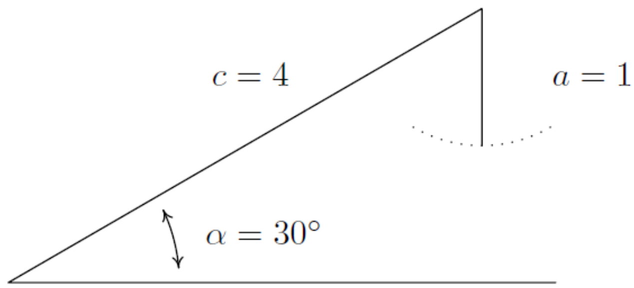
Knowing the measurement of two sides and an angle that is not between the known sides proves to be more complex than our first two scenarios. While we can use the Law of Sines to solve any oblique triangle, some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an **ambiguous case**. Triangles classified as ASS, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solutions.

Example 7.1.4. Solve the triangle: $\alpha = 30^\circ$, $a = 1$ unit, $c = 4$ units.

Solution. Since we are given (α, a) and c, we use the Law of Sines to find the measure of γ.

$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{1}$$
$$\sin(\gamma) = 4\sin(30^\circ)$$
$$\sin(\gamma) = 2$$

Since the range of the sine function is $[-1,1]$, there is no real number with $\sin(\gamma) = 2$. Geometrically, we see that side a is just too short to make a triangle.



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The following examples keep the same value for the measure of α and the length of c while varying the length of a. We will discuss the preceding case in more detail after we see what happens in the next three examples.

Example 7.1.5. Solve the triangle: $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units.

Solution. Using the Law of Sines, we get

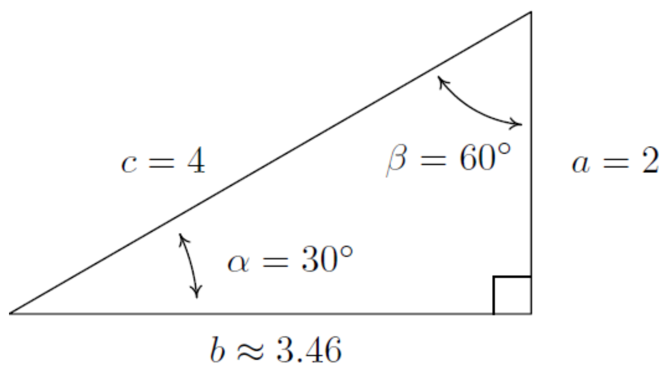
$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}$$
$$\sin(\gamma) = 2\sin(30^\circ)$$
$$\sin(\gamma) = 1$$

Now γ is an angle in a triangle which also contains the angle $\alpha = 30^\circ$. This means that γ must measure between 0° and 150° in order to fit inside the triangle with α. The only angle that satisfies this requirement and has $\sin(\gamma) = 1$ is $\gamma = 90^\circ$. In other words, we have a right triangle.

We find the measure of β to be $\beta = 180^\circ - 30^\circ - 90^\circ = 60^\circ$ and then determine b using the Law of Sines.

$$\frac{\sin(30^\circ)}{2} = \frac{\sin(60^\circ)}{b}$$
$$b = \frac{2\sin(60^\circ)}{\sin(30^\circ)}$$
$$b = \frac{2\left(\sqrt{3}/2\right)}{1/2}$$
$$b = 2\sqrt{3} \approx 3.46 \text{ units}$$

In this case, the side *a* is precisely long enough to form a unique right triangle.



□

Example 7.1.6. Solve the triangle: $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units.

Solution. Proceeding as we have in the previous two examples, we use the Law of Sines to find γ. In this case, we have

$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{3}$$
$$\sin(\gamma) = \frac{4\sin(30^\circ)}{3}$$
$$\sin(\gamma) = \frac{2}{3}$$

Since γ lies in a triangle with $\alpha = 30^\circ$, we must have $0^\circ < \gamma < 150^\circ$. There are two angles γ that fall in this range and have $\sin(\gamma) = \frac{2}{3}$: $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians, approximately 41.81°, and $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians, approximately 138.19°.

- In the case $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians ≈ 41.81°, we find[5] $\beta \approx 180^\circ - 30^\circ - 41.81^\circ = 108.19^\circ$.

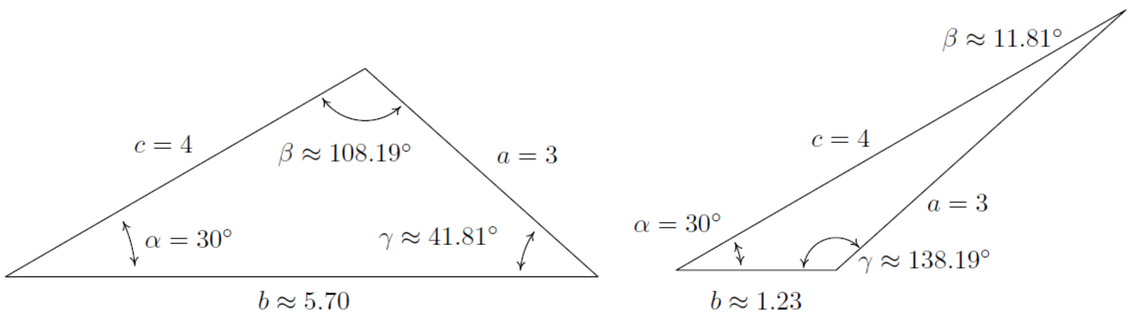
  Using the Law of Sines with the angle-side opposite pair (α, a) and β, we find

  $b \approx \frac{3\sin(108.19^\circ)}{\sin(30^\circ)} \approx 5.70$ units.

- In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians ≈ 138.19°, we repeat the exact same steps and find

  $\beta \approx 11.81°$ and $b \approx 1.23$ units.[6]

Both triangles are drawn below.



□

Example 7.1.7. Solve the triangle: $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units.

Solution. For this last problem, we repeat the usual Law of Sines routine to find that

$$\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$$
$$\sin(\gamma) = \sin(30^\circ)$$
$$\sin(\gamma) = \frac{1}{2}$$

---

[5] To find an exact expression for β, we convert everything back to radians: $\alpha = 30^\circ = \frac{\pi}{6}$ radians, $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians and $180^\circ = \pi$ radians. Hence, $\beta = \pi - \frac{\pi}{6} - \arcsin\left(\frac{2}{3}\right) = \frac{5\pi}{6} - \arcsin\left(\frac{2}{3}\right)$ radians ≈ 108.19°.

[6] An exact answer for β in this case is $\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{6}$ radians ≈ 11.81°.

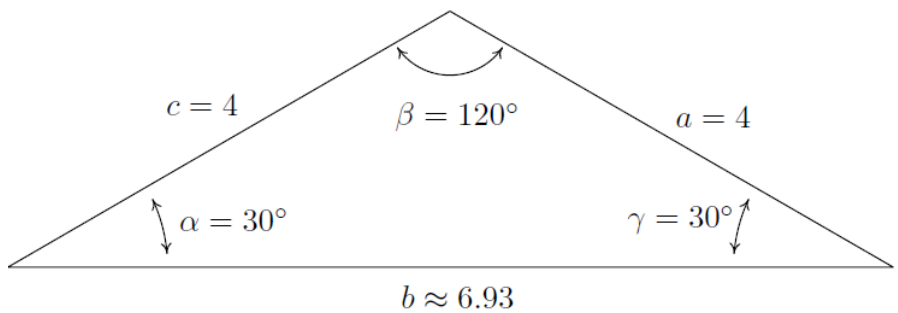
Then γ must inhabit a triangle with $\alpha = 30^\circ$, so we must have $0^\circ < \gamma < 150^\circ$. Since the measure of γ must be *strictly* less than 150°, there is just one angle which satisfies both required conditions, namely $\gamma = 30^\circ$. So $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$ and, using the Law of Sines one last time,

$$\frac{\sin(30^\circ)}{4} = \frac{\sin(120^\circ)}{b}$$

$$b = \frac{4\sin(120^\circ)}{\sin(30^\circ)}$$

$$b = \frac{4\left(\sqrt{3}/2\right)}{1/2}$$

$$b = 4\sqrt{3} \approx 6.93 \text{ units}$$



□

Some remarks are in order.

1. If we are given the measures of two of the angles in a triangle, say α and β, the measure of the third angle γ is uniquely determined using the equation $\gamma = 180^\circ - \alpha - \beta$. Knowing the measures of all three angles of a triangle completely determines its *shape*.
2. If, in addition to being given the measures of two angles, we are given the length of one of the sides of the triangle, we can then use the Law of Sines to find the lengths of the remaining two sides in order to determine the *size* of the triangle. This is true for the previously described AAS and ASA cases.
3. If we are given the measure of just one of the angles along with the lengths of two sides, only one of which is adjacent to the given angle, we have the ASS case.[7] As we saw in Examples 7.14 – 7.17, this information may describe one right triangle, one oblique triangle, two oblique triangles, or no triangle.

[7] In more reputable books, this is called the 'Side-Side-Angle' or SSA case.

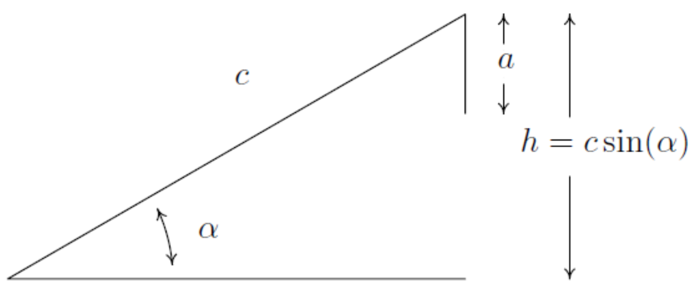
The four possibilities in the ASS case are summarized in the following theorem.

Theorem 7.2. Suppose (α, a) and (γ, c) are intended to be angle-side pairs in a triangle where α, a and c are given. Let $h = c\sin(\alpha)$.

- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists which satisfies the given criteria.

Theorem 7.2 is proved on a case-by-case basis.

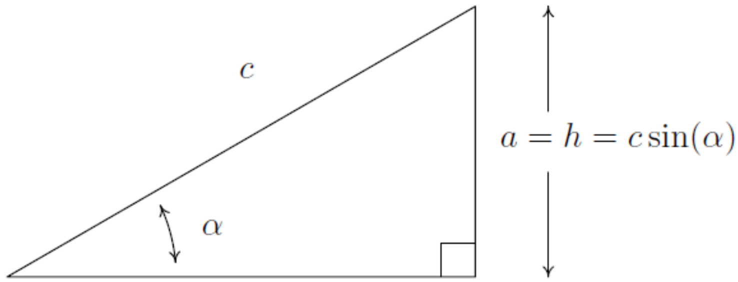
- If $a < h$ then $a < c\sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible. In the figure below, we see geometrically why this is the case.



$a < h$, no triangle

Simply put, if $a < h$ the side *a* is too short to connect to form a triangle. This means if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem tell us what kind and how many triangles to expect in each case.

- If $a = h$, then $a = c\sin(\alpha)$ and the Law of Sines gives $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$ so that $\sin(\gamma) = \frac{c\sin(\alpha)}{a} = \frac{a}{a} = 1$. Here, $\gamma = 90^\circ$ as required.



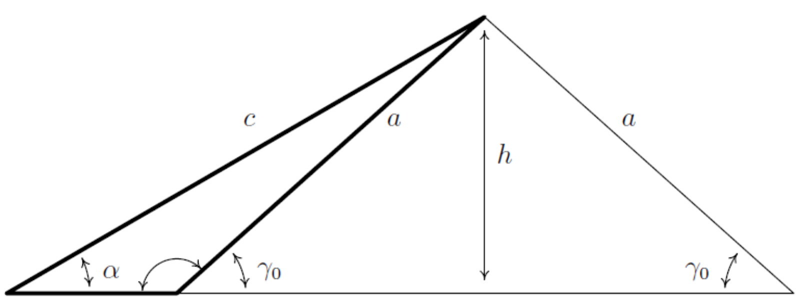
$a = h$, $\gamma = 90^\circ$

- Moving along, now suppose $h < a < c$. As before, the Law of Sines gives $\sin(\gamma) = \dfrac{c\sin(\alpha)}{a}$.

  Since $h < a$, $c\sin(\alpha) < a$ or $\dfrac{c\sin(\alpha)}{a} < 1$ which means there are two solutions to

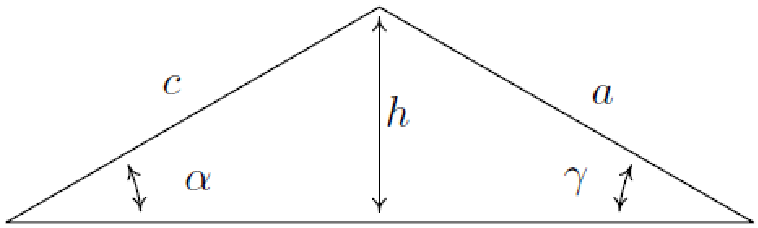
  $\sin(\gamma) = \dfrac{c\sin(\alpha)}{a}$: an acute angle which we'll call γ_0, and its supplement $180^\circ - \gamma_0$. We need to argue that each of these angles 'fit' into a triangle with α.

  - Since (α, a) and (γ_0, c) are angle-side opposite pairs, the assumption $c > a$ in this case gives $\gamma_0 > \alpha$. Since γ_0 is acute, we must have that α is acute as well. This means that one triangle can contain both α and γ_0, giving us one of the triangles promised in the theorem.
  - If we manipulate the inequality $\gamma_0 > \alpha$ a bit, we have $180^\circ - \gamma_0 < 180^\circ - \alpha$, which gives $(180^\circ - \gamma_0) + \alpha < 180^\circ$. This proves a triangle can contain both of the angles α and $(180^\circ - \gamma_0)$, giving us the second triangle predicted in the theorem.



$h < a < c$, two triangles

- To prove the last case in the theorem, we assume $a \geq c$. Then $\alpha \geq \gamma$, which forces γ to be an acute angle. Hence, we get only one triangle in this case, completing the proof.



$a \geq c$, one triangle

One last comment before we end this discussion. In the Angle-Side-Side case, if you are given an obtuse angle to begin with then it is impossible to have the two triangle case. We will next move on to **Section 7.2** where we use the Law of Sines to solve application problems.

7.1 Exercises

In Exercises 1 – 12, find the length of side *x*. Round to the nearest tenth.

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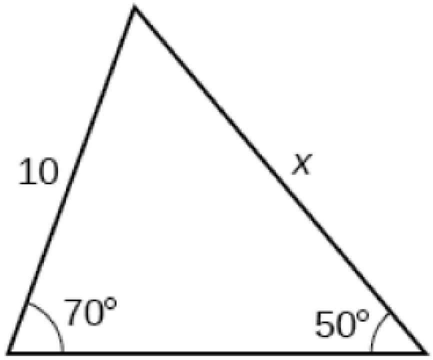

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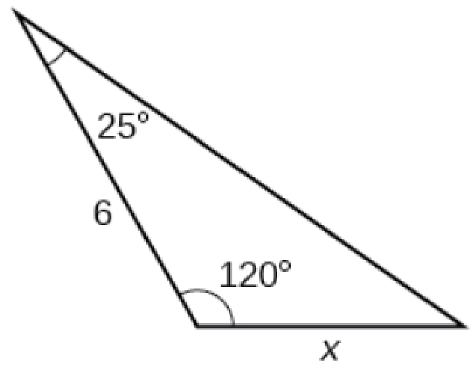

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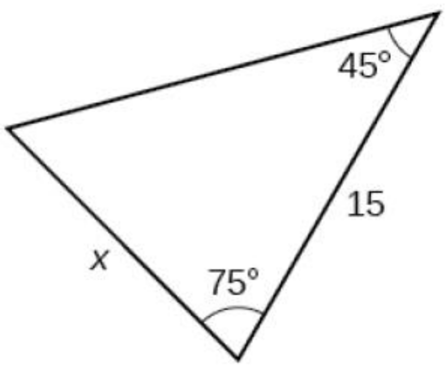

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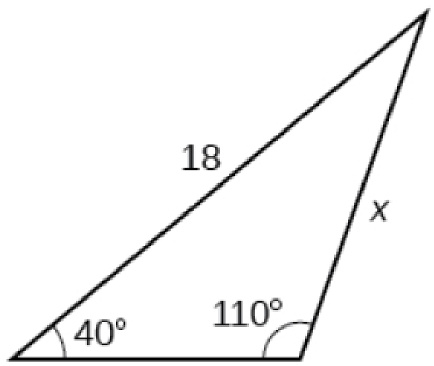

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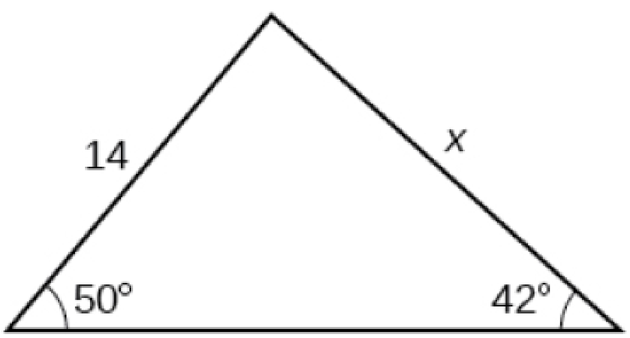

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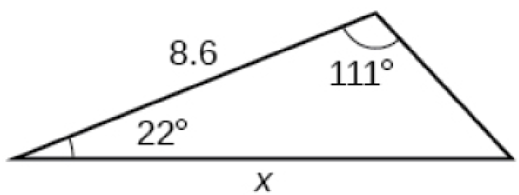

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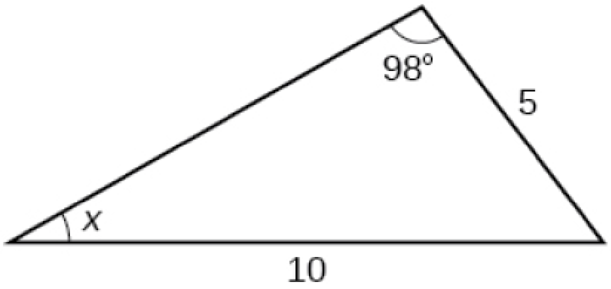

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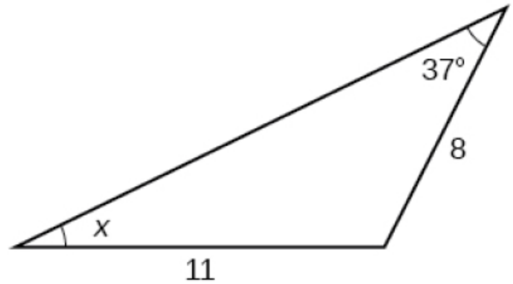

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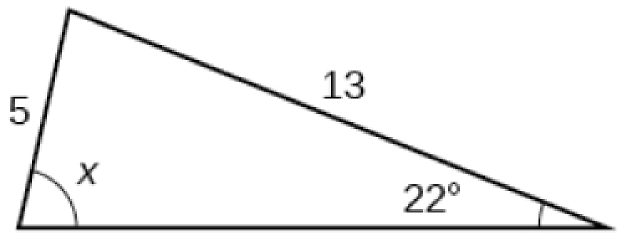

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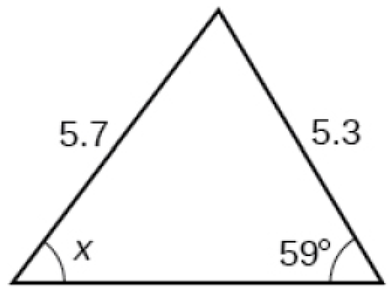
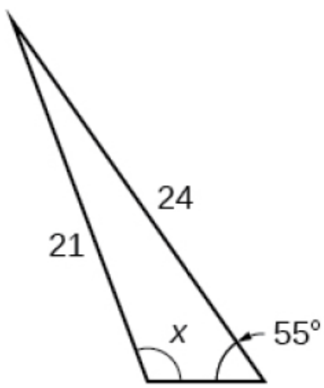
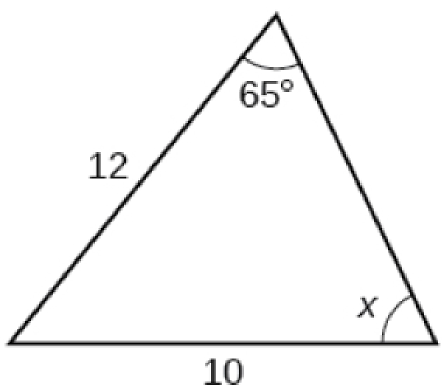


11. Notice that x is an obtuse angle.



12.



In Exercises 13 – 32, solve for the remaining side(s) and angle(s) if possible. As in the text, (α, a), (β, b) and (γ, c) are angle-side opposite pairs.

13. $\alpha = 13^\circ$, $\beta = 17^\circ$, $a = 5$

14. $\alpha = 73.2^\circ$, $\beta = 54.1^\circ$, $a = 117$

15. $\alpha = 95^\circ$, $\beta = 85^\circ$, $a = 33.33$

16. $\alpha = 95^\circ$, $\beta = 62^\circ$, $a = 33.33$

17. $\alpha = 117^\circ$, $a = 35$, $b = 42$

18. $\alpha = 117^\circ$, $a = 45$, $b = 42$

19. $\alpha = 68.7^\circ$, $a = 88$, $b = 92$

20. $\alpha = 42^\circ$, $a = 17$, $b = 23.5$

21. $\alpha = 68.7^\circ$, $a = 70$, $b = 90$

22. $\alpha = 30^\circ$, $a = 7$, $b = 14$

23. $\alpha = 42^\circ$, $a = 39$, $b = 23.5$

24. $\gamma = 53^\circ$, $\alpha = 53^\circ$, $c = 28.01$

25. $\alpha = 6^\circ$, $a = 57$, $b = 100$

26. $\gamma = 74.6^\circ$, $c = 3$, $a = 3.05$

27. $\beta = 102^\circ$, $b = 16.75$, $c = 13$

28. $\beta = 102^\circ$, $b = 16.75$, $c = 18$

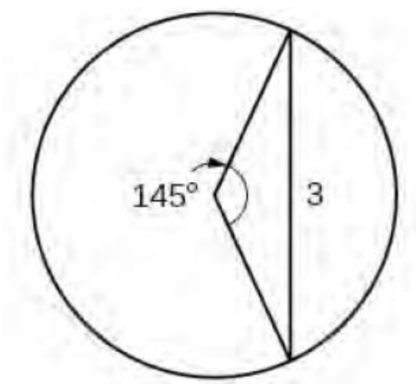
29. $\beta = 102^\circ$, $\gamma = 35^\circ$, $b = 16.75$

30. $\beta = 29.13^\circ$, $\gamma = 83.95^\circ$, $b = 314.15$

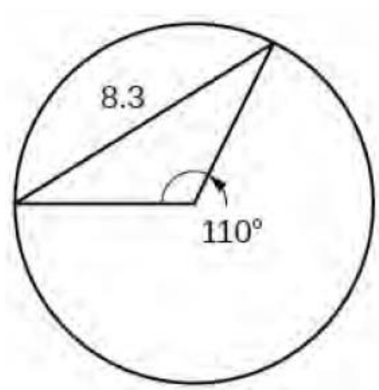
31. $\gamma = 120^\circ$, $\beta = 61^\circ$, $c = 4$

32. $\alpha = 50^\circ$, $a = 25$, $b = 12.5$

33. Find the radius of the circle. Round to the nearest tenth.



34. Find the diameter of the circle. Round to the nearest tenth.



35. Prove that the Law of Sines holds when $\triangle ABC$ is a right triangle.

36. Discuss with your classmates why knowing only the three angles of a triangle is not enough to determine any of the sides.

37. Discuss with your classmates why the Law of Sines cannot be used to find the angles in a triangle when only the three sides are given. Also discuss what happens if only two sides and the angle between them are given. (Said another way, explain why the Law of Sines cannot be used in the SSS and SAS cases.)

36. Given $\alpha = 30^{\circ}$ and $b = 10$, choose four different values for *a* so that

a) the information yields no triangle

b) the information yields exactly one right triangle

c) the information yields two distinct triangles

d) the information yields exactly one obtuse triangle

Explain why you cannot choose *a* in such a way as to have $\alpha = 30^{\circ}$, $b = 10$, and your choice of *a* yield only one triangle where that unique triangle has three acute angles.

7.2 Applications of the Law of Sines

Learning Objectives

In this section you will:

- **Find the area of an oblique triangle using the sine function.**
- **Solve applied problems using the Law of Sines.**

Following our practice with solving triangles for missing values in **Section 7.1**, we begin **Section 7.2** by using some of those values to find the area of an oblique triangle.

Finding the Area of an Oblique Triangle

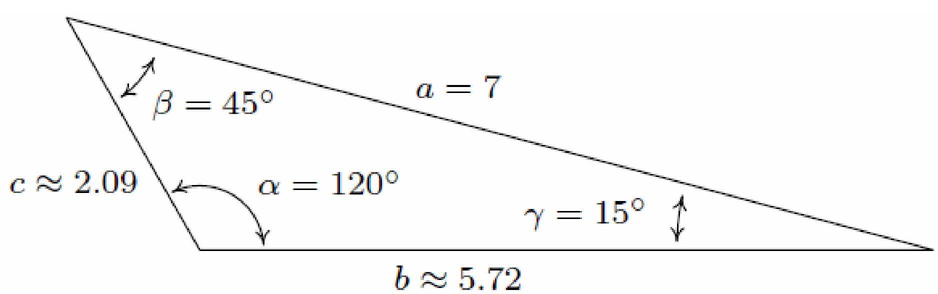
The following theorem introduces a new formula to compute the area enclosed by a triangle. Its proof uses the same cases and diagrams as the proof of the Law of Sines and is left as an exercise.

Theorem 7.3. Suppose (α, a), (β, b) and (γ, c) are the angle-side opposite pairs of a triangle. Then the area enclosed by the triangle is given by

$$A = \frac{1}{2}bc\sin(\alpha) = \frac{1}{2}ac\sin(\beta) = \frac{1}{2}ab\sin(\gamma)$$

Example 7.2.1. Find the area of the triangle in which $\alpha = 120^\circ$, $a = 7$ units, and $\beta = 45^\circ$.

Solution. This is the triangle from **Example 7.1.2** in which we found all three angles and all three sides.



To minimize propagated error, we choose $A = \frac{1}{2}ac\sin(\beta)$, from **Theorem 7.3**, because it uses the most pieces of given information.

We are given $a = 7$ and $\beta = 45^\circ$, and we calculated $c = \frac{7\sin(15^\circ)}{\sin(120^\circ)}$ in **Example 7.1.2**. Using these values, we find

$$A = \frac{1}{2}(7)\left(\frac{7\sin(15^\circ)}{\sin(120^\circ)}\right)\sin(45^\circ)$$
$$\approx 5.18 \text{ square units}$$

The reader is encouraged to check this answer against the results obtained using the other formulas in **Theorem 7.3**.

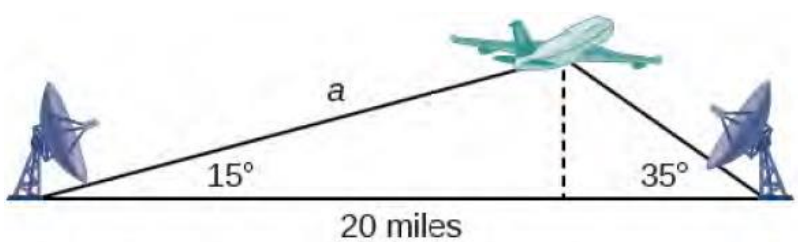
□

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering and physics involve three dimensions and motion.

Example 7.2.2. Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 15 degrees, whereas the angle of elevation measured by the second station is 35 degrees. Find the altitude of the aircraft and round your answer to the nearest tenth of a mile.

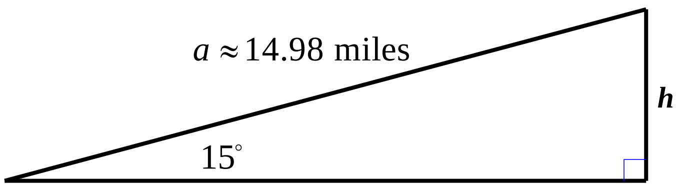
Solution. To find the altitude, or height, of the aircraft, we first sketch a triangle which reflects the information given to us in the problem. We then use the triangle to determine the distance from one station to the aircraft.



Letting *a* represent the distance from the first station to the aircraft, we look for an angle-side opposite pair from which we can determine the distance *a*. We know the measure of two angles in the triangle, but the measure of the angle opposite the side of length 20 miles is missing. Noting that the angles in a triangle add up to 180 degrees, we find the unknown angle measure to be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This gives us an angle-side opposite pair with known values and allows us to set up a Law of Sines relationship.

$$\frac{\sin(130^\circ)}{20} = \frac{\sin(35^\circ)}{a}$$
$$a\sin(130^\circ) = 20\sin(35^\circ)$$
$$a = \frac{20\sin(35^\circ)}{\sin(130^\circ)}$$
$$a \approx 14.98$$

The distance a, from the first station to the aircraft, is about 14.98 miles. Now that we know a, we can use right triangle relationships to solve for the height, h, of the aircraft.



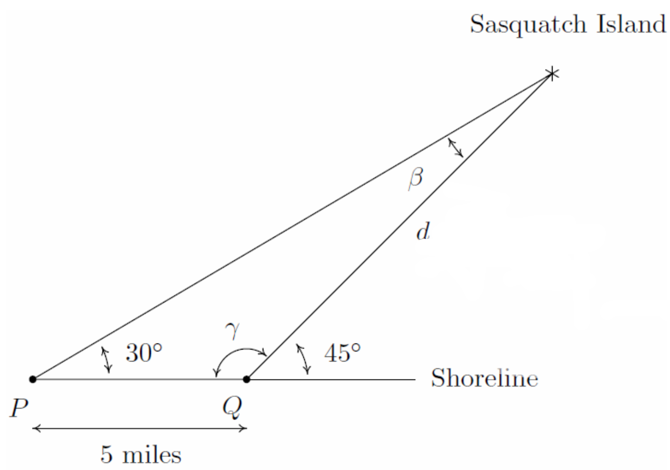
$$\sin(15^\circ) = \frac{h}{a}$$
$$h = a\sin(15^\circ)$$
$$h \approx 14.98\sin(15^\circ)$$
$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

□

Example 7.2.3. Sasquatch Island lies off the coast of Ippizuti Lake. Two sightings, taken 5 miles apart, are made to the island. The angle between the shore and the island at the first observation point is 30° and at the second point is 45°. Assuming a straight coastline, find the distance from the second observation to the island. What point on the shore is closest to the island? How far is the island from this point?

Solution. We sketch the problem below with the first observation point labeled as P and the second as Q.



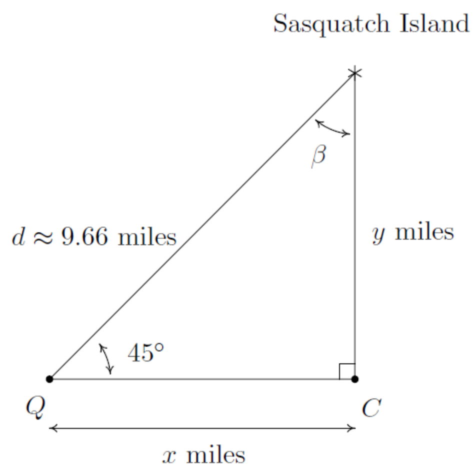
In order to use the Law of Sines to find the distance d from Q to the island, we first need to find the measure of β which is the angle opposite the side of length 5 miles. To that end, we note that the angles γ and 45° are supplemental, so that $\gamma = 180^\circ - 45^\circ = 135^\circ$. We can now find β.

$$\begin{aligned}\beta &= 180^\circ - 30^\circ - \gamma \\ &= 180^\circ - 30^\circ - 135^\circ \\ &= 15^\circ\end{aligned}$$

By the Law of Sines, we have

$$\begin{aligned}\frac{d}{\sin(30^\circ)} &= \frac{5}{\sin(15^\circ)} \\ d &= \frac{5\sin(30^\circ)}{\sin(15^\circ)} \\ d &\approx 9.66 \text{ miles}\end{aligned}$$

Next, to find the point on the coast closest to the island, which we've labeled as C, we need to find the perpendicular distance from the island to the coast.[1]



Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the island. Using the right triangle definition of sine, we get

[1] Do you see why C must lie to the right of Q?

$$\sin(45^\circ) = \frac{y}{d}$$
$$y = d\sin(45^\circ)$$
$$y \approx 9.66\left(\frac{\sqrt{2}}{2}\right)$$
$$y \approx 6.83 \text{ miles}$$

Hence, the island is approximately 6.83 miles from the coast. To find the distance from Q to C, we note that $\beta = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ so by symmetry , we get $x = y \approx 6.83$ miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point.

□

We close this section with the encouragement that, by working through the many problems in the Exercises, you will become proficient in applying the Law of Sines to real-world applications, and will be ready to move on to the Law of Cosines in **Section 7.3**.

7.2 Exercises

In Exercises 1 – 6, find the area of each triangle. Round each answer to the nearest tenth.

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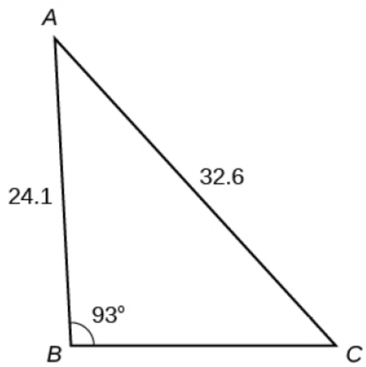

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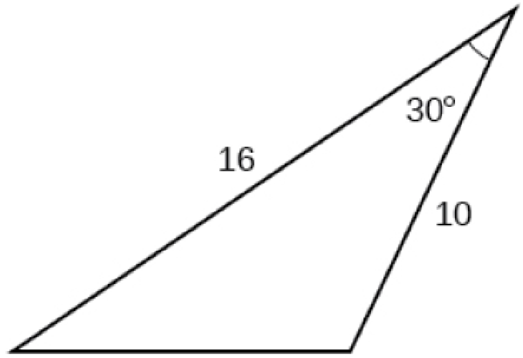

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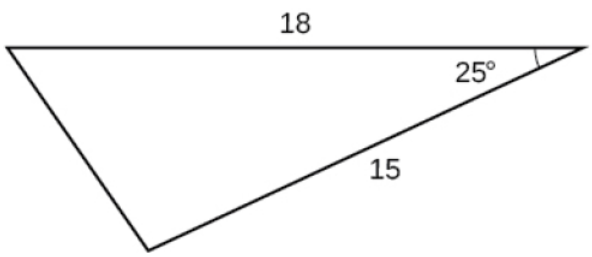

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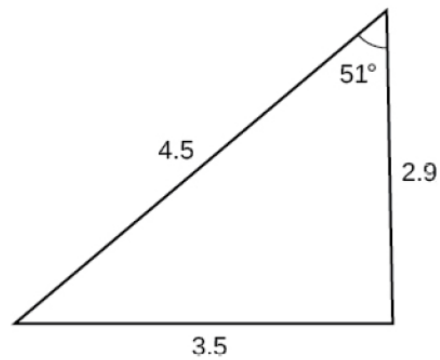

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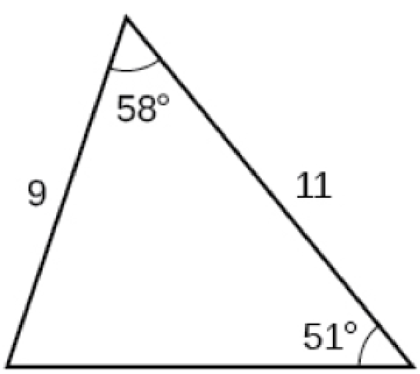

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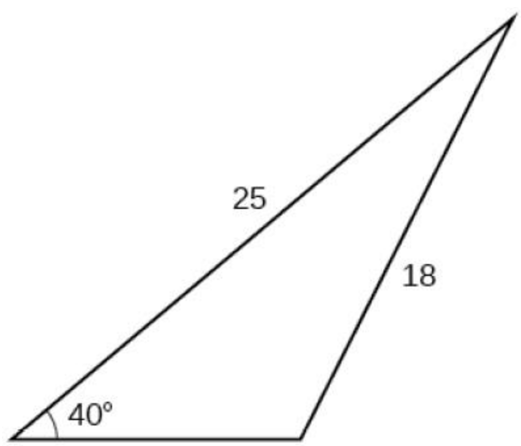


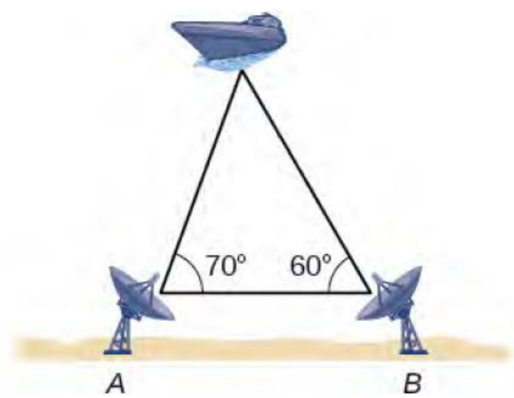
7. Find the area of the triangles. As in the text, (α,a), (β,b) and (γ,c) are angle-side opposite pairs.

(a) $\alpha = 13^\circ$, $\beta = 17^\circ$, $a = 5$ units

(b) $\gamma = 53^\circ$, $\alpha = 53^\circ$, $c = 28.01$ units

(c) $\alpha = 50^\circ$, $a = 25$ units, $b = 12.5$ units

8. To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in the figure. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.



9. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

10. A man and a woman standing 3.5 miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

11. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

12. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1040 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

13. Skippy and Sally decide to hunt UFOs. One night, they position themselves 2 miles apart on an abandoned stretch of desert runway. An hour into their investigation, Skippy spies a UFO hovering over a spot on the runway directly between him and Sally. He records the angle of inclination from the ground to the craft to be 75° and radios Sally immediately to find the angle of inclination from her position to the craft is 50°. How high off the ground is the UFO at this point? Round your answer to the nearest foot. (Recall: 1 mile is 5280 feet.)

14. A yield sign measures 30 inches on all three sides. What is the area of the sign?

Grade: The grade of a road is much like the pitch of a roof in that it expresses the ratio of rise/run. In the case of a road, this ratio is always positive because it is measured going uphill and it is usually given as a percentage. For example, a road which rises 7 feet for every 100 feet of (horizontal) forward progress is said to have a 7% grade. However, if we want to apply any trigonometry to a story problem involving roads going uphill or downhill, we need to view the grade as an angle with respect to the horizontal.

In Exercises 15 – 17, we first have you change road grades into angles and then use the Law of Sines in an application.

15. Using a right triangle with a horizontal leg of length 100 and vertical leg with length 7, show that a 7% grade means that the road (hypotenuse) makes about a 4° angle with the horizontal. (It will not be exactly 4°, but it is pretty close.

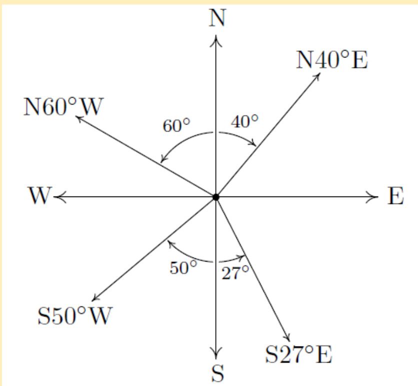
16. What grade is given by a 9.65° angle made by the road and the horizontal?

17. Along a long, straight stretch of mountain road with a 7% grade, you see a tall tree standing perfectly plumb alongside the road.[1] From a point 500 feet downhill from the tree, the angle of inclination from the road to the top of the tree is 6°. Use the Law of Sines to find the height of the tree. (Hint: First show that the tree makes a 94° angle with the road.)

[1] The word 'plumb' here means that the tree is perpendicular to the horizontal.

Bearings (Another Classic Application): In the next several exercises we introduce and work with the navigation tool known as bearings. Simply put, a bearing is the direction you are heading according to a compass. The classic nomenclature for bearings, however, is not given as an angle in standard position, so we must first understand the notation.

A bearing is given as an acute angle of rotation (to the east or to the west) away from the north-south (up and down) line of a compass rose. For example, N40°E (read "40° east of north") is a bearing which is rotated clockwise 40° from due north. If we imagine standing at the origin in the Cartesian Plane, this bearing would have us heading into Quadrant I along the terminal side of $\theta = 50^\circ$. Similarly, S50°W would point into Quadrant III along the terminal side of $\theta = 220^\circ$ because we started out pointing due south (along $\theta = 270^\circ$) and rotated clockwise 50° back to 220°. Counter-clockwise rotations would be found in the bearings N60°W (which is on the terminal side of $\theta = 150^\circ$) and S27°E (which lies along the terminal side of $\theta = 297^\circ$). These four bearings are drawn in the plane below.



The cardinal directions north, south, east and west are usually not given as bearings in the fashion described above, but rather, one just refers to them as 'due north', 'due south', 'due east' and 'due west', respectively, and it is assumed that you know which quadrantal angle goes with each cardinal direction. (Hint: Look at the diagram above.)

18. Find the angle θ in standard position with $0^\circ \le \theta < 360^\circ$ which corresponds to each of the bearings given below.

(a) due west	(b) S83°E	(c) N5.5°E	(d) due south
(e) N31.25°W	(f) S72°41'12"W	(g) N45°E	(h) S45°W

19. The Colonel spots a campfire at a bearing N42°E from his current position. Sarge, who is positioned 3000 feet due east of the Colonel, reckons the bearing to the fire to be N20°W from his current position. Determine the distance from the campfire to each man, rounded to the nearest foot.

20. A hiker starts walking due west from Sasquatch Point and gets to the Chupacabra Trailhead before she realizes that she hasn't reset her pedometer. From the Chupacabra Trailhead she hikes for 5 miles along a bearing of N53°W which brings her to the Muffin Ridge Observatory. From there, she knows a bearing of S65°Ewill take her straight back to Sasquatch Point. How far will she have to walk to get from the Muffin Ridge Observatory to Sasquatch Point? What is the distance between Sasquatch Point and the Chupacabra Trailhead?

21. The captain of the SS Bigfoot sees a signal flare at a bearing of N15°E from her current location. From his position, the captain of the HMS Sasquatch finds the signal flare to be at a bearing of N75°W. If the SS Bigfoot is 5 miles from the HMS Sasquatch and the bearing from the SS Bigfoot to the HMS Sasquatch is N50°E, find the distances from the flare to each vessel, rounded to the nearest tenth of a mile.

22. Carl spies a potential Sasquatch nest at a bearing of N10°E and radios Jeff, who is at a bearing of N50°E from Carl's position. From Jeff's position, the nest is at a bearing of S70°W. If Jeff and Carl are 500 feet apart, how far is Jeff from the Sasquatch nest? Round your answer to the nearest foot.

23. A hiker determines the bearing to a lodge from her current position is S40°W. She proceeds to hike 2 miles at a bearing of S20°E at which point she determines the bearing to the lodge is S75°W. How far is she from the lodge at this point? Round your answer to the nearest hundredth of a mile.

24. A watchtower spots a ship off shore at a bearing of N70°E. A second tower, which is 50 miles from the first at a bearing of S80°E from the first tower, determines the bearing to the ship to be N25°W. How far is the boat from the second tower? Round your answer to the nearest tenth of a mile.

25. The angle of depression from an observer in an apartment complex to a gargoyle on the building next door is 55°. From a point five stories below the original observer, the angle of inclination to the gargoyle is 20°. Find the distance from each observer to the gargoyle and the distance from the gargoyle to the apartment complex. Round your answers to the nearest foot. (Use the rule of thumb that one story of a building is 9 feet.)

26. Use the cases and diagrams in the proof of the Law of Sines (**Theorem 7.1**) to prove the area formulas given in **Theorem 7.3**. Why do those formulas yield square units when four quantities are being multiplied together?

7.3 The Law of Cosines

Learning Objectives

In this section you will:

- **Use The Law of Cosines to solve oblique triangles.**
- **Solve SAS and SSS Triangles.**
- **Use Heron's Formula to find the area of a triangle.**
- **Solve applied problems using the Law of Cosines.**

In **Section 7.1**, we developed the Law of Sines (**Theorem 7.1**) to enable us to solve triangles in the 'Angle-Side-Angle' (ASA), the 'Angle-Angle-Side' (AAS) and the ambiguous 'Angle-Side-Side' (ASS) cases. In this section, we develop the Law of Cosines which handles solving triangles in the 'Side-Angle-Side' (SAS) and 'Side-Side-Side' (SSS) cases.[1]

The Law of Cosines

We state and prove the Law of Cosines theorem below.

Theorem 7.4. The Law of Cosines. Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following equations hold

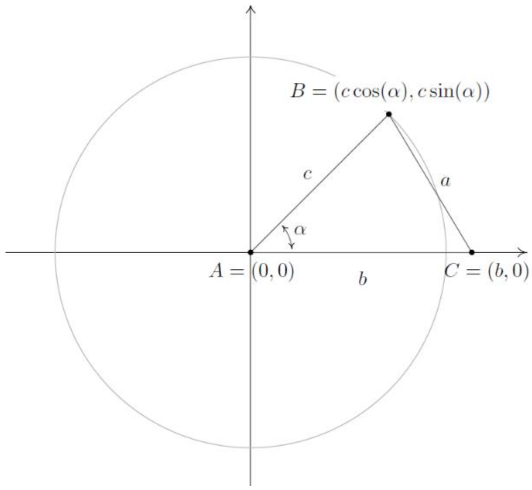
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha) \qquad b^2 = a^2 + c^2 - 2ac\cos(\beta) \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

To prove the theorem, we consider a generic triangle with the vertex of angle α at the origin, and with side b positioned along the positive x-axis.

[1] Here, 'Side-Angle-Side' means that we are given two sides and the included angle – that is, the given angle adjacent to both of the given sides.



From this set-up, we immediately find that the coordinates of *A* and *C* are $A(0,0)$ and $C(b,0)$. From **Theorem 2.6**, we know that since the point $B(x,y)$ lies on a circle of radius *c*, the coordinates of *B* are $B(x,y) = B(c\cos(\alpha), c\sin(\alpha))$. (This would be true even if *α* were an obtuse or right angle so although we have drawn the case where *α* is acute, the following computations hold for any angle *α* drawn in standard position where $0^\circ < \alpha < 180^\circ$.) We note that the distance between the points *B* and *C* is none other than the length of side *a*. Using the distance formula, we get

$$a = \sqrt{(c\cos(\alpha) - b)^2 + (c\sin(\alpha) - 0)^2}$$
$$a^2 = \left(\sqrt{(c\cos(\alpha) - b)^2 + c^2\sin^2(\alpha)}\right)^2$$
$$a^2 = (c\cos(\alpha) - b)^2 + c^2\sin^2(\alpha)$$
$$a^2 = c^2\cos^2(\alpha) - 2bc\cos(\alpha) + b^2 + c^2\sin^2(\alpha)$$
$$a^2 = b^2 + c^2\left(\cos^2(\alpha) + \sin^2(\alpha)\right) - 2bc\cos(\alpha)$$
$$a^2 = b^2 + c^2(1) - 2bc\cos(\alpha) \qquad \textbf{since } \cos^2(\alpha) + \sin^2(\alpha) = 1$$
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

The remaining formulas given in **Theorem 7.4** can be shown by simply reorienting the triangle to place a different vertex at the origin. We leave these details to the reader. What's important about *a* and *α* in the above proof is that (α, a) is an angle-side opposite pair and *b* and *c* are the sides adjacent to *α*. The same can be said of any other angle-side opposite pair in the triangle.

Notice that the proof of the Law of Cosines relies on the distance formula, which has its roots in the Pythagorean Theorem. That being said, the Law of Cosines can be thought of as a generalization of the

Pythagorean Theorem. If we have a triangle in which $\gamma = 90^\circ$, then $\cos(\gamma) = \cos(90^\circ) = 0$ so we get the familiar relationship $c^2 = a^2 + b^2$. What this means is that in the larger mathematical sense, the Law of Cosines and the Pythagorean Theorem amount to pretty much the same thing.[2]

Example 7.3.1. Solve the triangle in which $\beta = 50^\circ$, $a = 7$ units, and $c = 2$ units. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

Solution. We are given the lengths of two sides, $a = 7$ and $c = 2$, and the measure of the included angle, $\beta = 50^\circ$. With no angle-side opposite pair to use, we apply the Law of Cosines to find *b*.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\ b^2 &= 7^2 + 2^2 - 2(7)(2)\cos(50^\circ) \\ b &= \sqrt{53 - 28\cos(50^\circ)} \\ b &\approx 5.92 \text{ units} \end{aligned}$$

In order to determine the measures of the remaining angles α and γ, we are forced to use the derived value for *b*. There are two ways to proceed at this point. We could use the Law of Cosines again, or, since we have the angle-side opposite pair (β, b) we could use the Law of Sines. The advantage to using the Law of Cosines over the Law of Sines in cases like this is that, unlike the sine function, the cosine function distinguishes between acute and obtuse angles. The cosine of an acute angle is positive, whereas the cosine of an obtuse angle is negative. Since the sine of both acute and obtuse angles is positive, the sine of an angle alone is not enough to determine if the angle in question is acute or obtuse.

We proceed with the Law of Cosines. When using the Law of Cosines, It's always best to find the measure of the largest unknown angle first, since this will give us the obtuse angle of the triangle if there is one. Since the largest angle is opposite the longest side, we choose to find α first. To that end, we use the formula for $\cos(\alpha)$ from **Theorem 7.4** and substitute $a = 7$, $b = \sqrt{53 - 28\cos(50^\circ)}$ and $c = 2$.

[2] This shouldn't come as too much of a shock. All of the theorems in Trigonometry can ultimately be traced back to the definition of the circular functions along with the distance formula and, hence, the Pythagorean Theorem.

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos(\alpha) = \frac{\left(\sqrt{53 - 28\cos(50^\circ)}\right)^2 + (2)^2 - (7)^2}{2\left(\sqrt{53 - 28\cos(50^\circ)}\right)(2)}$$

$$\cos(\alpha) = \frac{2 - 7\cos(50^\circ)}{\sqrt{53 - 28\cos(50^\circ)}}$$ **after simplifying**

Since α is an angle in a triangle, we know the radian measure of α must lie between 0 and π radians. This matches the range of the arccosine function, so we have

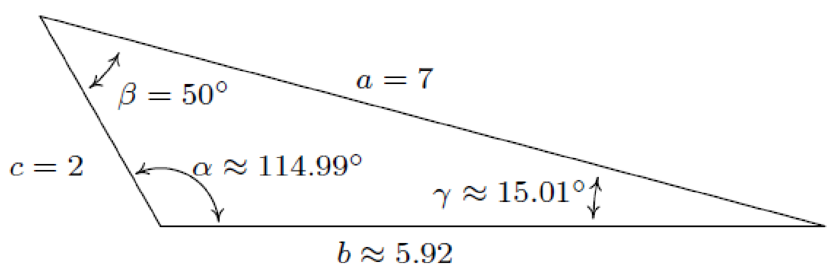
$$\alpha = \arccos\left(\frac{2 - 7\cos(50^\circ)}{\sqrt{53 - 28\cos(50^\circ)}}\right) \text{ radians}$$

$$\alpha \approx 114.99^\circ$$

At this point, we could find γ using the Law of Cosines, to minimize the propagation of error. Or we can take the shortcut of using angle measures that have already been determined.

$$\gamma = 180^\circ - \alpha - \beta$$
$$\approx 180^\circ - 114.99^\circ - 50^\circ$$
$$\approx 15.01^\circ$$

We sketch the triangle below.



□

As mentioned earlier in the preceding example, once we'd determined b it was possible to use the Law of Sines to determine the remaining angles. However, noting that this was the ambiguous ASS case, proceeding with the Law of Sines would require caution. It is advisable to first find the *smallest* of the unknown angles, since we are guaranteed it will be acute.[3] In this case, we would find γ since the side

[3] There can only be one obtuse angle in the triangle, and if there is one, it must be the largest.

opposite γ is smaller than the side opposite the other unknown angle, α. Using the Law of Sines with the angle-side opposite pair (β, b), we get

$$\frac{\sin(\gamma)}{2} = \frac{\sin(50^\circ)}{\sqrt{53 - 28\cos(50^\circ)}}$$

The usual calculations produce $\gamma \approx 15.01^\circ$ and we find

$$\begin{aligned} \alpha &= 180^\circ - \beta - \gamma \\ &\approx 180^\circ - 50^\circ - 15.01^\circ \\ &= 114.99^\circ \end{aligned}$$

Example 7.3.2. Solve for the angles in the triangle with sides of length $a = 4$ units, $b = 7$ units and $c = 5$ units.

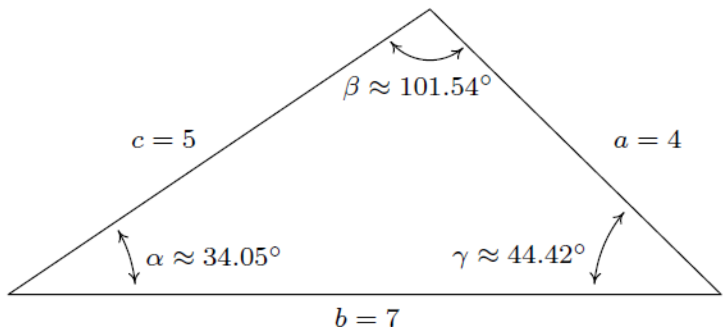
Solution. Since all three sides and no angles are given, we are forced to use the Law of Cosines. Following our discussion in the previous problem, we find β first, since it is opposite the longest side, b.

$$\begin{aligned} \cos(\beta) &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{(4)^2 + (5)^2 - (7)^2}{2(4)(5)} \\ &= -\frac{1}{5} \end{aligned}$$

We then determine β through finding the arccosine of $-\frac{1}{5}$.

$$\begin{aligned} \beta &= \arccos\left(-\frac{1}{5}\right) \text{ radians} \\ &\approx 101.54^\circ \end{aligned}$$

As in the previous problem, now that we have obtained an angle-side opposite pair (β, b) we could proceed using the Law of Sines. The Law of Cosines, however, offers us a rare opportunity to find the remaining angles using only the data given to us in the statement of the problem. Using the problem data, we get $\gamma = \arccos\left(\frac{5}{7}\right)$ radians $\approx 44.42^\circ$ and $\alpha = \arccos\left(\frac{29}{35}\right)$ radians $\approx 34.05^\circ$.



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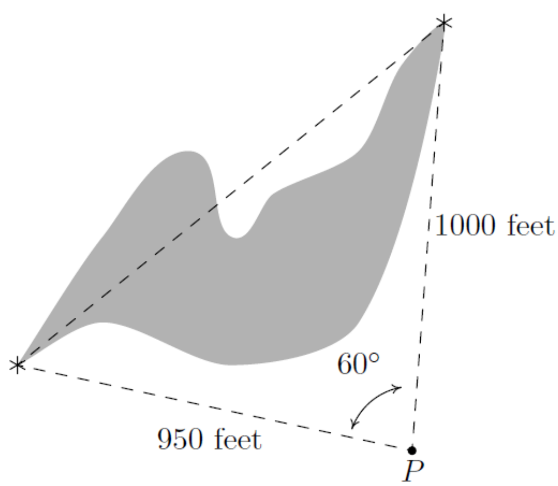
We note that, depending on how many decimal places are carried through successive calculations, and depending on which approach is used to solve the problem, the approximate answers you obtain may differ slightly from those posted in the Examples and the Exercises. **Example 7.3.2** is a great example of this, in that the approximate values we record for the measures of the angles sum to 180.01°, which is geometrically impossible.

Solving Applied Problems Using the Law of Cosines

Next, we have an application of the Law of Cosines.

Example 7.3.3. A researcher wishes to determine the width of a vernal pond as drawn below. From a point *P*, he finds the distance to the western-most point of the pond to be 950 feet, while the distance to the northern-most point of the pond from *P* is 1000 feet. If the angle between the two lines of sight is 60°, find the width of the pond.



Solution. We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle. Calling this length w (for width), we get

$$\begin{aligned} w^2 &= 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) \\ &= 952{,}500 \end{aligned}$$

We next take the square root to get

$$\begin{aligned} w &= \sqrt{952500} \\ &\approx 976 \text{ feet} \end{aligned}$$

□

Heron's Formula

In **Section 7.2**, we used the proof of the Law of Sines to develop **Theorem 7.3** as an alternate formula for the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula – Heron's Formula.

Theorem 7.5. Heron's Formula. Suppose a, b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a+b+c)$. Then the area A, enclosed by the triangle, is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We prove **Theorem 7.5** using **Theorem 7.3**. Using the convention that the angle γ is opposite the side c, we have $A = \frac{1}{2}ab\sin(\gamma)$ from **Theorem 7.3**. In order to simplify computations, we start by manipulating the expression for A^2.

$$\begin{aligned} A^2 &= \left(\frac{1}{2}ab\sin(\gamma)\right)^2 \\ &= \frac{1}{4}a^2b^2\sin^2(\gamma) \\ &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \quad \textbf{since } \cos^2(\gamma)+\sin^2(\gamma)=1 \end{aligned}$$

The Law of Cosines tells us $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$, so substituting this into our equation for A^2 gives

$$
\begin{aligned}
A^2 &= \frac{a^2b^2}{4}\left(1-\cos^2(\gamma)\right) \\
&= \frac{a^2b^2}{4}\left(1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right) && \textbf{Law of Cosines} \\
&= \frac{a^2b^2}{4}\left(1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right) \\
&= \frac{a^2b^2}{4}\left(\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right) \\
&= \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16} \\
&= \frac{\left(2ab-\left(a^2+b^2-c^2\right)\right)\left(2ab+\left(a^2+b^2-c^2\right)\right)}{16} && \textbf{difference of squares} \\
&= \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} \\
&= \frac{\left(c^2-\left(a^2-2ab+b^2\right)\right)\left(\left(a^2+2ab+b^2\right)-c^2\right)}{16} \\
&= \frac{\left(c^2-(a-b)^2\right)\left((a+b)^2-c^2\right)}{16} && \textbf{perfect square trinomials} \\
&= \frac{\left(c-(a-b)\right)\left(c+(a-b)\right)\left((a+b)-c\right)\left((a+b)+c\right)}{16} && \textbf{difference of squares} \\
&= \frac{(b+c-a)(a+c-b)(a+b-c)(a+b+c)}{16} \\
&= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2}
\end{aligned}
$$

At this stage, we recognize the last factor as the semiperimeter, $s=\frac{1}{2}(a+b+c)=\frac{a+b+c}{2}$. To complete the proof, we note that

$$
\begin{aligned}
(s-a) &= \frac{a+b+c}{2}-a \\
&= \frac{a+b+c-2a}{2} \\
&= \frac{b+c-a}{2}
\end{aligned}
$$

Similarly, we find $(s-b)=\dfrac{a+c-b}{2}$ and $(s-c)=\dfrac{a+b-c}{2}$. Hence, we get

$$\begin{aligned}A^2 &= \frac{(b+c-a)}{2}\cdot\frac{(a+c-b)}{2}\cdot\frac{(a+b-c)}{2}\cdot\frac{(a+b+c)}{2}\\ &=(s-a)(s-b)(s-c)s\end{aligned}$$

so that $A=\sqrt{s(s-a)(s-b)(s-c)}$ as required.

We close with an example of Heron's Formula.

Example 7.3.4. Find the area enclosed by the triangle in **Example 7.3.2**.

Solution. In **Example 7.3.2**, we are given a triangle with sides of length $a=4$ units, $b=7$ units and $c=5$ units. Using these values, we find $s=\frac{1}{2}(4+5+7)=8$. Using Heron's Formula, we get

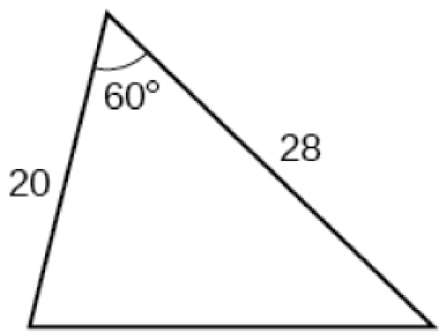
$$\begin{aligned}A &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{8(8-4)(8-7)(8-5)}\\ &= \sqrt{8(4)(1)(3)}\\ &= \sqrt{96}\\ &= 4\sqrt{6}\approx 9.80 \text{ square units}\end{aligned}$$

□

7.3 Exercises

In Exercises 1 – 4, solve for the length of the unknown side. Round to the nearest tenth.

1.



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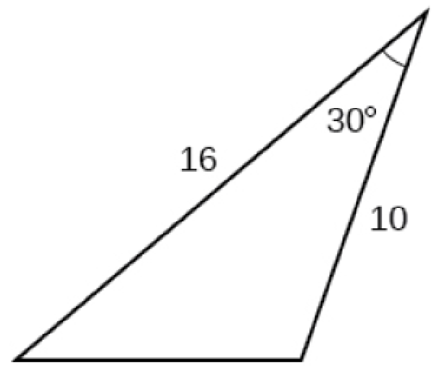

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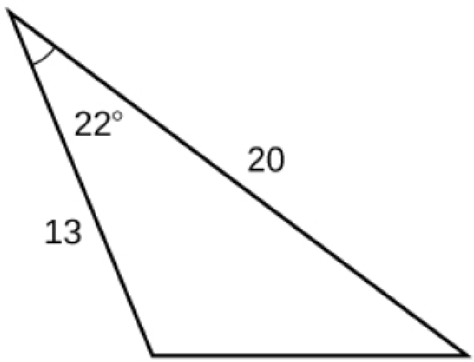

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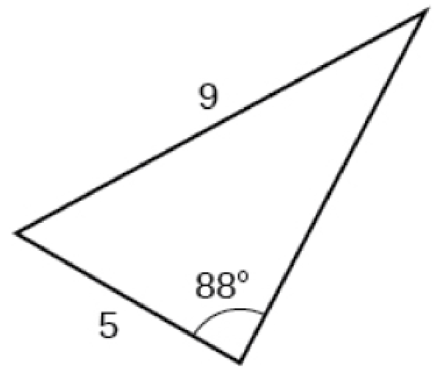
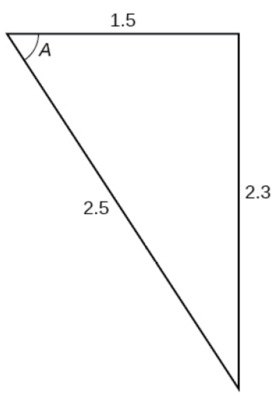


In Exercises 5 – 8, find the measure of angle *A*. Round to the nearest tenth.

5.



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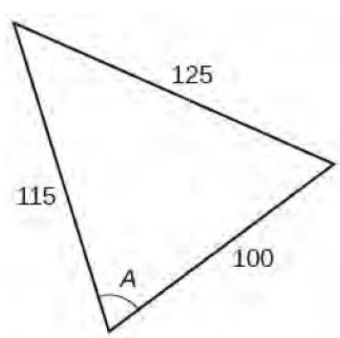

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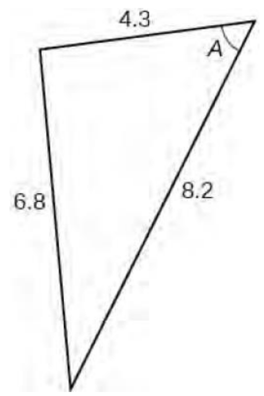

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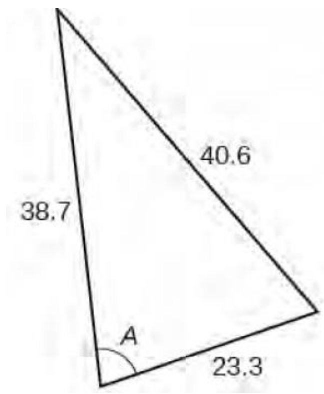
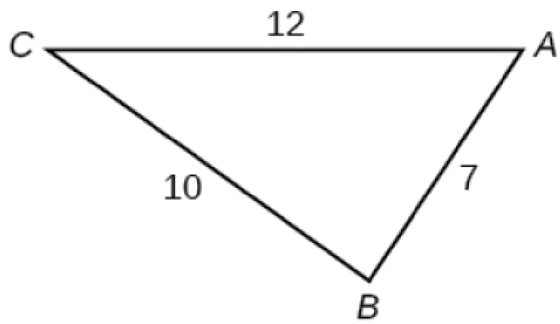


9. Find the measure of each angle in the triangle. Round to the nearest tenth.



In Exercises 10 – 19, use the Law of Cosines to find the remaining side(s) and angles(s) if possible.

10. $a = 7,\ b = 12,\ \gamma = 59.3^\circ$

11. $\alpha = 104^\circ,\ b = 25,\ c = 37$

12. $a = 153,\ \beta = 8.2^\circ,\ c = 153$

13. $a = 3,\ b = 4,\ \gamma = 90^\circ$

14. $\alpha = 120^\circ,\ b = 3,\ c = 4$

15. $a = 7,\ b = 10,\ c = 13$

16. $a = 1,\ b = 2,\ c = 5$

17. $a = 300,\ b = 302,\ c = 48$

18. $a = 5,\ b = 5,\ c = 5$

19. $a = 5,\ b = 12,\ c = 13$

In Exercises 20 – 25, solve for the remaining side(s) and angle(s), if possible, using any appropriate technique.

20. $a = 18,\ \alpha = 63^\circ,\ b = 20$

21. $a = 37,\ b = 45,\ c = 26$

22. $a = 16,\ \alpha = 63^\circ,\ b = 20$

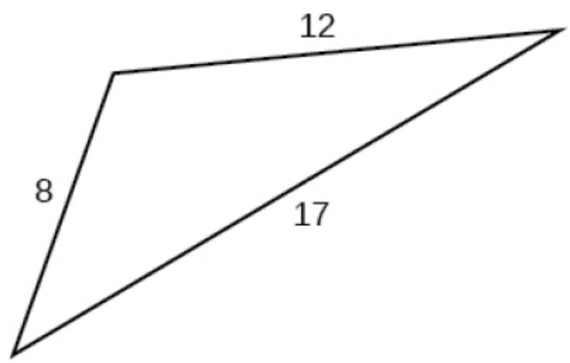
23. $a = 22,\ \alpha = 63^\circ,\ b = 20$

24. $\alpha = 42^\circ,\ b = 117,\ c = 88$

25. $\beta = 7^\circ,\ \gamma = 170^\circ,\ c = 98.6$

In Exercises 26 – 29, find the area of the triangle. Round to the nearest hundredth.

26.



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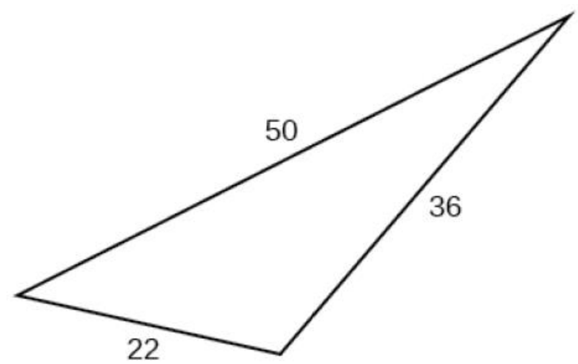

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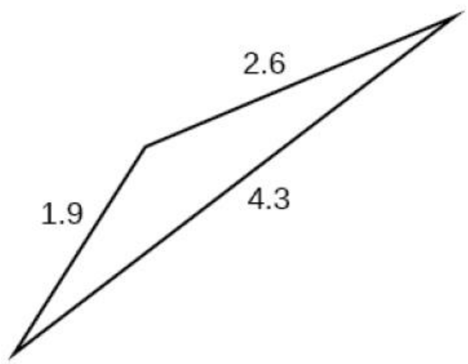

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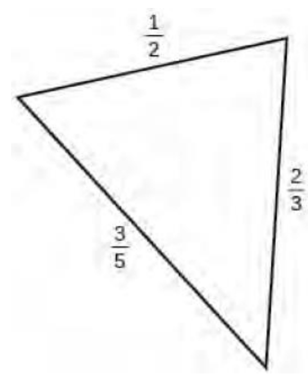


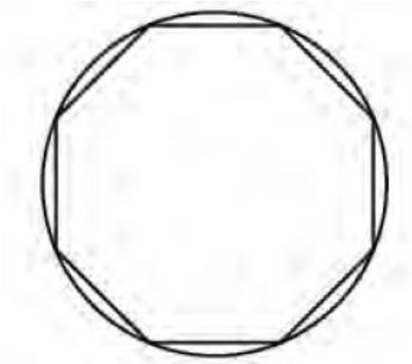
30. Find the area of the triangles.

(a) $a = 7,\ b = 10,\ c = 13$

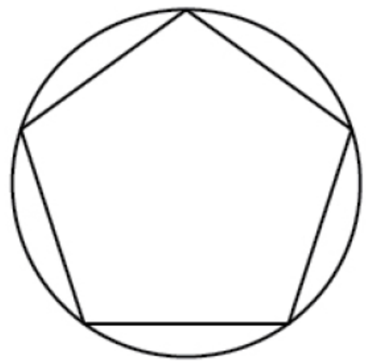
(b) $a = 300,\ b = 302,\ c = 48$

(c) $a = 5,\ b = 12,\ c = 13$

31. A rectangular octagon is inscribed in a circle with a radius of 8 inches. Find the perimeter of the octagon.



32. A rectangular pentagon is inscribed in a circle of radius 12 cm. Find the perimeter of the pentagon.



33. The hour hand on my antique Seth Thomas schoolhouse clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

34. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

35. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of N8.2°E for 192 miles and it can fly to Bigfoot Falls by following a bearing of S68.5°E for 207 miles.[1] Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

[1] Please refer to the **7.2 Exercises** for an introduction to bearings.

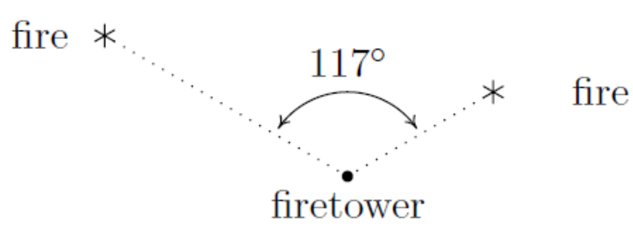
36. Cliffs of Insanity Point and Bigfoot Falls from **Exercise 35** both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

37. A naturalist sets off on a hike from a lodge on a bearing of S80°W. After 1.5 miles, she changes her bearing to S17°W and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

38. The HMS Sasquatch leaves port on a bearing of N23°E and travels for 5 miles. It then changes course and follows a heading of S41°E for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

39. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of N32°E. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now S70°W, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

40. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will give you a SAS case in which to apply the Law of Cosines.)



41. If you apply the Law of Cosines to the ambiguous ASS case, the result is a quadratic equation whose variable is that of the missing side. If the equation has no positive real zeros then the information given does not yield a triangle. If the equation has only one positive real zero then exactly one triangle is formed and if the equation has two distinct positive real zeros then two distinct triangles are formed. Apply the Law of Cosines to each of the following in order to demonstrate this result.

(a) $a = 18,\ \alpha = 63^\circ,\ b = 20$

(b) $a = 16,\ \alpha = 63^\circ,\ b = 20$

(c) $a = 22,\ \alpha = 63^\circ,\ b = 20$

42. Discuss with your classmates why Heron's Formula yields an area in square units even though four lengths are being multiplied together.

CHAPTER 8
POLAR COORDINATES AND APPLICATIONS

Chapter Outline

8.1 Polar Coordinates
8.2 Polar Equations
8.3 Graphing Polar Equations
8.4 Polar Representations for Complex Numbers
8.5 Complex Products, Powers, Quotients and Roots

Introduction

Chapter 8 takes us from graphing on the Cartesian coordinate plane to graphing on a polar grid, using the pole and polar axis for reference. We begin in Section 8.1 by plotting points defined by polar coordinates. The geometry and connection between the pole and origin, polar axis and positive *x*-axis, lead to the conversion of points between polar and rectangular coordinates. Section 8.2 continues this theme by converting equations back and forth between polar form and rectangular form. Graphing is the focus of Section 8.3, beginning with circles and lines in the coordinate plane, then moving on to more complicated polar graphs. Throughout Section 8.3, techniques are introduced and emphasized that enable the student to complete polar graphs by hand, without the aid of technology. The theme of Section 8.4 is complex numbers, as represented on the complex plane. A polar definition for complex numbers is introduced, and practice is provided for converting between rectangular and polar forms. This chapter ends with Section 8.5 and the introduction of ‘arithmetic‘ with complex numbers. DeMoivre’s Theorem is included as a means for determining powers and roots of complex numbers.

This chapter introduces new concepts that rely on the trigonometric tools and skills developed up to this point in the course. It provides valuable insight into polar graphs and complex numbers.

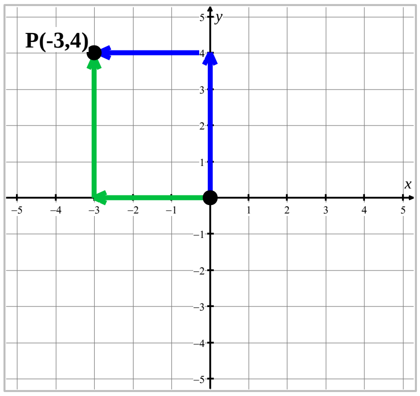
8.1 Polar Coordinates

Learning Objectives

In this section you will:

- **Graph points in polar coordinates.**
- **Convert points in polar coordinates to rectangular coordinates and vice versa.**

Up to this point, we have graphed points in the Cartesian coordinate plane by assigning ordered pairs of numbers to points in the plane. We defined the Cartesian coordinate plane using two number lines, one horizontal and one vertical, which intersect at right angles at a point called the origin. To plot a point, say $P(-3,4)$, we start at the origin, travel horizontally to the left 3 units, then up 4 units.



Alternatively, we could start at the origin, travel up 4 units, then to the left 3 units and arrive at the same location. For the most part, the motions of the Cartesian system (over and up) describe a rectangle, and most points can be thought of as the corner diagonally across the rectangle from the origin.[1] For this reason, the Cartesian coordinates of a point are often called rectangular coordinates.

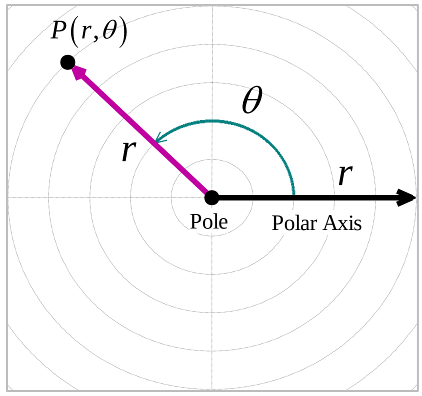
In this section, we introduce **polar coordinates**, a new system for assigning coordinates to points in the plane.

Plotting Polar Coordinates

We start with an origin point, called the **pole**, and a ray called the **polar axis**. We then locate a point *P* using two coordinates, (r,θ), where *r* represents a *directed* distance from the pole[2] and θ is a measure of

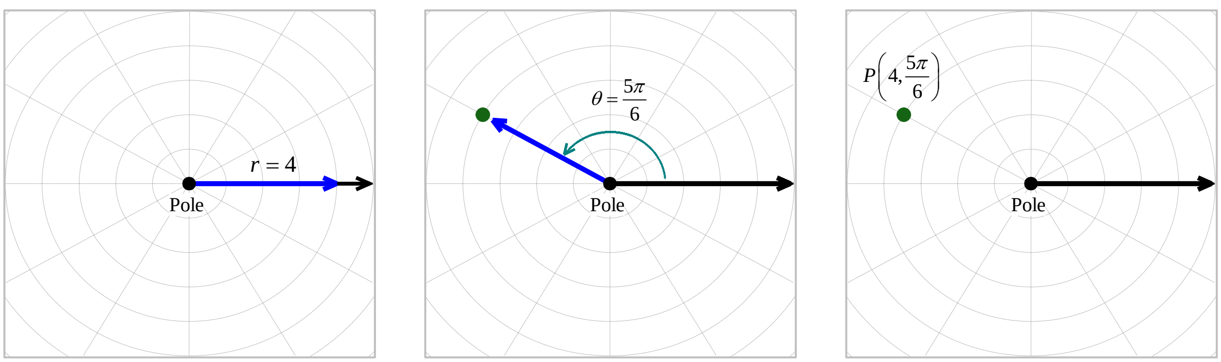
[1] Excluding, of course, the points in which one or both coordinates are 0.
[2] We will explain more about this momentarily.

rotation from the polar axis. Roughly speaking, the polar coordinates (r,θ) of a point measure how far out the point is from the pole (that's r) and how far to rotate from the polar axis (that's θ).



Example 8.1.1. Plot the point P with polar coordinates $\left(4, \frac{5\pi}{6}\right)$.

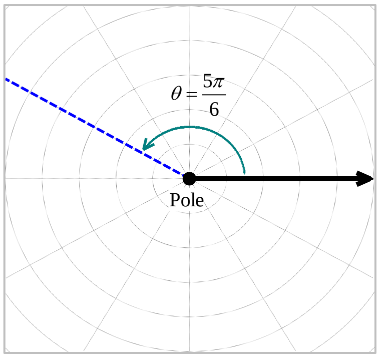
Solution. We start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.



First Second The Resulting Point

We may also visualize this process by thinking of the rotation first.[3] To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ radians counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially, we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole.

[3] As with anything in Mathematics, the more ways you have to look at something, the better. Take some time to think about both approaches to plotting points given in polar coordinates.



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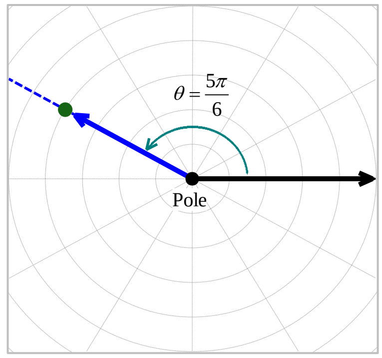

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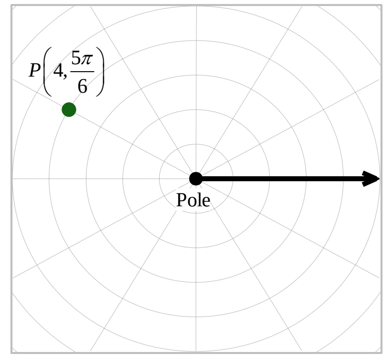

The Resulting Point

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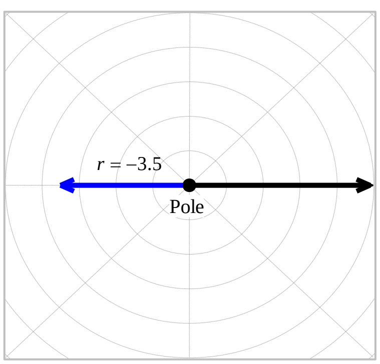
If $r < 0$, we begin by moving, from the pole, in the opposite direction of the polar axis.

Example 8.1.2. Plot $Q\left(-3.5, \frac{\pi}{4}\right)$.

Solution. We start at the pole, moving 3.5 units in the opposite direction of the polar axis. We then rotate $\frac{\pi}{4}$ units counter-clockwise.



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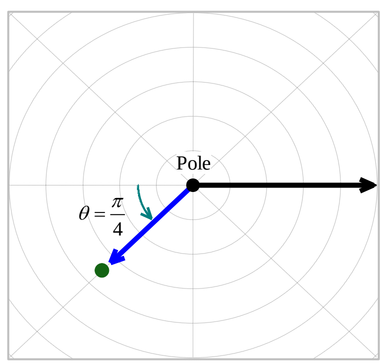

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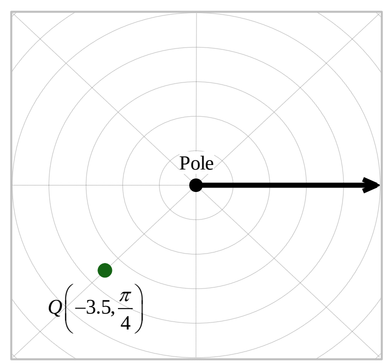
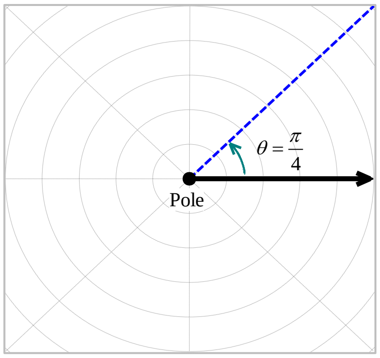


The Resulting Point

If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.



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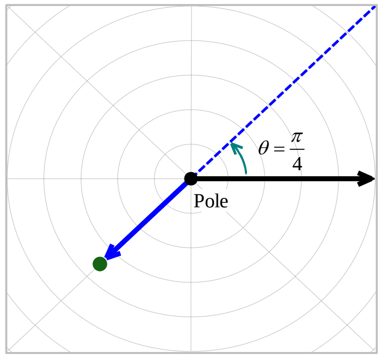

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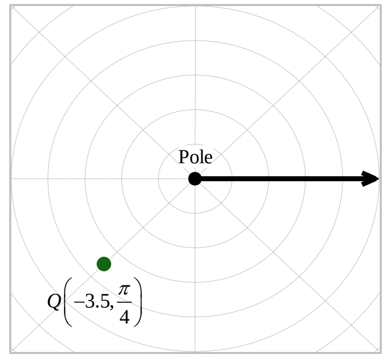

The Resulting Point

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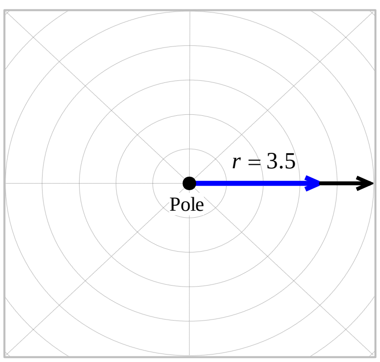
As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise.

Example 8.1.3. Plot $R\left(3.5,-\dfrac{3\pi}{4}\right)$.

Solution. To plot $R\left(3.5,-\dfrac{3\pi}{4}\right)$, we have the following.



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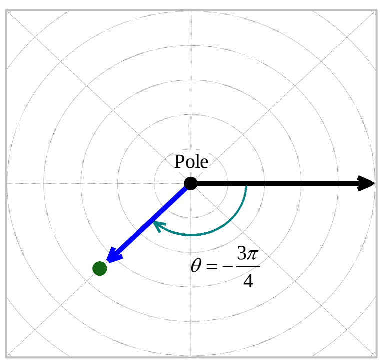

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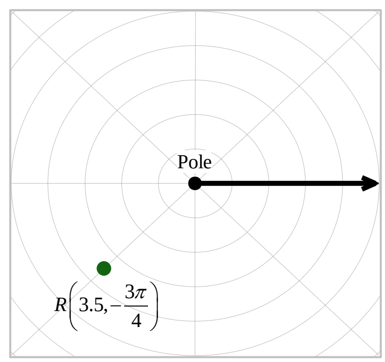
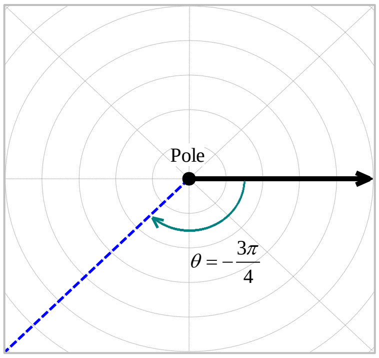


The Resulting Point

From an ‘angles first’ approach, we rotate $-\dfrac{3\pi}{4}$ then move out 3.5 units from the pole. We see that R is the point on the terminal side of $\theta = -\dfrac{3\pi}{4}$ which is 3.5 units from the pole.



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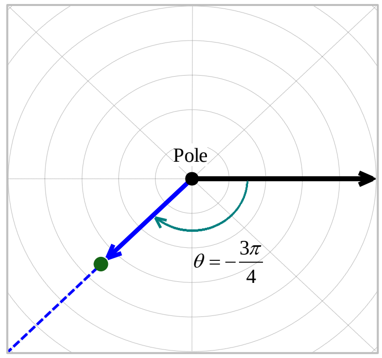

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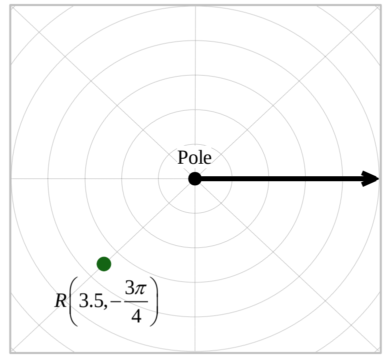

The Resulting Point

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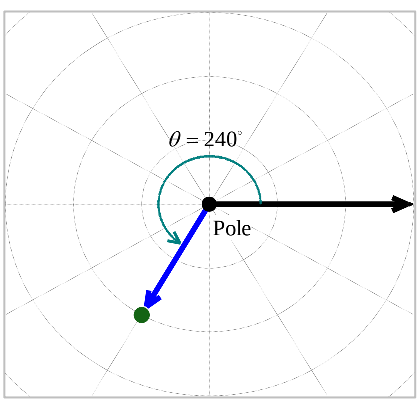
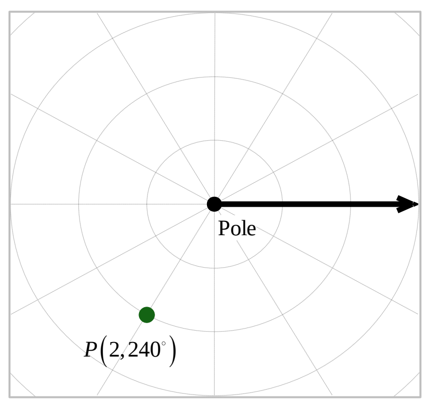
Multiple Representations for Polar Coordinates

The points Q and R in the above examples are, in fact, the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a,b) and (c,d) represent the same point if and only if $a = c$ and $b = d$, a point can be represented by infinitely many polar coordinate pairs. We explore this notion in the following examples.

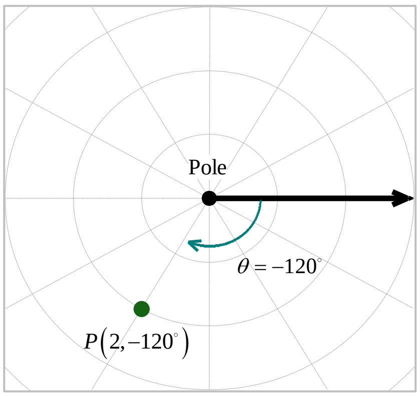
Example 8.1.4. Plot the point $P(2, 240^\circ)$, given in polar coordinates, and then give two additional expressions for the point, one of which has $r > 0$ and the other with $r < 0$.

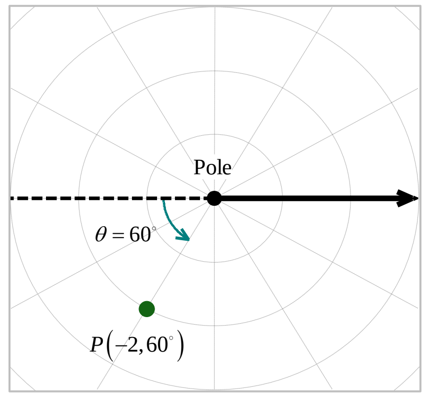
Solution. Whether we move 2 units along the polar axis and then rotate 240° or rotate 240° then move out 2 units from the pole, we plot $P(2, 240^\circ)$ below.





We now set about finding alternate descriptions (r, θ) for the point P. Since P is 2 units from the pole, $r = \pm 2$. Next, we choose angles θ for each of the r values. The given representation for P is $(2, 240^\circ)$ so the angle θ we choose for the $r = 2$ case must be coterminal with 240°. (Can you see why?) One such angle is $\theta = -120^\circ$ so one answer for this case is $(2, -120^\circ)$. For the case $r = -2$, we visualize our rotation starting 2 units to the left of the pole. From this position, we need only to rotate $\theta = 60^\circ$ to arrive at a location coterminal with 240°. Hence, our answer here is $(-2, 60^\circ)$. We check our answers by plotting them.





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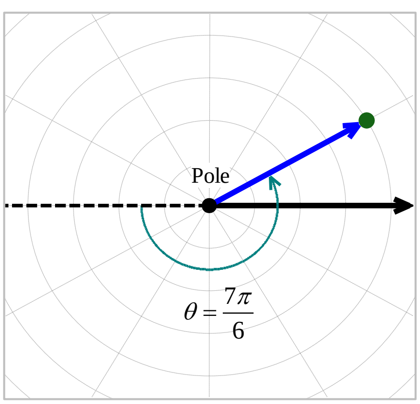
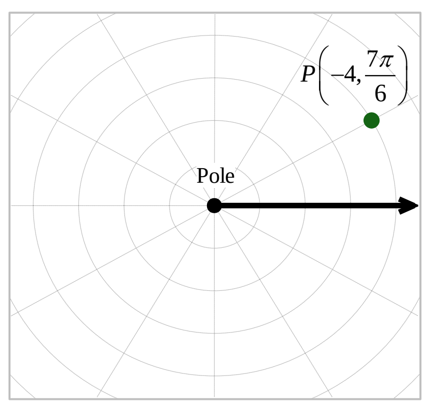
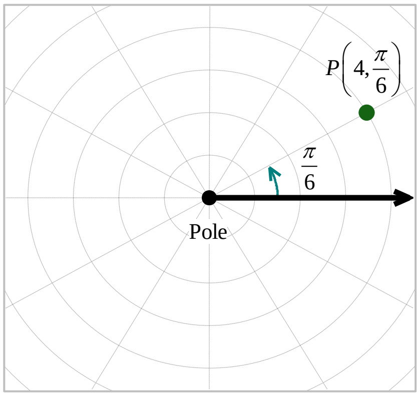
Example 8.1.5. Plot the point $P\left(-4, \frac{7\pi}{6}\right)$ and give two additional expressions for the point, one with $r > 0$ and the other with $r < 0$.

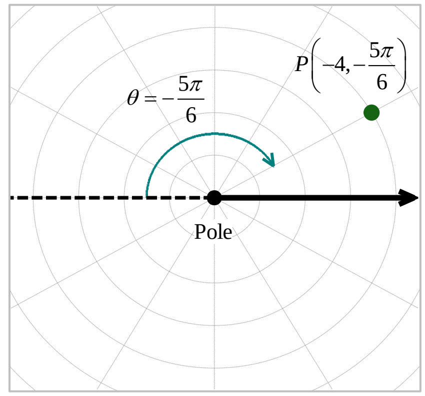
Solution. We plot $\left(-4, \frac{7\pi}{6}\right)$ by first moving 4 units to the left of the pole and then rotating $\frac{7\pi}{6}$ radians. We find our point lies 4 units from the pole on the terminal side of $\frac{\pi}{6}$.





To find alternate descriptions for *P*, we note that the distance from *P* to the pole is 4 units, so any representation (r, θ) for *P* must have $r = \pm 4$. As we noted above, *P* lies on the terminal side of $\frac{\pi}{6}$, so this, coupled with $r = 4$, gives us $\left(4, \frac{\pi}{6}\right)$ as one of our answers. To find a different representation for *P* with $r = -4$, we may choose any angle coterminal with the angle in the original representation of $P\left(-4, \frac{7\pi}{6}\right)$. We pick $-\frac{5\pi}{6}$ and get $\left(-4, -\frac{5\pi}{6}\right)$ as our second answer.





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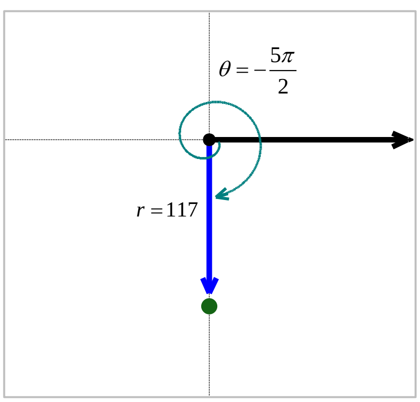
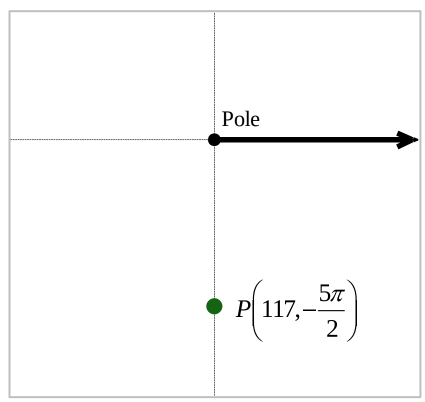
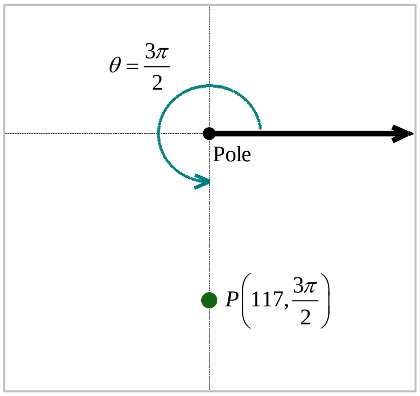
Example 8.1.6. Plot the point $P\left(117, -\frac{5\pi}{2}\right)$ and give two additional expressions for the point, one with $r > 0$ and the other with $r < 0$.

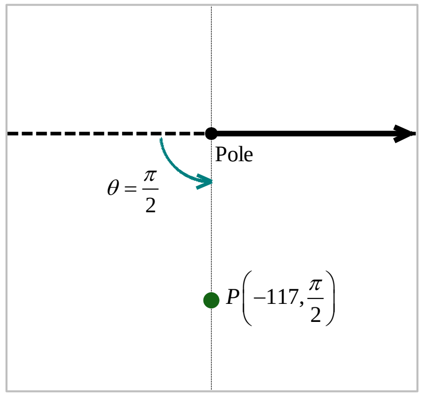
Solution. To plot $P\left(117, -\frac{5\pi}{2}\right)$, we move along the polar axis 117 units from the pole and rotate *clockwise* $\frac{5\pi}{2}$ radians as illustrated below.





Since P is 117 units from the pole, any representation (r, θ) for P satisfies $r = \pm 117$. For the $r = 117$ case, we can take θ to be any angle coterminal with $-\frac{5\pi}{2}$. In this case, we choose $\theta = \frac{3\pi}{2}$ and get $\left(117, \frac{3\pi}{2}\right)$ as one answer. For the $r = -117$ case, we visualize moving left 117 units from the pole and then rotating through an angle θ to reach P. We find that $\theta = \frac{\pi}{2}$ satisfies this requirement, so our second answer is $\left(-117, \frac{\pi}{2}\right)$.





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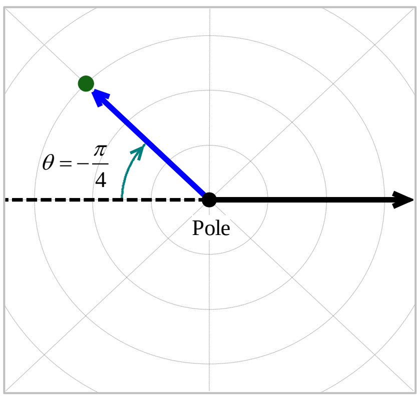
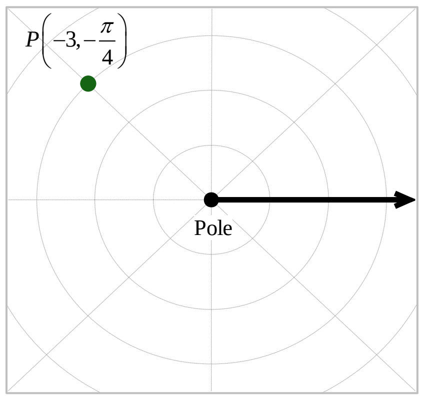
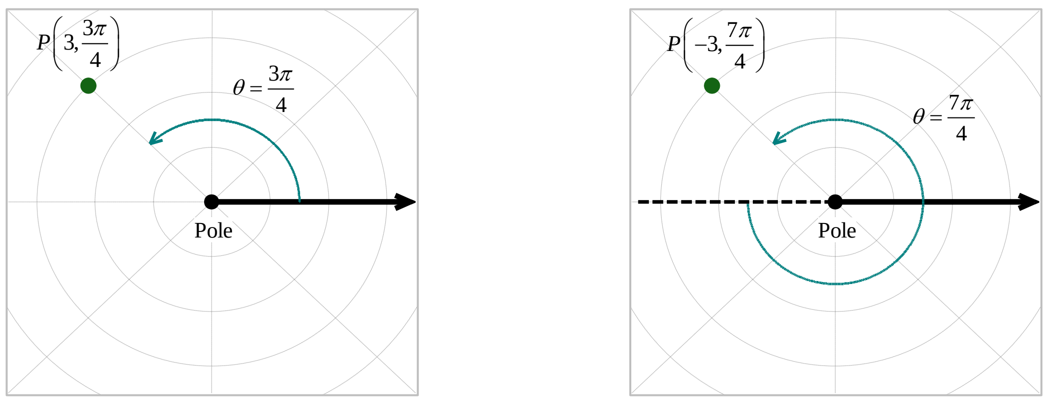
Example 8.1.7. Plot the point $P\left(-3, -\frac{\pi}{4}\right)$ and give two additional expressions for the point, one with $r > 0$ and the other with $r < 0$.

Solution. We move three units to the left of the pole and follow up with a clockwise rotation of $\frac{\pi}{4}$ radians to plot $P\left(-3, -\frac{\pi}{4}\right)$. We see that P lies on the terminal side of $\frac{3\pi}{4}$.





Since *P* lies on the terminal side of $\frac{3\pi}{4}$, one alternative representation for *P* is $\left(3,\frac{3\pi}{4}\right)$. To find a different representation for *P* with $r=-3$, we may choose any angle coterminal with $-\frac{\pi}{4}$. We choose $\theta=\frac{7\pi}{4}$ for our final answer of $\left(-3,\frac{7\pi}{4}\right)$.



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Now that we have had some practice with plotting points in polar coordinates, it should come as no surprise that any given point expressed in polar coordinates has infinitely many other representations in polar coordinates. The following property of the polar coordinate system summarizes characteristics of different polar coordinates that determine the same point in the plane.

Equivalent Representations of Points in Polar Coordinates

Suppose (r,θ) and (r',θ') are polar coordinates where $r\neq 0$, $r'\neq 0$ and the angles are measured in radians. Then (r,θ) and (r',θ') determine the same point *P* if and only if one of the following is true:

- $r'=r$ and $\theta'=\theta+2\pi k$ for some integer *k*
- $r'=-r$ and $\theta'=\theta+(2k+1)\pi$ for some integer *k*

All polar coordinates of the form $(0,\theta)$ represent the pole regardless of the value of *θ*.

The key to understanding this result, and indeed the whole polar coordinate system, is to keep in mind that (r,θ) means (directed distance from pole, angle of rotation). If $r=0$, then no matter how much rotation is performed, the point never leaves the pole. Thus, $(0,\theta)$ is the pole for all values of *θ*.

Now let's assume that neither *r* nor *r*' is zero. If (r,θ) and (r',θ') determine the same point *P* then the (non-zero) distance from *P* to the pole in each case must be the same. Since this distance is controlled by the first coordinate, we have either $r'=r$ or $r'=-r$.

1. If $r'=r$ then, when plotting (r,θ) and (r',θ'), the angles θ and θ' have the same initial side. Hence, if (r,θ) and (r',θ') determine the same point, we must have that θ' is coterminal with θ. We know that this means $\theta'=\theta+2\pi k$ for some integer *k*, as required.

2. If, on the other hand, $r'=-r$, then when plotting (r,θ) and (r',θ') the initial side of θ' is rotated π radians away from the initial side of θ. In this case, θ' must be coterminal with $\pi+\theta$. Hence, $\theta'=\pi+\theta+2\pi k$ which we rewrite as $\theta'=\theta+(2k+1)\pi$ for some integer *k*.

Conversely,

1. If $r'=r$ and $\theta'=\theta+2\pi k$ for some integer *k*, then the points $P(r,\theta)$ and $P'(r',\theta')$ lie the same (directed) distance from the pole on the terminal sides of coterminal angles, and hence are the same point.

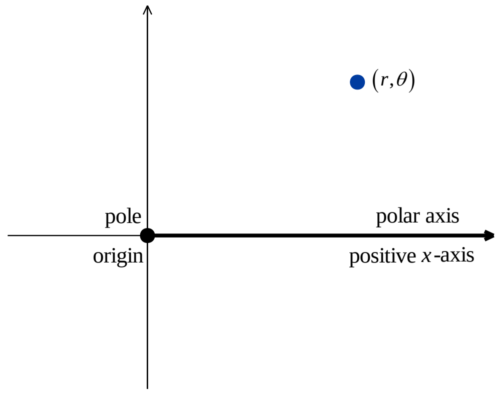
2. Suppose that $r'=-r$ and $\theta'=\theta+(2k+1)\pi$ for some integer *k*. To plot *P*, we first move a directed distance *r* from the pole; to plot P', our first step is to move the same distance from the pole as *P*, but in the opposite direction. At this intermediate stage, we have two points equidistant from the pole rotated exactly *π* radians apart. Since

$$\begin{aligned}\theta' &= \theta+(2k+1)\pi \\ &= (\theta+\pi)+2\pi k\end{aligned}$$

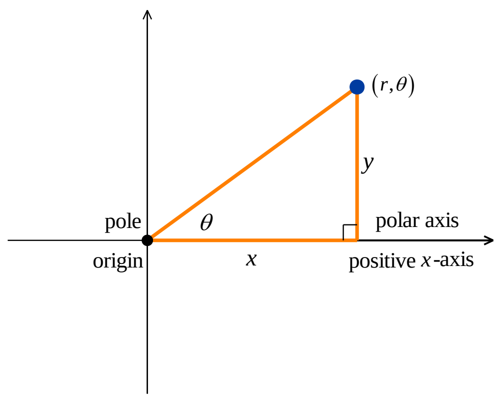
   for some integer *k*, we see that θ' is coterminal to $(\theta+\pi)$ and it is this extra *π* radians of rotation which aligns the points P and P'.

Converting Between Rectangular and Polar Coordinates

To move points from the polar coordinate system to the Cartesian (rectangular) coordinate system, or vice versa, we identify the pole and polar axis in the polar system to the origin and positive *x*-axis, respectively, in the rectangular system.



If we have a polar point (r,θ) in Quadrant I, we can form a right triangle by first dropping a perpendicular line segment from the point (r,θ) to the point $(r,0)$, on the positive x-axis, to form a vertical leg. To form a horizontal leg, we sketch the line segment from the origin to the point $(r,0)$. Finally, the hypotenuse is the line segment from the origin to the polar point (r,θ).



The lengths of the legs of this triangle, x and y, are the corresponding rectangular coordinates (x,y) for the polar point (r,θ). Using right triangle trigonometry, we can express x and y in terms of r and θ:

$$\cos(\theta)=\frac{x}{r} \qquad \sin(\theta)=\frac{y}{r}$$

$$x=r\cos(\theta) \qquad y=r\sin(\theta)$$

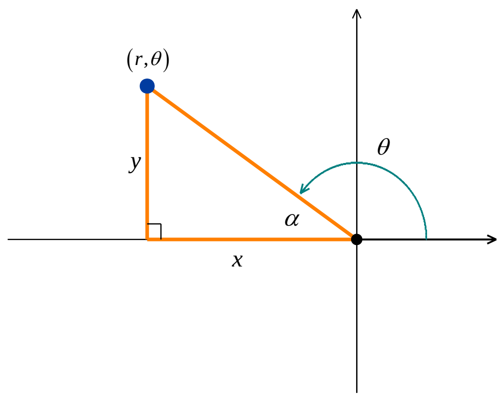
Translating a polar point to its rectangular representation is fairly straightforward using these two formulas for x and y. Suppose, on the other hand, we want to translate a rectangular point (x,y) to its polar representation (r,θ). We can find r using the Pythagorean Theorem.

$$x^2+y^2=r^2$$

Then, to determine θ, we use the tangent.

$$\tan(\theta) = \frac{y}{x}$$

For Quadrant II, III or IV, we can use a reference angle, α, and include the appropriate signs on *x* and *y*, as determined by the quadrant in which they lie.



Recall that, since $\arctan(\tan(\theta)) = \theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, to find an angle θ in Quadrant II or III it will be necessary to add π to obtain the correct angle. Also keep in mind that a point in polar coordinates can be expressed in many ways since $(r,\theta) = (r,\theta + 2\pi k)$ where *k* is an integer.

We get the following result.

Theorem 8.1. Conversion Between Rectangular and Polar Coordinates: Suppose *P* is represented in rectangular coordinates as (x, y) and in polar coordinates as (r,θ). Then

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ and $\tan(\theta) = \frac{y}{x}$ (provided $x \neq 0$)

To verify this result, we check out the three cases: $r > 0$, $r < 0$ and $r = 0$.

1. In the case $r > 0$, the theorem is an immediate consequence of **Theorem 2.6**. Recall that

$$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta)$$

$$\sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$$

We apply the quotient identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to verify that $\tan(\theta) = \frac{y}{x}$.

2. If $r < 0$, then we know an alternate representation for (r,θ) is $(-r,\theta+\pi)$. Using $\cos(\theta+\pi) = -\cos(\theta)$ and $\sin(\theta+\pi) = -\sin(\theta)$, we apply **Theorem 2.6** as follows.

$$\begin{aligned} x &= (-r)\cos(\theta+\pi) \\ &= (-r)(-\cos(\theta)) \\ &= r\cos(\theta) \end{aligned} \qquad \begin{aligned} y &= (-r)\sin(\theta+\pi) \\ &= (-r)(-\sin(\theta)) \\ &= r\sin(\theta) \end{aligned}$$

Moreover, $x^2 + y^2 = (-r)^2 = r^2$ and $\dfrac{y}{x} = \tan(\theta+\pi) = \tan(\theta)$, so the theorem is true in this case too.

3. The remaining case is $r = 0$, in which case $(r,\theta) = (0,\theta)$ is the pole. Since the pole is identified with the origin $(0,0)$ in rectangular coordinates, the theorem in this case amounts to checking '$0 = 0$'.

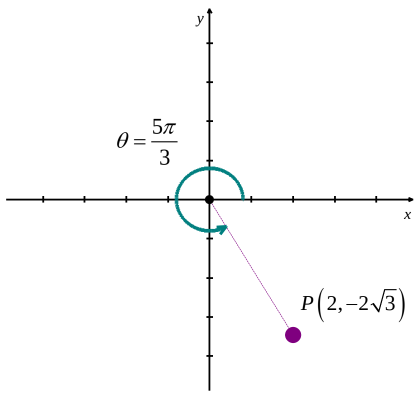
The following example puts **Theorem 8.1** to good use.

Example 8.1.8. Convert each point in rectangular coordinates given below into polar coordinates with $r \ge 0$ and $0 \le \theta < 2\pi$. Use exact values if possible and round any approximate values to two decimal places. Check your answer by converting back to rectangular coordinates.

1. $P\left(2,-2\sqrt{3}\right)$ 2. $Q(-3,-3)$ 3. $R(0,-3)$ 4. $S(-3,4)$

Solution.

1. Even though we are not explicitly told to do so, we can avoid many common mistakes by taking the time to plot the point $P\left(2,-2\sqrt{3}\right)$ before we do any calculations.



Plotting $P\left(2,-2\sqrt{3}\right)$ shows that it lies in Quadrant IV.

With $x = 2$ and $y = -2\sqrt{3}$, we get

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (2)^2 + \left(-2\sqrt{3}\right)^2 \\ &= 4 + 12 \\ &= 16 \end{aligned}$$

So $r = \pm 4$ and, since we are asked for $r \geq 0$, we choose $r = 4$. To find θ, we have that

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{-2\sqrt{3}}{2} \\ &= -\sqrt{3} \end{aligned}$$

This tells us θ has a reference angle of $\frac{\pi}{3}$, and since P lies in Quadrant IV, we know θ is a Quadrant IV angle. We are asked to have $0 \leq \theta < 2\pi$, so we choose $\theta = \frac{5\pi}{3}$. Hence, our answer is $\left(4, \frac{5\pi}{3}\right)$.

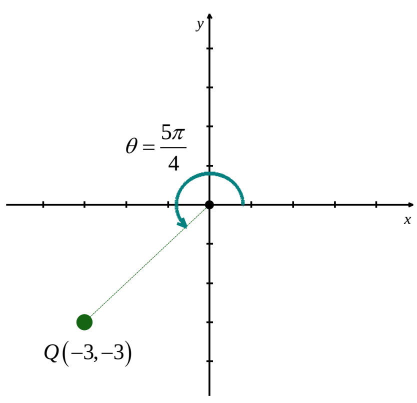
To check, we convert $(r,\theta) = \left(4, \frac{5\pi}{3}\right)$ back to rectangular coordinates and we find

$$\begin{aligned} x &= r\cos(\theta) \\ &= 4\cos\left(\frac{5\pi}{3}\right) \\ &= 4\left(\frac{1}{2}\right) \\ &= 2 \end{aligned} \qquad\qquad \begin{aligned} y &= r\sin(\theta) \\ &= 4\sin\left(\frac{5\pi}{3}\right) \\ &= 4\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

The result is the point $\left(2, -2\sqrt{3}\right)$ in rectangular coordinates, as required.

2. The point $Q(-3,-3)$ is in Quadrant III. Using $x = y = -3$, we get $r^2 = (-3)^2 + (-3)^2 = 18$ so

$$\begin{aligned} r &= \pm\sqrt{18} \\ &= \pm 3\sqrt{2} \\ &= 3\sqrt{2} \quad \textbf{since we are asked for } r \geq 0 \end{aligned}$$



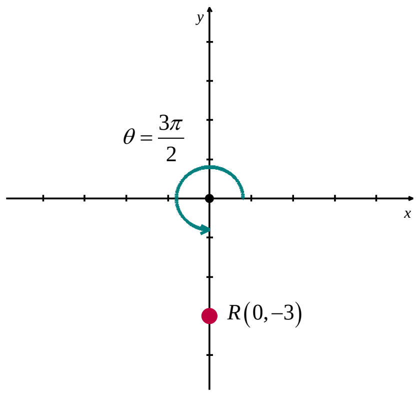
We find $\tan(\theta) = \frac{-3}{-3} = 1$, which means θ has a reference angle of $\frac{\pi}{4}$. Since Q lies in Quadrant III, we choose $\theta = \frac{5\pi}{4}$, which satisfies the requirement that $0 \leq \theta < 2\pi$. Our final answer is $(r,\theta) = \left(3\sqrt{2}, \frac{5\pi}{4}\right)$.

To check, we find

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$
$$= \left(3\sqrt{2}\right)\cos\left(\frac{5\pi}{4}\right) \qquad\qquad = \left(3\sqrt{2}\right)\sin\left(\frac{5\pi}{4}\right)$$
$$= \left(3\sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}\right) \qquad\qquad = \left(3\sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= -3 \qquad\qquad = -3$$

The resulting point $(-3,-3)$ verifies our solution.

3. The point $R(0,-3)$ lies along the negative *y*-axis. While we could go through the usual computation[4] to find the polar form of *R*, in this case we can find the polar coordinates of *R* using the definition. Since the pole is identified with the origin, we can easily tell the point *R* is 3 units from the pole, which means in the polar representation (r,θ) of *R* we know $r = \pm 3$. Since we require $r \geq 0$, we choose $r = 3$.

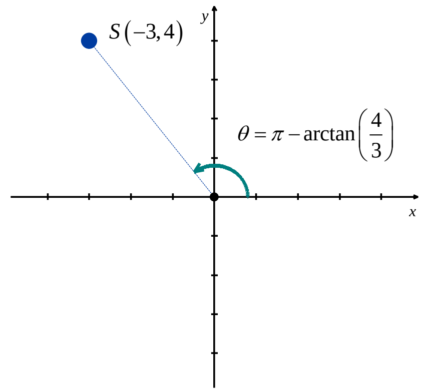


Concerning *θ*, the angle $\theta = \frac{3\pi}{2}$ satisfies $0 \leq \theta < 2\pi$ with its terminal side along the negative *y*-axis, so our answer is $\left(3, \frac{3\pi}{2}\right)$.

To check, we note

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$
$$= 3\cos\left(\frac{3\pi}{2}\right) \qquad\qquad = 3\sin\left(\frac{3\pi}{2}\right)$$
$$= (3)(0) \qquad\qquad = 3(-1)$$
$$= 0 \qquad\qquad = -3$$

[4] Since *x* = 0, we would have to determine *θ* geometrically.



4. The point $S(-3,4)$ lies in Quadrant II. With $x=-3$ and $y=4$, we get $r^2=(-3)^2+(4)^2=25$ so $r=\pm 5$. As usual, we choose $r=5\geq 0$ and proceed to determine θ. We have

$$\begin{aligned}\tan(\theta)&=\frac{y}{x}\\&=\frac{4}{-3}\\&=-\frac{4}{3}\end{aligned}$$

Since this isn't the tangent of one of the common angles, we resort to using the arctangent function. Since θ lies in Quadrant II and must satisfy $0\leq\theta<2\pi$, we choose $\theta=\pi-\arctan\left(\frac{4}{3}\right)$ radians. Hence, our answer is $(r,\theta)=\left(5,\pi-\arctan\left(\frac{4}{3}\right)\right)\approx(5,2.21)$.

To check our answers requires a bit of tenacity since we need to simplify expressions of the form $\cos\left(\pi-\arctan\left(\frac{4}{3}\right)\right)$ and $\sin\left(\pi-\arctan\left(\frac{4}{3}\right)\right)$. These are good review exercises and are hence left to the reader. We find $\cos\left(\pi-\arctan\left(\frac{4}{3}\right)\right)=-\frac{3}{5}$ and $\sin\left(\pi-\arctan\left(\frac{4}{3}\right)\right)=\frac{4}{5}$, so that

$$\begin{aligned}x&=r\cos(\theta)\\&=(5)\left(-\frac{3}{5}\right)\\&=-3\end{aligned}\qquad\qquad\begin{aligned}y&=r\sin(\theta)\\&=(5)\left(\frac{4}{5}\right)\\&=4\end{aligned}$$

□

Now that we've had practice converting representations of *points* between the rectangular and polar coordinate systems, we move on to the next section where we will convert *equations* from one system to another.

8.1 Exercises

In Exercises 1 – 16, plot the point given in polar coordinates and then give three different expressions for the point such that

(a) $r < 0$ and $0 \le \theta \le 2\pi$ (b) $r > 0$ and $\theta \le 0$ (c) $r > 0$ and $\theta \ge 2\pi$

1. $\left(2, \frac{\pi}{3}\right)$
2. $\left(5, \frac{7\pi}{4}\right)$
3. $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$
4. $\left(\frac{5}{2}, \frac{5\pi}{6}\right)$
5. $\left(12, -\frac{7\pi}{6}\right)$
6. $\left(3, -\frac{5\pi}{4}\right)$
7. $\left(2\sqrt{2}, -\pi\right)$
8. $\left(\frac{7}{2}, -\frac{13\pi}{6}\right)$
9. $\left(-20, 3\pi\right)$
10. $\left(-4, \frac{5\pi}{4}\right)$
11. $\left(-1, \frac{2\pi}{3}\right)$
12. $\left(-3, \frac{\pi}{2}\right)$
13. $\left(-3, -\frac{11\pi}{6}\right)$
14. $\left(-2.5, -\frac{\pi}{4}\right)$
15. $\left(-\sqrt{5}, -\frac{4\pi}{3}\right)$
16. $\left(-\pi, -\pi\right)$

In Exercises 17 – 36, convert the point from polar coordinates into rectangular coordinates.

17. $\left(5, \frac{7\pi}{4}\right)$
18. $\left(2, \frac{\pi}{3}\right)$
19. $\left(11, -\frac{7\pi}{6}\right)$
20. $\left(-20, 3\pi\right)$
21. $\left(\frac{3}{5}, \frac{\pi}{2}\right)$
22. $\left(-4, \frac{5\pi}{6}\right)$
23. $\left(9, \frac{7\pi}{2}\right)$
24. $\left(-5, -\frac{9\pi}{4}\right)$
25. $\left(42, \frac{13\pi}{6}\right)$
26. $\left(-117, 117\pi\right)$
27. $\left(6, \arctan(2)\right)$
28. $\left(10, \arctan(3)\right)$
29. $\left(-3, \arctan\left(\frac{4}{3}\right)\right)$
30. $\left(5, \arctan\left(-\frac{4}{3}\right)\right)$
31. $\left(2, \pi - \arctan\left(\frac{1}{2}\right)\right)$
32. $\left(-\frac{1}{2}, \pi - \arctan(5)\right)$
33. $\left(-1, \pi + \arctan\left(\frac{3}{4}\right)\right)$
34. $\left(\frac{2}{3}, \pi + \arctan\left(2\sqrt{2}\right)\right)$

35. $\left(\pi, \arctan(\pi)\right)$

36. $\left(13, \arctan\left(\frac{12}{5}\right)\right)$

In Exercises 37 – 56, convert the point from rectangular coordinates into polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$.

37. $(0,5)$

38. $\left(3,\sqrt{3}\right)$

39. $(7,-7)$

40. $\left(-3,-\sqrt{3}\right)$

41. $(-3,0)$

42. $\left(-\sqrt{2},\sqrt{2}\right)$

43. $\left(-4,-4\sqrt{3}\right)$

44. $\left(\frac{\sqrt{3}}{4},-\frac{1}{4}\right)$

45. $\left(-\frac{3}{10},-\frac{3\sqrt{3}}{10}\right)$

46. $\left(-\sqrt{5},-\sqrt{5}\right)$

47. $(6,8)$

48. $\left(\sqrt{5},2\sqrt{5}\right)$

49. $(-8,1)$

50. $\left(-2\sqrt{10},6\sqrt{10}\right)$

51. $(-5,-12)$

52. $\left(-\frac{\sqrt{5}}{15},-\frac{2\sqrt{5}}{15}\right)$

53. $(24,-7)$

54. $(12,-9)$

55. $\left(\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4}\right)$

56. $\left(-\frac{\sqrt{65}}{5},\frac{2\sqrt{65}}{5}\right)$

8.2 Polar Equations

Learning Objectives

In this section you will:

- **Convert an equation from rectangular coordinates into polar coordinates.**
- **Convert an equation from polar coordinates into rectangular coordinates.**

Just as we've used equations in x and y to represent relations in rectangular coordinates, equations in the variables r and θ represent relations in polar coordinates. We use **Theorem 8.1** to convert equations between the two systems.

Converting from Rectangular to Polar Coordinates

One strategy to convert an equation from rectangular to polar coordinates is to replace every occurrence of x with $r\cos(\theta)$ and every occurrence of y with $r\sin(\theta)$, and use identities to simplify. This is the technique we employ in the following three examples.

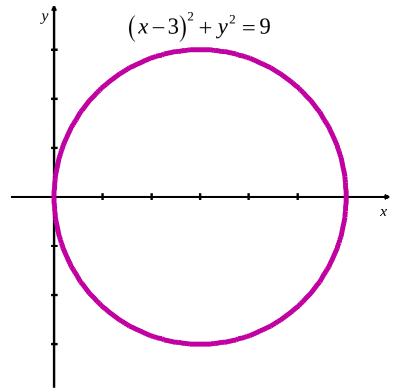
Example 8.2.1. Convert $(x-3)^2+y^2=9$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We start by substituting $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $(x-3)^2+y^2=9$ and then simplify. With no real direction in which to proceed, we follow our mathematical instincts and see where they take us.[1]

$$
\begin{aligned}
(x-3)^2+y^2&=9\\
(r\cos(\theta)-3)^2+(r\sin(\theta))^2&=9\\
r^2\cos^2(\theta)-6r\cos(\theta)+9+r^2\sin^2(\theta)&=9\\
r^2\cos^2(\theta)+r^2\sin^2(\theta)-6r\cos(\theta)&=0 && \text{after subtracting 9 from both sides}\\
r^2\left(\cos^2(\theta)+\sin^2(\theta)\right)-6r\cos(\theta)&=0\\
r^2-6r\cos(\theta)&=0 && \text{since } \cos^2(\theta)+\sin^2(\theta)=1\\
r(r-6\cos(\theta))&=0 && \text{after factoring}
\end{aligned}
$$

[1] Study this example and see what techniques are employed, then try your best to apply these techniques in the Exercises.

We get $r=0$ or $r=6\cos(\theta)$. Recognizing the equation $(x-3)^2+y^2=9$ as describing a circle, we exclude the first since $r=0$ describes only a point (namely the pole/origin). We choose $r=6\cos(\theta)$ for our final answer.



Note that when we substitute $\theta=\frac{\pi}{2}$ into $r=6\cos(\theta)$, we recover the point $r=0$, so we aren't losing anything by disregarding $r=0$.

□

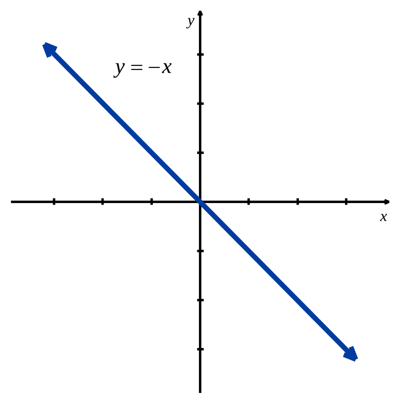
Example 8.2.2. Convert $y=-x$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We substitute $x=r\cos(\theta)$ and $y=r\sin(\theta)$ into $y=-x$.

$$\begin{aligned} y &= -x \\ r\sin(\theta) &= -r\cos(\theta) \\ r\cos(\theta)+r\sin(\theta) &= 0 && \textbf{after rearranging} \\ r\big(\cos(\theta)+\sin(\theta)\big) &= 0 && \textbf{after factoring} \end{aligned}$$

This gives $r=0$ or $\cos(\theta)+\sin(\theta)=0$. Solving the latter for θ, we get $\theta=-\frac{\pi}{4}+\pi k$ for integers k.

As we did in the previous example, we take a step back and think geometrically. We know $y=-x$ describes a line through the origin. As before, $r=0$ describes the origin but nothing else. Consider the equation $\theta=-\frac{\pi}{4}$. In this equation, the variable r is free, meaning it can assume any and all values including $r=0$. If we imagine plotting points $\left(r,-\frac{\pi}{4}\right)$ for all conceivable values of r (positive, negative and zero), we are essentially drawing the line containing the terminal side of $\theta=-\frac{\pi}{4}$ which is none other than $y=-x$.



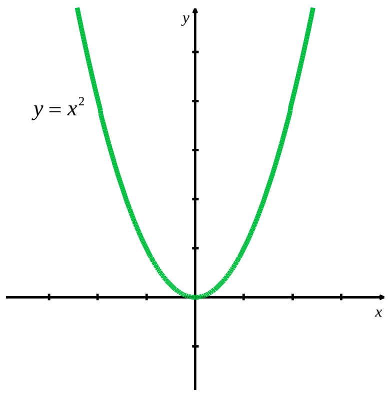
Hence, we can take as our final answer $\theta = -\dfrac{\pi}{4}$.[2]

□

Example 8.2.3. Convert $y = x^2$ from an equation in rectangular coordinates into an equation in polar coordinates.

Solution. We substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $y = x^2$.

$$\begin{aligned} y &= x^2 \\ r\sin(\theta) &= \left(r\cos(\theta)\right)^2 \\ r\sin(\theta) &= r^2\cos^2(\theta) \\ 0 &= r^2\cos^2(\theta) - r\sin(\theta) \\ 0 &= r\left(r\cos^2(\theta) - \sin(\theta)\right) \end{aligned}$$



Either $r = 0$ or $r\cos^2(\theta) = \sin(\theta)$. We can solve the latter equation for *r* by dividing both sides of the equation by $\cos^2(\theta)$.

As a general rule we never divide through by a quantity that may be equal to 0. In this particular case, we are safe since if $\cos^2(\theta) = 0$ then $\cos(\theta) = 0$ and, for the equation $r\cos^2(\theta) = \sin(\theta)$ to hold, then $\sin(\theta)$ would also have to be 0. Since there are no angles with both $\cos(\theta) = 0$ and $\sin(\theta) = 0$, we are not losing any information by dividing both sides of $r\cos^2(\theta) = \sin(\theta)$ by $\cos^2(\theta)$. Doing so, we get

$$\begin{aligned} r &= \frac{\sin(\theta)}{\cos^2(\theta)} \\ &= \frac{1}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)} \\ &= \sec(\theta)\tan(\theta) \end{aligned}$$

[2] We could take it to be $\theta = -\pi/4 + \pi k$ for *any* integer *k*.

As before, the $r = 0$ case is recovered in the solution $r = \sec(\theta)\tan(\theta)$ when $\theta = 0$. So we state our final solution as $r = \sec(\theta)\tan(\theta)$.

□

Converting from Polar to Rectangular Coordinates

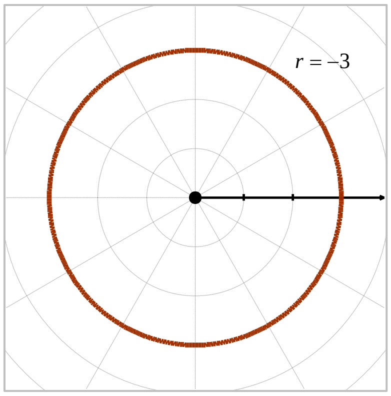
As a general rule, converting equations from polar to rectangular coordinates isn't as straight forward as the reverse process. We will begin with the strategy of rearranging the given polar equations so that the expressions $r^2 = x^2 + y^2$, $r\cos(\theta) = x$, $r\sin(\theta) = y$ and/or $\tan(\theta) = \dfrac{y}{x}$ present themselves.

Example 8.2.4. Convert $r = -3$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. Starting with $r = -3$, we can square both sides.

$$r = -3$$
$$r^2 = (-3)^2$$
$$r^2 = 9$$

We may now substitute $r^2 = x^2 + y^2$ to get the equation $x^2 + y^2 = 9$.



As we have seen, squaring an equation does not, in general, produce an equivalent equation. The concern here is that the equation $r^2 = 9$ might be satisfied by more points than $r = -3$. On the surface, this appears to be the case since $r^2 = 9$ is equivalent to $r = \pm 3$, not just $r = -3$. However, any point with polar coordinates $(3, \theta)$ can be represented as $(-3, \theta + \pi)$, which means any point (r, θ) whose polar coordinates satisfy the relation $r = \pm 3$ has an equivalent[3] representation which satisfies $r = -3$.

[3] Here, 'equivalent' means they represent the same point in the plane. As ordered pairs, (3,0) and (–3,π) are different, but when interpreted as polar coordinates, they correspond to the same point in the plane. Mathematically speaking, relations are sets of ordered pairs, so the equations $r^2 = 9$ and $r = -3$ represent different relations since they correspond to different sets of ordered pairs. Since polar coordinates were defined geometrically to describe the location of points in the plane, however, we concern ourselves only with ensuring that the sets of *points* in the plane generated by two equations are the same. This was not an issue, by the way, in algebra when we first defined relations as sets of points in the plane. Back then, a point in the plane was identified with a unique ordered pair given by its Cartesian coordinates.

Thus, we state our final solution as $x^2 + y^2 = 9$.

□

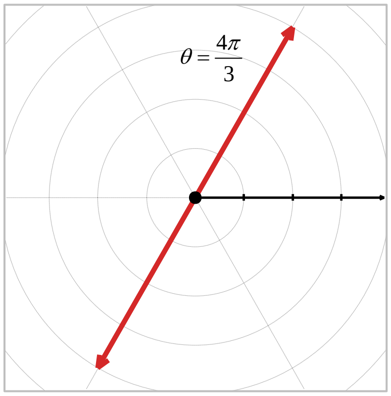
Example 8.2.5. Convert $\theta = \frac{4\pi}{3}$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. We begin by taking the tangent of both sides of the equation.

$$\theta = \frac{4\pi}{3}$$
$$\tan(\theta) = \tan\left(\frac{4\pi}{3}\right)$$
$$\tan(\theta) = \sqrt{3}$$

Since $\tan(\theta) = \frac{y}{x}$, we get the following.

$$\frac{y}{x} = \sqrt{3}$$
$$y = x\sqrt{3}$$



Of course, we pause a moment to wonder if, geometrically, the equations $\theta = \frac{4\pi}{3}$ and $y = x\sqrt{3}$ generate the same set of points.[4] The same argument presented in **Example 8.2.4** applies equally well here.

We conclude that our answer of $y = x\sqrt{3}$ is correct.

□

Example 8.2.6. Convert $r = 1 - \cos(\theta)$ from an equation in polar coordinates into an equation in rectangular coordinates.

Solution. Once again, we need to manipulate $r = 1 - \cos(\theta)$ a bit before using the conversion formulas given in **Theorem 8.1**. We could square both sides of this equation like we did in **Example 8.2.4** to

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[4] There are infinitely many solutions to $\tan(\theta) = \sqrt{3}$, and $\theta = \frac{4\pi}{3}$ is only one of them. Additionally, we went from $\frac{y}{x} = \sqrt{3}$, in which *x* cannot be 0, to $y = x\sqrt{3}$ in which we assume *x* can be 0.

obtain an r^2 on the left hand side, but that does nothing helpful for the right hand side. Instead, we multiply both sides by *r* and continue manipulating the equation so that we can apply the conversion formulas from **Theorem 8.1**.

$$r = 1 - \cos(\theta)$$

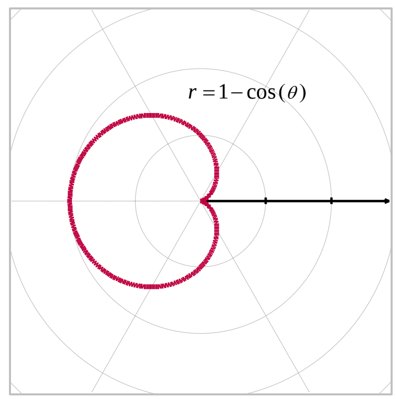
$$r^2 = r - r\cos(\theta) \quad \textbf{multiplying through by r}$$

$$r^2 + r\cos(\theta) = r \quad \textbf{adding } r\cos(\theta) \textbf{ to both sides}$$

$$\left(r^2 + r\cos(\theta)\right)^2 = r^2 \quad \textbf{squaring both sides}$$

$$\left(x^2 + y^2 + x\right)^2 = x^2 + y^2 \quad \textbf{substituting } r^2 = x^2 + y^2 \textbf{ and } r\cos(\theta) = x$$

In the last step, we applied **Theorem 8.1** and we now have the equation $\left(x^2 + y^2 + x\right)^2 = x^2 + y^2$ as a solution.



It can be shown that this is a legitimate solution by confirming the results when $r = 0$. We will forego this verification for now, as well as the verification that points with coordinates (r, θ) which satisfy $r^2 = \left(r^2 + r\cos(\theta)\right)^2$ will also satisfy $r = r^2 + r\cos(\theta)$.

To the right is a graph of the polar equation $r = 1 - \cos(\theta)$, from this example. This curve is referred to as a cardioid. In the next section, we will graph cardioids, along with other polar equations.

□

In practice, much of the pedantic verification of the equivalence of equations is left unsaid. Indeed, in most textbooks, squaring equations like $r = -3$ to arrive at $r^2 = 9$ happens without a second thought. Your instructor will ultimately decide how much, if any, justification is warranted.

8.2 Exercises

In Exercises 1 – 20, convert the equation from rectangular coordinates into polar coordinates. Solve for r in all but #4 through #7. In Exercises 4 – 7, you need to solve for θ.

1. $x = 6$
2. $x = -3$
3. $y = 7$
4. $y = 0$
5. $y = -x$
6. $y = x\sqrt{3}$
7. $y = 2x$
8. $x^2 + y^2 = 25$
9. $x^2 + y^2 = 117$
10. $y = 4x - 19$
11. $x = 3y + 1$
12. $y = -3x^2$
13. $4x = y^2$
14. $x^2 + y^2 - 2y = 0$
15. $x^2 - 4x + y^2 = 0$
16. $x^2 + y^2 = x$
17. $y^2 = 7y - x^2$
18. $(x+2)^2 + y^2 = 4$
19. $x^2 + (y-3)^2 = 9$
20. $4x^2 + 4\left(y - \frac{1}{2}\right)^2 = 1$

In Exercises 21 – 40, convert the equation from polar coordinates into rectangular coordinates.

21. $r = 7$
22. $r = -3$
23. $r = \sqrt{2}$
24. $\theta = \frac{\pi}{4}$
25. $\theta = \frac{2\pi}{3}$
26. $\theta = \pi$
27. $\theta = \frac{3\pi}{2}$
28. $r = 4\cos(\theta)$
29. $5r = \cos(\theta)$
30. $r = 3\sin(\theta)$
31. $r = -2\sin(\theta)$
32. $r = 7\sec(\theta)$
33. $12r = \csc(\theta)$
34. $r = -2\sec(\theta)$
35. $r = -\sqrt{5}\csc(\theta)$
36. $r = 2\sec(\theta)\tan(\theta)$
37. $r^2 = \sin(2\theta)$
38. $r = 1 - 2\cos(\theta)$
39. $r = 1 + \sin(\theta)$
40. $r = -\csc(\theta)\cot(\theta)$

41. Convert the origin $(0,0)$ into polar coordinates in four different ways.

42. With the help of your classmates, use the Law of Cosines to develop a formula for the distance between two points in polar coordinates.

8.3 Graphing Polar Equations

Learning Objectives

In this section you will:

- **Learn techniques for graphing polar equations.**
- **Graph polar equations.**

In this section, we discuss how to graph equations in polar coordinates on the rectangular coordinate plane. Since any given point in the plane has infinitely many different representations in polar coordinates, practice with graphing polar equations will be an essential part of the learning process. We begin with the Fundamental Graphing Principle for polar equations.

The Fundamental Graphing Principle for Polar Equations

The graph of an equation in polar coordinates is the set of points which satisfy the equation. That is, a point $P(r,\theta)$ is on the graph of an equation if and only if there is a representation of P, say (r',θ'), such that r' and θ' satisfy the equation.

Graphing a Simple Polar Equation – Constant Radius or Constant Angle

Our first example focuses on some of the more structurally simple polar equations.

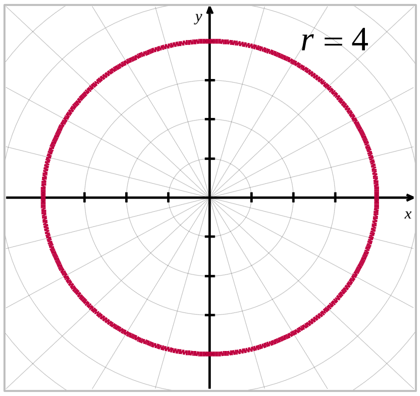
Example 8.3.1. Graph the following polar equations.

1. $r = 4$ 2. $r = -3\sqrt{2}$ 3. $\theta = \dfrac{5\pi}{4}$ 4. $\theta = -\dfrac{3\pi}{2}$

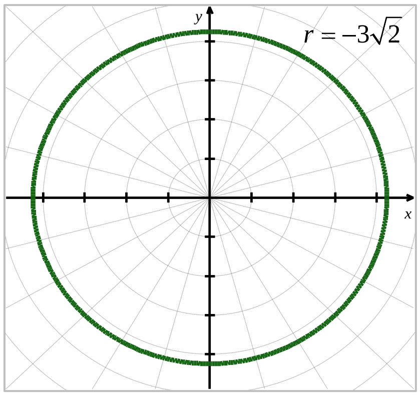
Solution. In each of these equations, only one of the variables r and θ is present, resulting in the missing variable taking on all values without restriction. This makes these graphs easier to visualize that others.

1. In the equation $r = 4$, θ is missing. The graph of this equation is, therefore, all points which have a polar coordinate representation $(4,\theta)$, for any choice of θ.

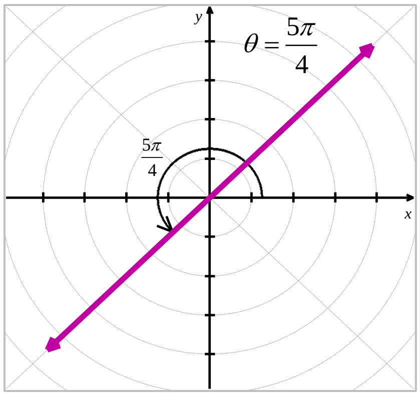
   Graphically, this translates into tracing out all of the points 4 units away from the origin. This is exactly the definition of circle, centered at the origin, with a radius of 4.



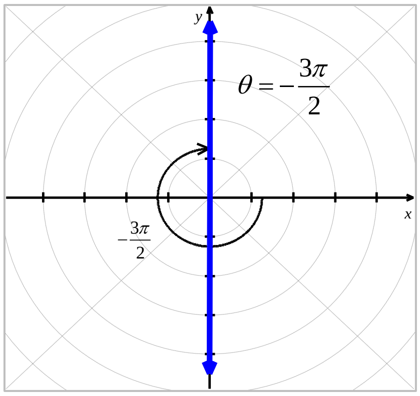
2. Once again, we have θ missing in the equation $r=-3\sqrt{2}$. Plotting all of the points of the form $\left(-3\sqrt{2},\theta\right)$ gives us a circle of radius $3\sqrt{2}$ centered at the origin.



3. In the equation $\theta=\dfrac{5\pi}{4}$, r is missing, so we plot all of the points with polar representations $\left(r,\dfrac{5\pi}{4}\right)$. What we find is that we are tracing out the line which contains the terminal side of $\theta=\dfrac{5\pi}{4}$ when plotted in standard position.



4. As in the previous problem, the variable r is missing in the equation, $\theta=-\dfrac{3\pi}{2}$. Plotting $\left(r,-\dfrac{3\pi}{2}\right)$ for various values of r shows us that we are tracing out the y-axis.



□

Hopefully, our experience in **Example 8.3.1** makes the following result clear.

Theorem 8.2. Graphs of Constant *r* and *θ*: Suppose a and α are constants, $a \neq 0$.

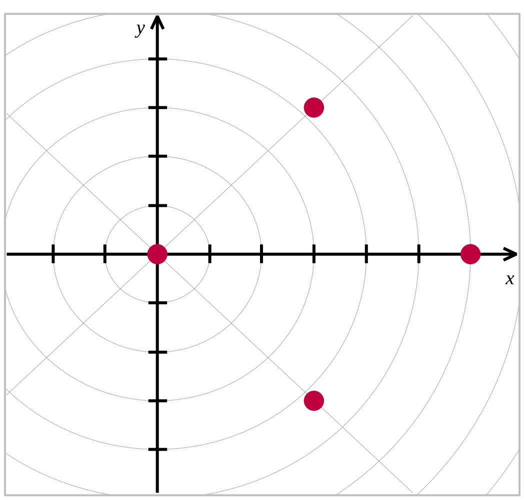
- The graph of the polar equation $r = a$ on the Cartesian plane is a circle centered at the origin of radius $|a|$.
- The graph of the polar equation $\theta = \alpha$ on the Cartesian plane is the line containing the terminal side of α when plotted in standard position.

Graphing Polar Equations Containing Variables *r* and *θ*

Suppose we wish to graph $r = 6\cos(\theta)$. A reasonable way to start is to treat θ as the independent variable, r as the dependent variable, evaluate $r = f(\theta)$ at some 'friendly' values of θ and plot the resulting points. We generate the table below, followed by a graph of the resulting points in the *xy*-plane.

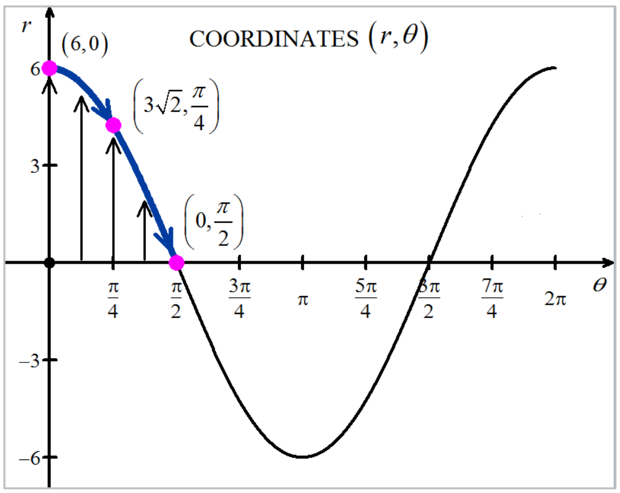
θ	$r = 6\cos(\theta)$	(r,θ)
0	6	$(6,0)$
$\frac{\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2},\frac{\pi}{4}\right)$
$\frac{\pi}{2}$	0	$\left(0,\frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2},\frac{3\pi}{4}\right)$
π	-6	$(-6,\pi)$

θ	$r = 6\cos(\theta)$	(r,θ)
$\frac{5\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2},\frac{5\pi}{4}\right)$
$\frac{3\pi}{2}$	0	$\left(0,\frac{3\pi}{2}\right)$
$\frac{7\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2},\frac{7\pi}{4}\right)$
2π	6	$(6,2\pi)$

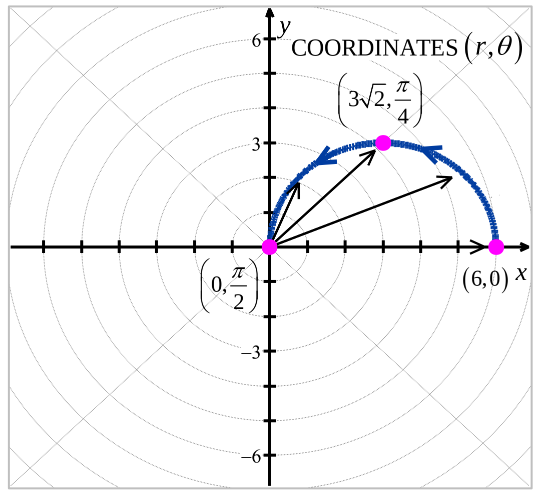


Despite having nine ordered pairs, we only get four distinct points on the graph. For this reason, we employ a slightly different strategy. We graph $r = 6\cos(\theta)$ on the θr-plane[1] and use it as a guide for graphing the equation on the *xy*-plane.

We first see that as θ ranges from 0 to $\frac{\pi}{2}$, r ranges from 6 to 0. In the *xy*-plane, this means that the curve starts 6 units from the origin on the positive *x*-axis, when $\theta = 0$, and gradually returns to the origin, at $\theta = \frac{\pi}{2}$.



$r = 6\cos(\theta)$ in the θr-plane

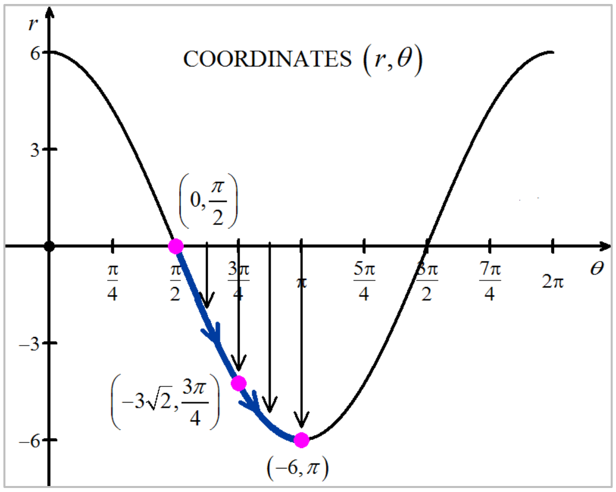


$r = 6\cos(\theta)$ in the *xy*-plane

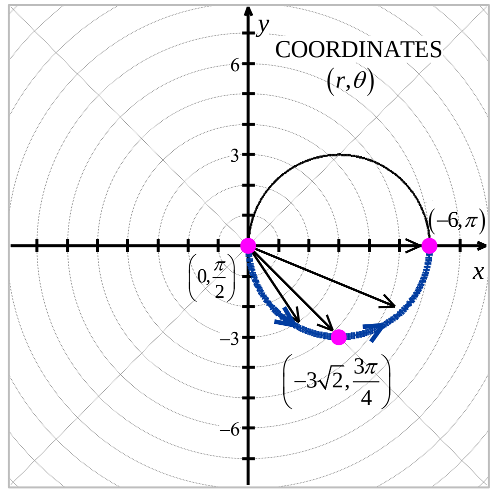
The arrows drawn in the above figures are meant to help you visualize this process. In the θr-plane, the arrows are drawn from the θ-axis to the curve $r = 6\cos(\theta)$. In the *xy*-plane, each of these arrows starts at the origin and is rotated through the corresponding angle θ, in accordance with how we plot polar coordinates. It is a less-precise way to generate the graph than computing the actual function values, but is markedly faster.

Next, we repeat the process as θ ranges from $\frac{\pi}{2}$ to π. Here, the r-values are all negative. This means that in the *xy*-plane, instead of graphing in Quadrant II, we graph in Quadrant IV, with all of the angle rotations starting from the negative *x*-axis.

[1] The graph looks exactly like $y = 6\cos(x)$ in the *xy*-plane, and for good reason. At this stage, we are just graphing the relationship between r and θ before we interpret them as polar coordinates (r,θ) on the *xy*-plane.



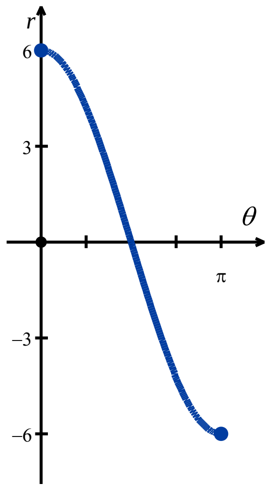
$r = 6\cos(\theta)$ in the θr-plane



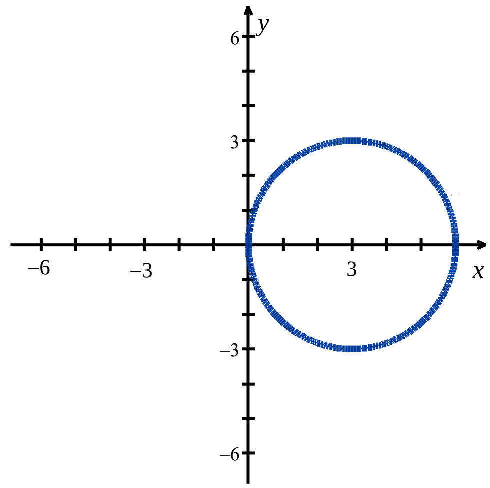
$r = 6\cos(\theta)$ in the xy-plane

As θ ranges from π to $\frac{3\pi}{2}$, the r values are still negative, which means the graph is traced out in Quadrant I instead of Quadrant III. Since the $|r|$ for these values of θ match the r values for θ in $\left[\theta, \frac{\pi}{2}\right]$, we have that the curve begins to retrace itself at this point. Proceeding further, we find that when $\frac{3\pi}{2} \le \theta \le 2\pi$, we retrace the part of the curve in Quadrant IV that we first traced out as $\frac{\pi}{2} \le \theta \le \pi$. The reader is invited to verify that plotting any range of θ outside the interval $[0,\pi]$ results in retracing some portion of the curve.[2] We present the final graph below.

[2] The graph of $r = 6\cos(\theta)$ looks suspiciously like a circle, for good reason. See **Example 8.2.1**.



$r = 6\cos(\theta)$ in the θr-plane

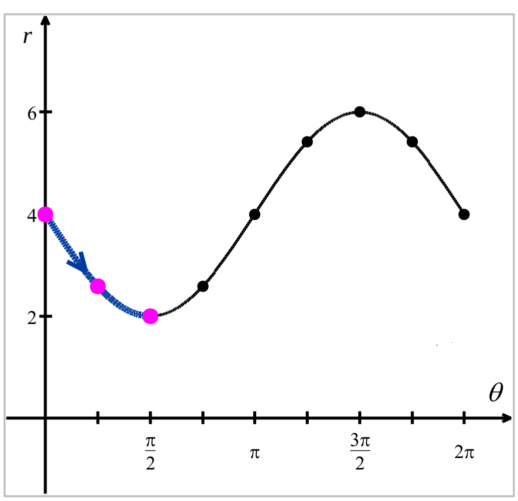


$r = 6\cos(\theta)$ in the xy-plane

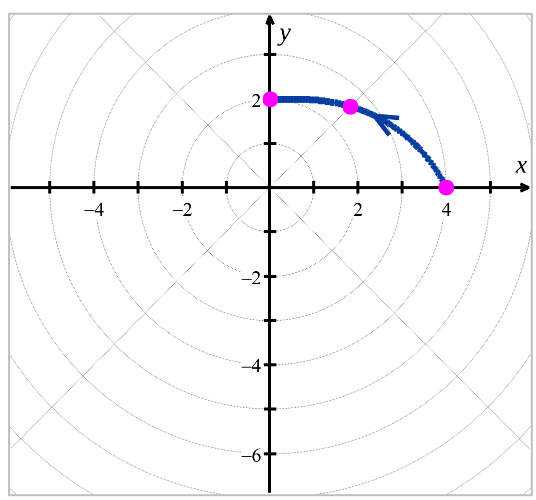
Example 8.3.2. Graph the polar equation $r = 4 - 2\sin(\theta)$.

Solution. We first plot the fundamental cycle of $r = 4 - 2\sin(\theta)$ on the θr-axes. To help us visualize what is going on graphically, we divide up $[0, 2\pi]$ into the usual four subintervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$, and proceed as we did above.

1. As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 4 to 2. This means that the curve in the xy-plane starts 4 units from the origin on the positive x-axis and gradually pulls in toward a point 2 units from the origin on the positive y-axis.

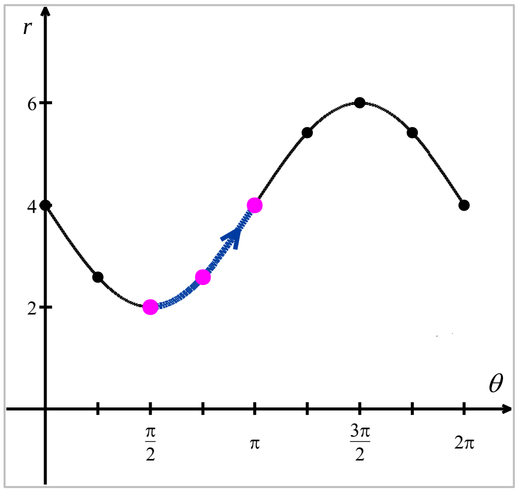


$r = 4 - 2\sin(\theta)$ in the θr-plane

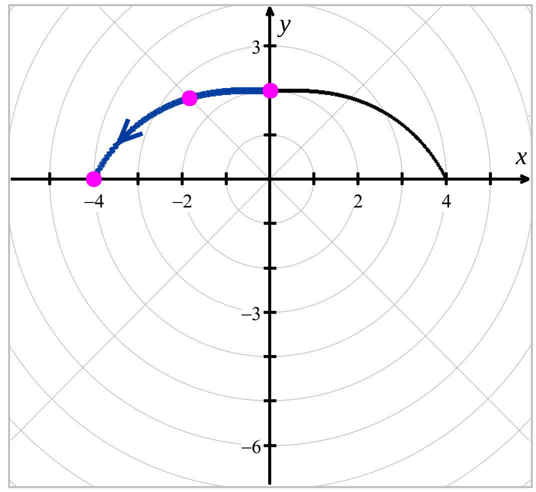


$r = 4 - 2\sin(\theta)$ in the xy-plane

2. Next, as θ runs from $\frac{\pi}{2}$ to π, we see that r increases from 2 to 4. In the *xy*-plane, picking up where we left off, we gradually pull the graph toward the point 4 units away from the origin on the negative *x*-axis.

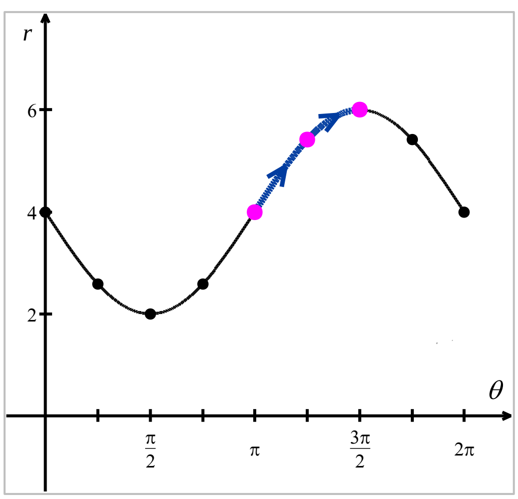


$r = 4 - 2\sin(\theta)$ in the θr-plane

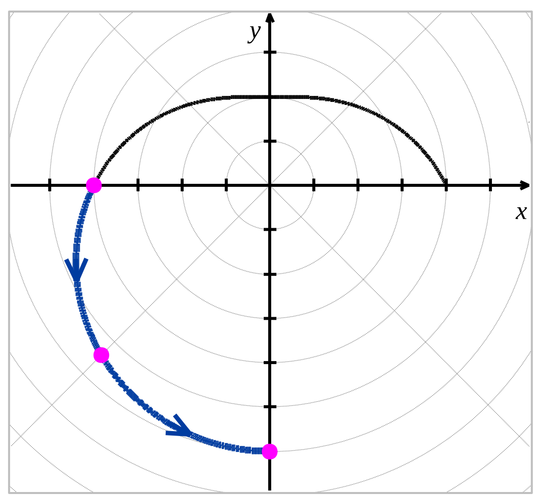


$r = 4 - 2\sin(\theta)$ in the *xy*-plane

3. Over the interval $\left[\pi, \frac{3\pi}{2}\right]$, we see that r increases from 4 to 6. On the *xy*-plane, the curve sweeps out away from the negative *x*-axis toward the negative *y*-axis.

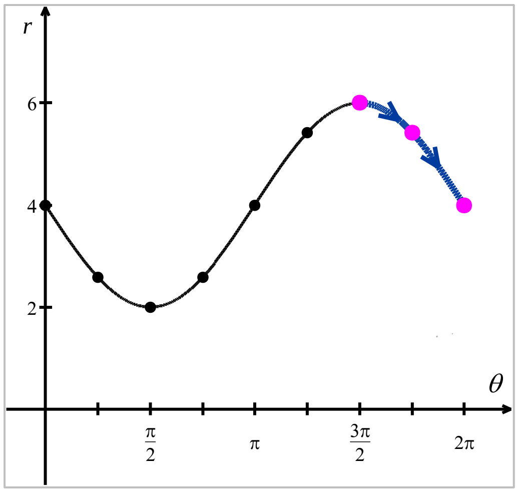


$r = 4 - 2\sin(\theta)$ in the θr-plane

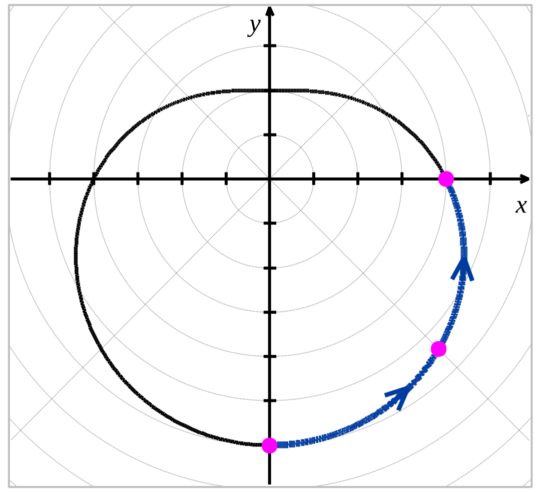


$r = 4 - 2\sin(\theta)$ in the *xy*-plane

4. Finally, as θ takes on values from $\frac{3\pi}{2}$ to 2π, r decreases from 6 back to 4. The graph on the *xy*-plane pulls in from the negative *y*-axis to finish where we started.

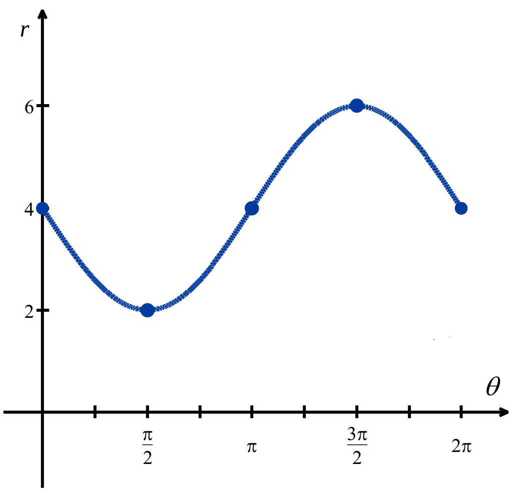


$r = 4 - 2\sin(\theta)$ in the θr-plane

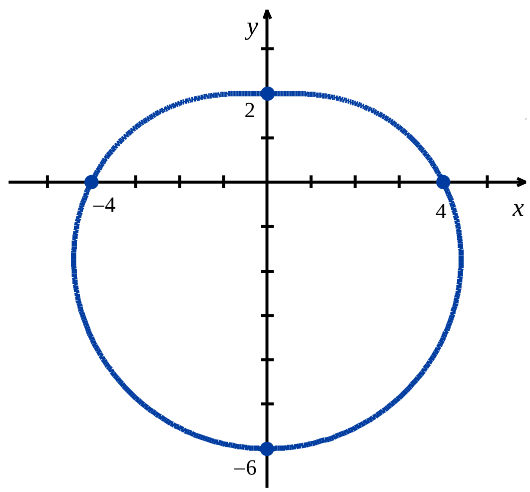


$r = 4 - 2\sin(\theta)$ in the xy-plane

We leave it to the reader to verify that plotting points corresponding to values of θ outside the interval $[0, 2\pi]$ results in retracing portions of the curve, so we are finished.



$r = 4 - 2\sin(\theta)$ in the θr-plane



$r = 4 - 2\sin(\theta)$ in the xy-plane

□

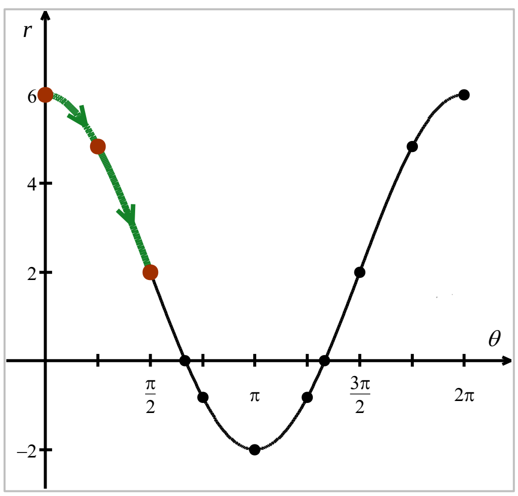
Example 8.3.3. Graph the polar equation $r = 2 + 4\cos(\theta)$.

Solution. The first thing to note when graphing $r = 2 + 4\cos(\theta)$ on the θr-plane over the interval $[0, 2\pi]$ is that the graph crosses through the θ-axis. This corresponds to the graph of the curve passing through the origin in the xy-plane, and our first task is to determine when this happens by determining the values of θ for which $r = 0$.

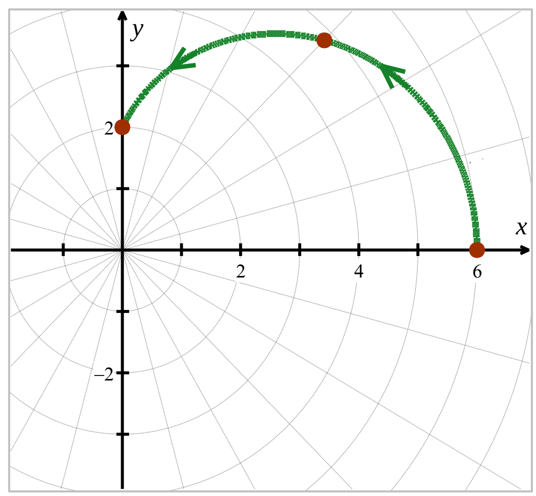
$$r = 0$$
$$2 + 4\cos(\theta) = 0$$
$$\cos(\theta) = -\frac{1}{2}$$

Solving for θ in $[0, 2\pi]$ gives $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Since these values of θ are important geometrically, we break the interval $[0, 2\pi]$ into six subintervals: $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, $\left[\frac{2\pi}{3}, \pi\right]$, $\left[\pi, \frac{4\pi}{3}\right]$, $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

1. As θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 6 to 2. Plotting this on the xy-plane, we start 6 units out from the origin on the positive x-axis and slowly pull in towards the positive y-axis.

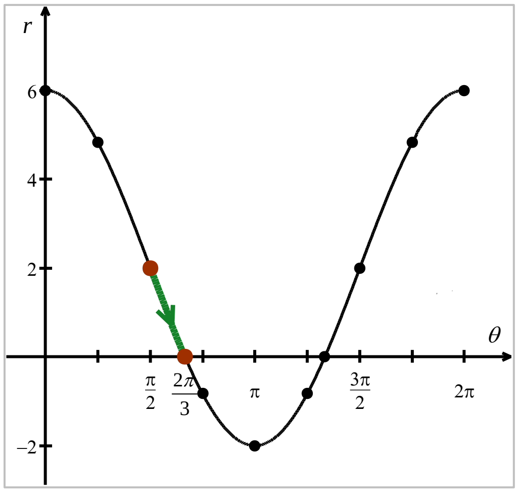


$r = 2 + 4\cos(\theta)$ in the θr-plane

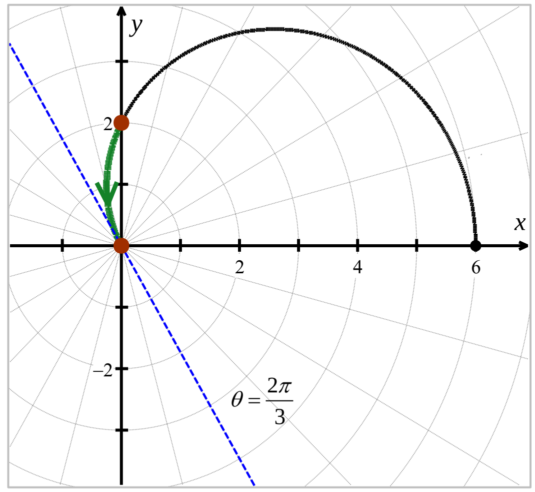


$r = 2 + 4\cos(\theta)$ in the xy-plane

2. On the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, r decreases from 2 to 0, which means the graph is heading into (and will eventually cross through) the origin. Not only do we reach the origin when $\theta = \frac{2\pi}{3}$, the curve hugs the line $\theta = \frac{2\pi}{3}$ as it approaches the origin.

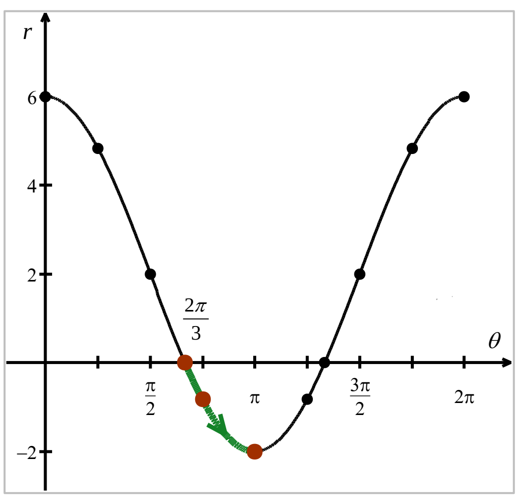


$r = 2 + 4\cos(\theta)$ in the θr-plane

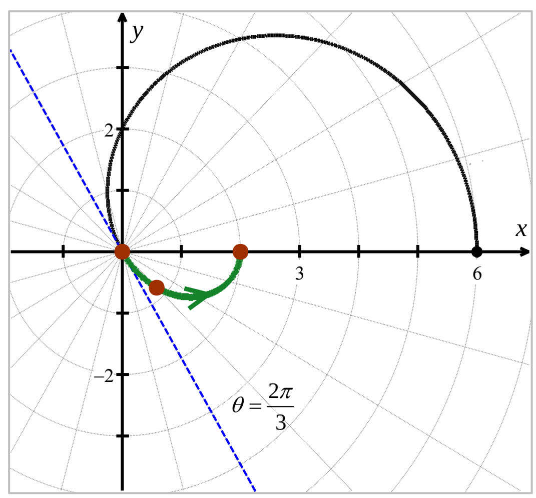


$r = 2 + 4\cos(\theta)$ in the xy-plane

3. On the interval $\left[\frac{2\pi}{3}, \pi\right]$, r ranges from 0 to -2. Since $r \le 0$, the curve passes through the origin in the xy-plane, following the line $\theta = \frac{2\pi}{3}$ and continues upwards through Quadrant IV toward the positive x-axis. With $|r|$ increasing from 0 to 2, the curve pulls away from the origin to finish at a point on the positive x-axis.

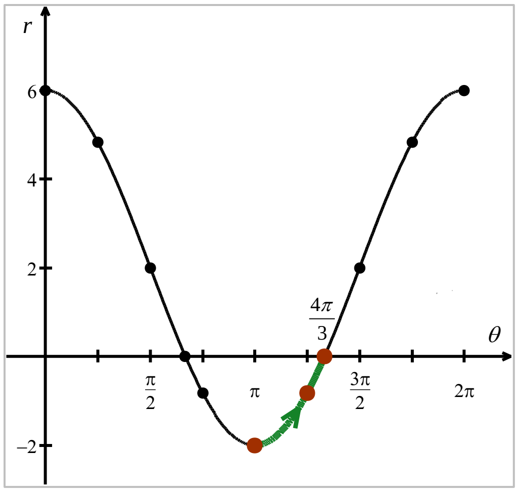


$r = 2 + 4\cos(\theta)$ in the θr-plane

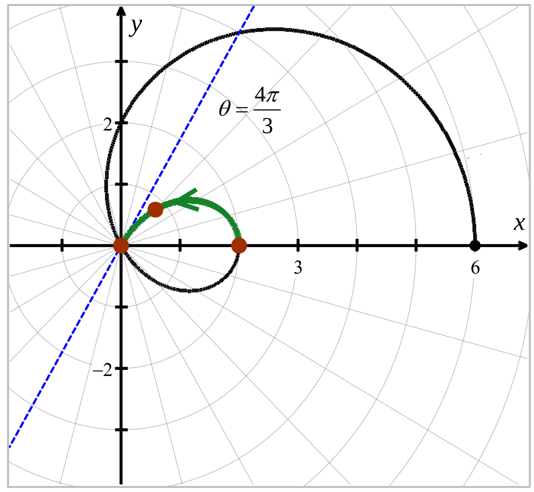


$r = 2 + 4\cos(\theta)$ in the xy-plane

4. Next, as θ progresses from π to $\frac{4\pi}{3}$, r ranges from -2 to 0. Since $r \le 0$, we continue our graph in the first quadrant, heading into the origin along the line $\theta = \frac{4\pi}{3}$.

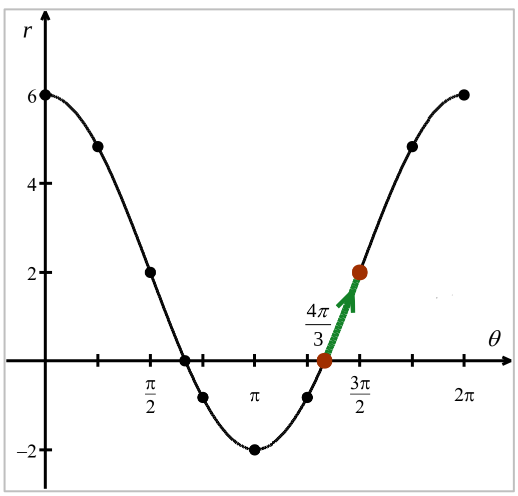


$r = 2 + 4\cos(\theta)$ in the *θr*-plane

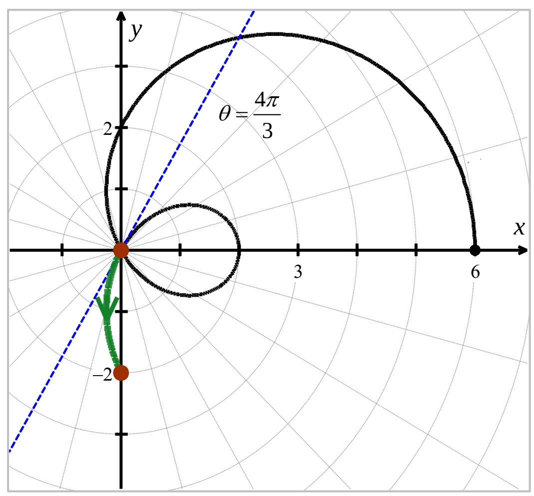


$r = 2 + 4\cos(\theta)$ in the *xy*-plane

5. On the interval $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$, *r* returns to positive values and increases from 0 to 2. We hug the line $\theta = \frac{4\pi}{3}$ as we move through the origin and head toward the negative *y*-axis.

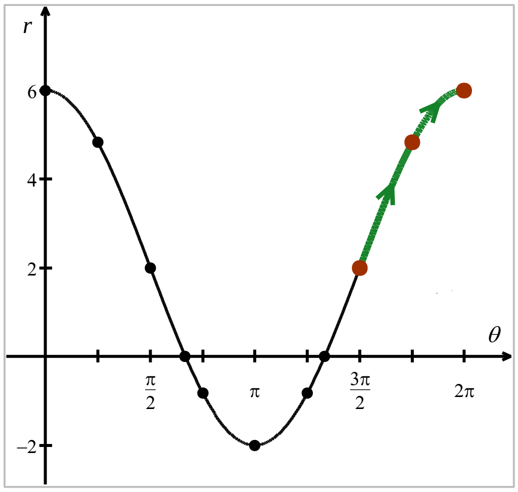


$r = 2 + 4\cos(\theta)$ in the *θr*-plane

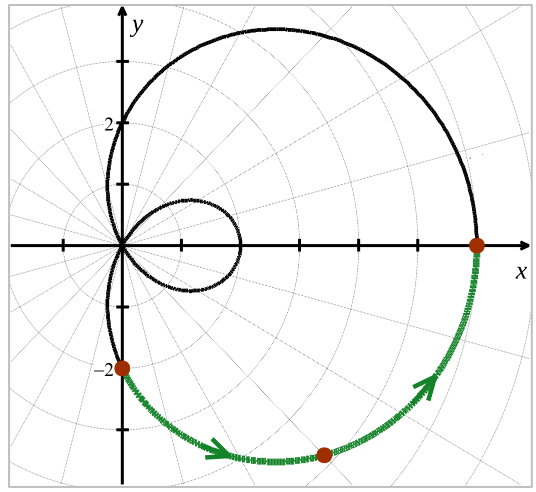


$r = 2 + 4\cos(\theta)$ in the *xy*-plane

6. In the last step, we find that as θ runs through $\frac{3\pi}{2}$ to 2π, *r* increases from 2 to 6, and we end up back where we started, 6 units from the origin on the positive x-axis.

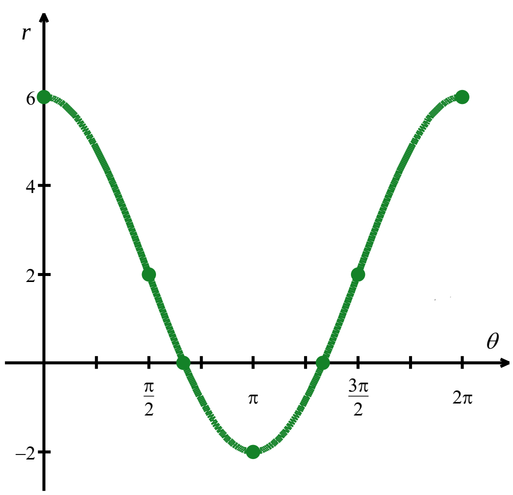


$r = 2 + 4\cos(\theta)$ in the θr-plane

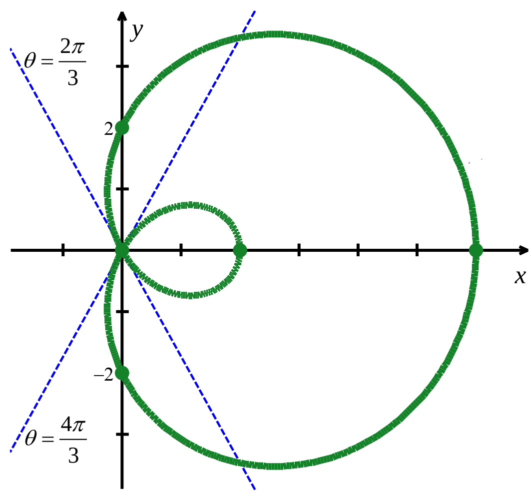


$r = 2 + 4\cos(\theta)$ in the xy-plane

Again, we invite the reader to show that plotting the curve for values of θ outside $[0, 2\pi]$ results in retracing a portion of the curve already traced. Our final graph is below.



$r = 2 + 4\cos(\theta)$ in the θr-plane



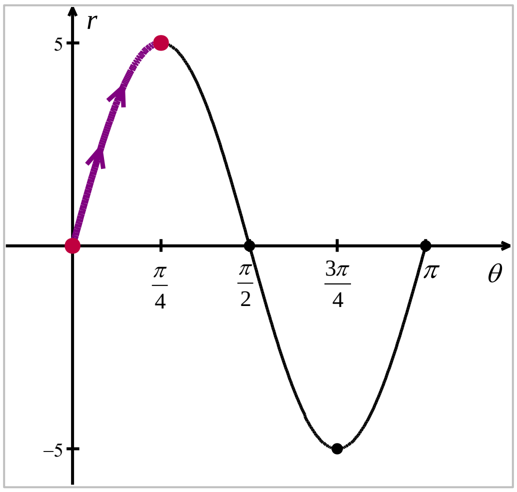
$r = 2 + 4\cos(\theta)$ in the xy-plane

□

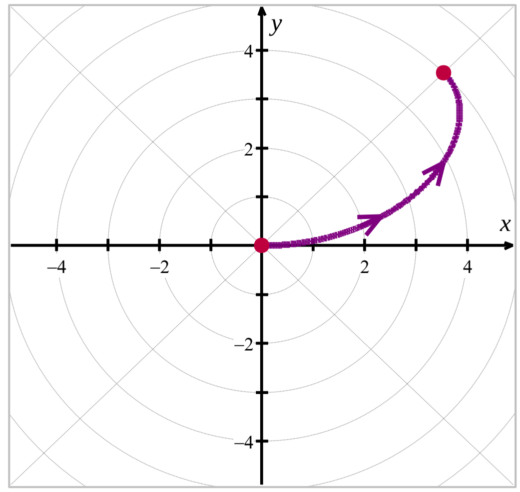
Example 8.3.4. Graph the polar equation $r = 5\sin(2\theta)$.

Solution. As usual, we start by graphing a fundamental cycle of $r = 5\sin(2\theta)$ in the θr-plane, which in this case occurs as θ ranges from 0 to π. We partition our interval into subintervals to help us with the graphing, namely $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\left[\frac{3\pi}{4}, \pi\right]$.

1. As θ ranges from 0 to $\frac{\pi}{4}$, r increases from 0 to 5. This means that the graph of $r = 5\sin(2\theta)$ in the *xy*-plane starts at the origin and gradually sweeps our so it is 5 units away from the origin on the line $\theta = \frac{\pi}{4}$.

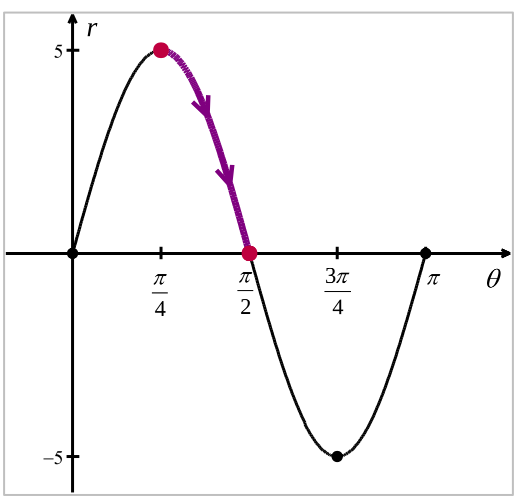


$r = 5\sin(2\theta)$ in the θr-plane

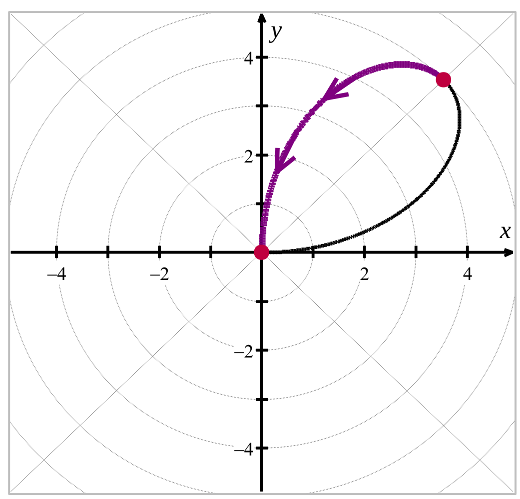


$r = 5\sin(2\theta)$ in the *xy*-plane

2. Next, we see that r decreases from 5 to 0 as θ runs through $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and, furthermore, r is heading negative as θ crosses $\frac{\pi}{2}$. Hence, we draw the curve hugging the line $\theta = \frac{\pi}{2}$ (the *y*-axis) as the curve heads to the origin.

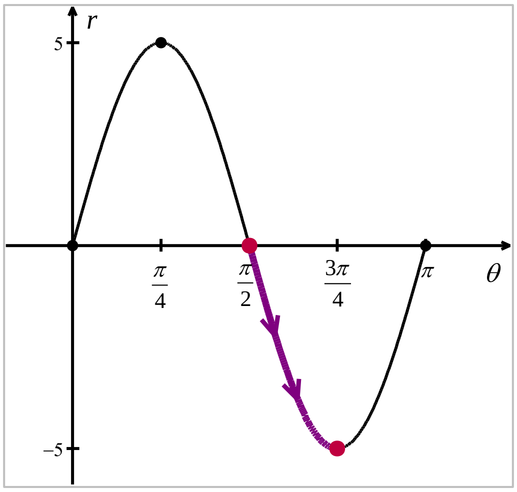


$r = 5\sin(2\theta)$ in the θr-plane

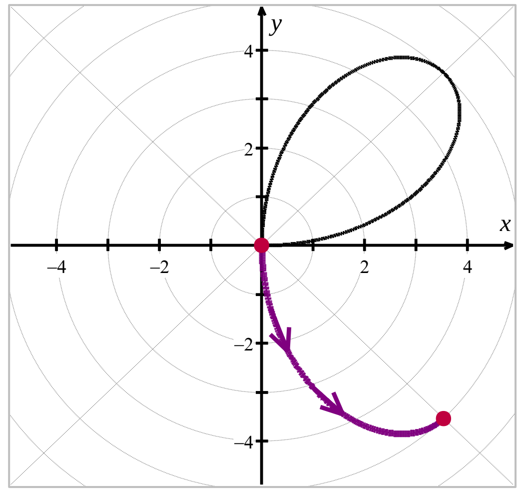


$r = 5\sin(2\theta)$ in the *xy*-plane

3. As θ runs from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r becomes negative and ranges from 0 to -5. Since $r \le 0$, the curve pulls away from the negative y-axis into Quadrant IV.

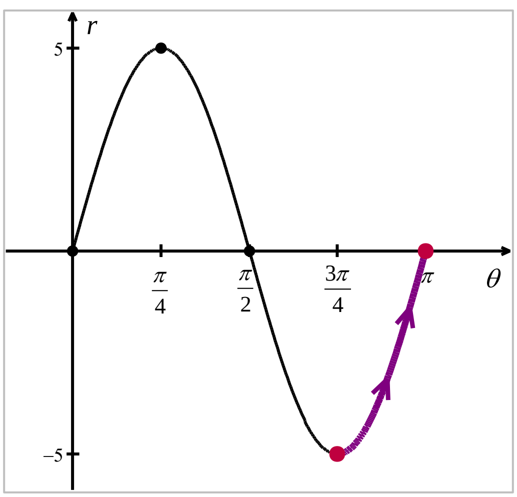


$r = 5\sin(2\theta)$ in the θr-plane

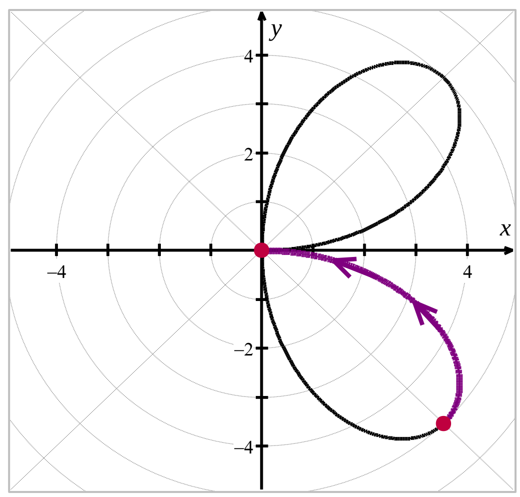


$r = 5\sin(2\theta)$ in the xy-plane

4. For $\frac{3\pi}{4} \le \theta \le \pi$, r increases from -5 to 0, so the curve pulls back to the origin.

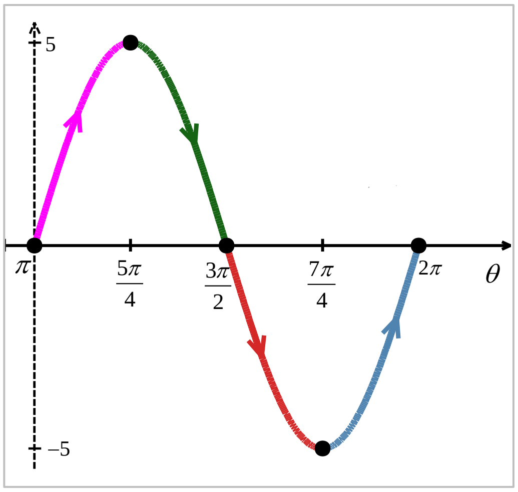


$r = 5\sin(2\theta)$ in the θr-plane

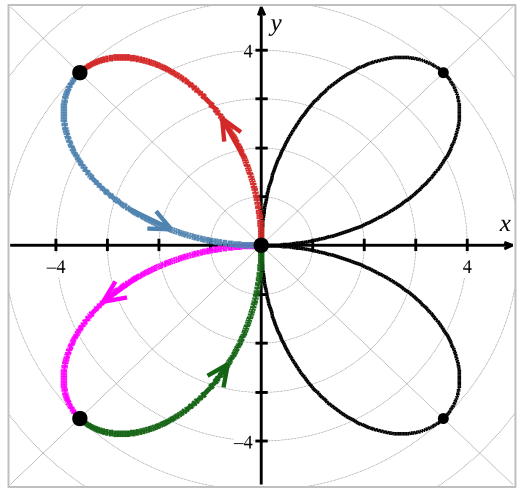


$r = 5\sin(2\theta)$ in the xy-plane

Even though we have finished with one complete cycle of $r = 5\sin(2\theta)$, if we continue plotting beyond $\theta = \pi$, we find that the curve continues into the third quadrant! Below we present a graph of a second cycle of $r = 5\sin(2\theta)$ which continues on from the first.

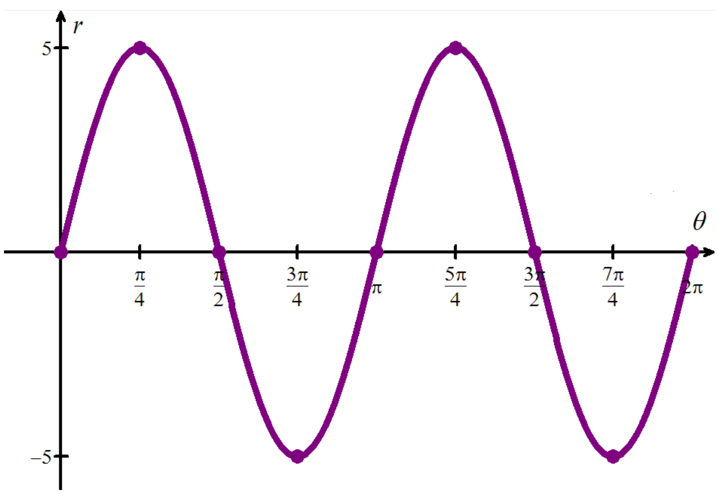


$r = 5\sin(2\theta)$ in the θr-plane

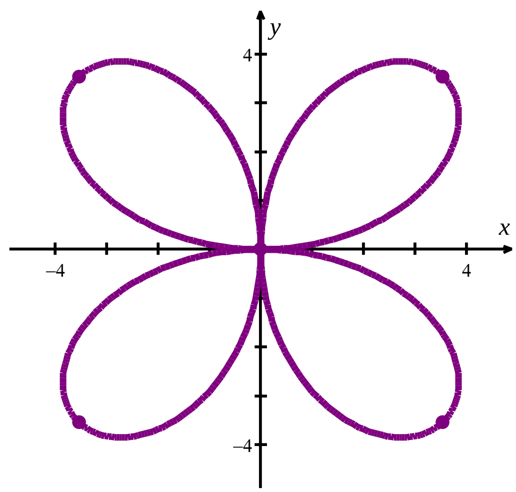


$r = 5\sin(2\theta)$ in the xy-plane

We have the final graph below.



$r = 5\sin(2\theta)$ in the θr-plane



$r = 5\sin(2\theta)$ in the xy-plane

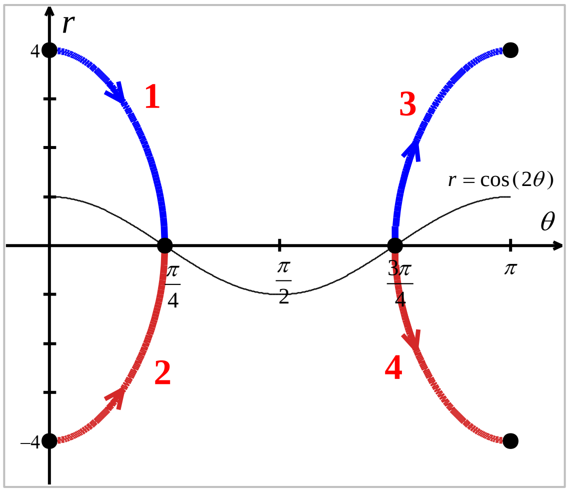
□

Example 8.3.5. Graph $r^2 = 16\cos(2\theta)$.

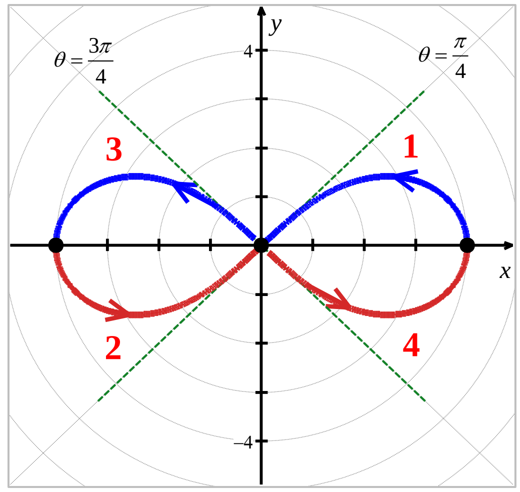
Solution. Graphing $r^2 = 16\cos(2\theta)$ is complicated by the r^2, so we solve to get

$$r = \pm\sqrt{16\cos(2\theta)}$$
$$r = \pm 4\sqrt{\cos(2\theta)}$$

How do we sketch such a curve? First off, we sketch a fundamental period of $r = \cos(2\theta)$, which is in the figure below. When $\cos(2\theta) < 0$, $\sqrt{\cos(2\theta)}$ is undefined, so we don't have any values on the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. On the intervals which remain, $\cos(2\theta)$ ranges from 0 to 1, inclusive. Hence, $\sqrt{\cos(2\theta)}$ ranges from 0 to 1 as well.[3] From this, we know $r = \pm 4\sqrt{\cos(2\theta)}$ ranges continuously from 0 to ±4, respectively. Below we graph both $r = 4\sqrt{\cos(2\theta)}$ and $r = -4\sqrt{\cos(2\theta)}$ on the θr-plane and use them to sketch the corresponding pieces of the curve $r^2 = 16\cos(2\theta)$ in the xy-plane.



$r = 4\sqrt{\cos(2\theta)}$ and $r = -4\sqrt{\cos(2\theta)}$
in the θr-plane



$r = 4\sqrt{\cos(2\theta)}$ and $r = -4\sqrt{\cos(2\theta)}$
in the xy-plane

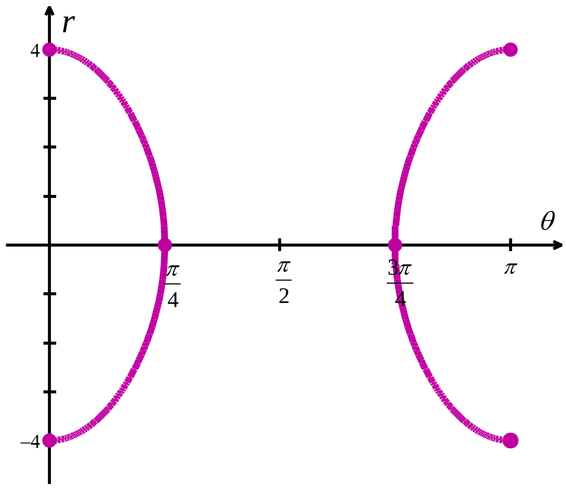
As we have seen in earlier examples, the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, which are the zeros of the functions $r = \pm 4\sqrt{\cos(2\theta)}$, serve as guides for us to draw the curve as it passes through the origin.

As we plot points corresponding to values of θ outside of the interval $[0, \pi]$, we find ourselves retracing parts of the curve,[4] so our final answer is below.

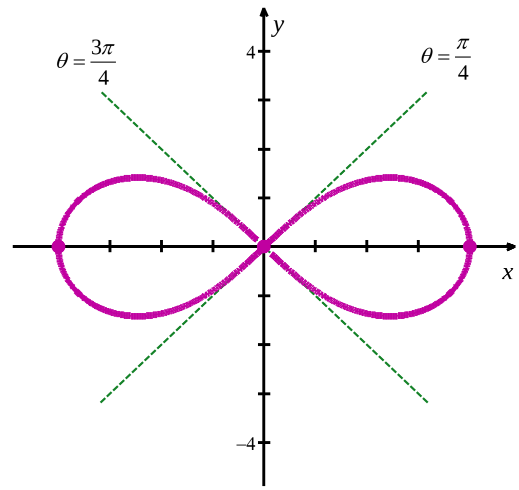
---

[3] Owing to the relationship between $y = x$ and $y = \sqrt{x}$ over $[0,1]$, we also know $\sqrt{\cos(2\theta)} \geq \cos(2\theta)$ wherever the former is defined.

[4] In this case, we could have generated the entire graph by using just the plot $r = 4\sqrt{\cos(2\theta)}$, but graphed over the interval $[0, 2\pi]$ in the θr-plane. We leave the details to the reader.



$r = \pm 4\sqrt{\cos(2\theta)}$ in the θr-plane



$r^2 = 16\cos(2\theta)$ in the xy-plane

□

A few remarks are in order.

1. There is no relation, in general, between the period of the function $f(\theta)$ and the length of the interval required to sketch the complete graph of $r = f(\theta)$ in the *xy*-plane. As we saw at the beginning of this section, despite the fact that the period of $f(\theta) = 6\cos(\theta)$ is 2π, we sketched the complete graph of $r = 6\cos(\theta)$ in the *xy*-plane just using the values of θ as θ ranged from 0 to π. In **Example 8.3.4**, the period of $f(\theta) = 5\sin(2\theta)$ is π, but in order to obtain the complete graph of $r = 5\sin(2\theta)$ we needed to run θ from 0 to 2π. While many of the 'common' polar graphs can be grouped into families,[5] taking the time to work through each graph in the manner presented here is the best way to not only understand the polar coordinate system but also prepare you for what is needed in Calculus.

2. The symmetry seen in the examples is a common occurrence when graphing polar equations. In addition to the usual kinds of symmetry discussed up to this point in the text (symmetry about each axis and the origin), it is possible to talk about *rotational* symmetry. Keep rotational symmetry in mind as you work through the Exercises.

---

[5] **Example 8.3.2** and **Example 8.3.3** are examples of **limacons.** **Example 8.3.4** is an example of a **polar rose**, and **Example 8.3.5** is the famous **Lemniscate of Bernoulli**.

8.3 Exercises

In Exercises 1 – 20, plot the graph of the polar equation by hand, without the aid of a calculator. Carefully label your graphs.

1. Circle: $r = 6\sin(\theta)$
2. Circle: $r = 2\cos(\theta)$
3. Rose: $r = 2\sin(2\theta)$
4. Rose: $r = 4\cos(2\theta)$
5. Rose: $r = 5\sin(3\theta)$
6. Rose: $r = \cos(5\theta)$
7. Rose: $r = \sin(4\theta)$
8. Rose: $r = 3\cos(4\theta)$
9. Cardioid: $r = 3 - 3\cos(\theta)$
10. Cardioid: $r = 5 + 5\sin(\theta)$
11. Cardioid: $r = 2 + 2\cos(\theta)$
12. Cardioid: $r = 1 - \sin(\theta)$
13. Limacon: $r = 1 - 2\cos(\theta)$
14. Limacon: $r = 1 - 2\sin(\theta)$
15. Limacon: $r = 2\sqrt{3} + 4\cos(\theta)$
16. Limacon: $r = 3 - 5\cos(\theta)$
17. Limacon: $r = 3 - 5\sin(\theta)$
18. Limacon: $r = 2 + 7\sin(\theta)$
19. Lemniscate: $r^2 = \sin(2\theta)$
20. Lemniscate: $r^2 = 4\cos(2\theta)$

Exercises 21 – 30 give you some curves to graph using a graphing calculator or other form of technology. Notice that some of the curves have explicit bounds on θ and others do not.

21. $r = \theta$, $0 \le \theta \le 12\pi$
22. $r = \ln(\theta)$, $1 \le \theta \le 12\pi$
23. $r = e^{0.1\theta}$, $0 \le \theta \le 12\pi$
24. $r = \theta^3 - \theta$, $-1.2 \le \theta \le 1.2$
25. $r = \sin(5\theta) - 3\cos(\theta)$
26. $r = \sin^3\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{3}\right)$
27. $r = \arctan(\theta)$, $-\pi \le \theta \le \pi$
28. $r = \dfrac{1}{1 - \cos(\theta)}$
29. $r = \dfrac{1}{2 - \cos(\theta)}$
30. $r = \dfrac{1}{2 - 3\cos(\theta)}$

31. How many petals does the polar rose $r = \sin(2\theta)$ have? What about $r = \sin(3\theta)$, $r = \sin(4\theta)$ and $r = \sin(5\theta)$? With the help of your classmates, make a conjecture as to how many petals the polar rose $r = \sin(n\theta)$ has for any natural number *n*. Replace sine with cosine and repeat the investigation. How many petals does $r = \cos(n\theta)$ have for each natural number *n*?

32. In this exercise, we have you and your classmates explore transformations of polar graphs. For both parts (a) and (b), let $f(\theta) = \cos(\theta)$ and $g(\theta) = 2 - \sin(\theta)$.

(a) Using a graphing calculator or other form of technology, compare the graph of $r = f(\theta)$ to each of the graphs of $r = f\left(\theta + \frac{\pi}{4}\right)$, $r = f\left(\theta + \frac{3\pi}{4}\right)$, $r = f\left(\theta - \frac{\pi}{4}\right)$ and $r = f\left(\theta - \frac{3\pi}{4}\right)$. Repeat this process for $g(\theta)$. In general, how do you think the graph of $r = f(\theta + \alpha)$ compares with the graph of $r = f(\theta)$?

(b) Using a graphing calculator or other form of technology, compare the graph of $r = f(\theta)$ to each of the graphs of $r = 2f(\theta)$, $r = \frac{1}{2}f(\theta)$, $r = -f(\theta)$ and $r = -3f(\theta)$. Repeat this process for $g(\theta)$. In general, how do you think the graph of $r = k \cdot f(\theta)$ compares with the graph of $r = f(\theta)$? (Does it matter if $k > 0$ or $k < 0$?)

33. With the help of your classmates, research cardioid microphones.

8.4 Polar Representations for Complex Numbers

Learning Objectives

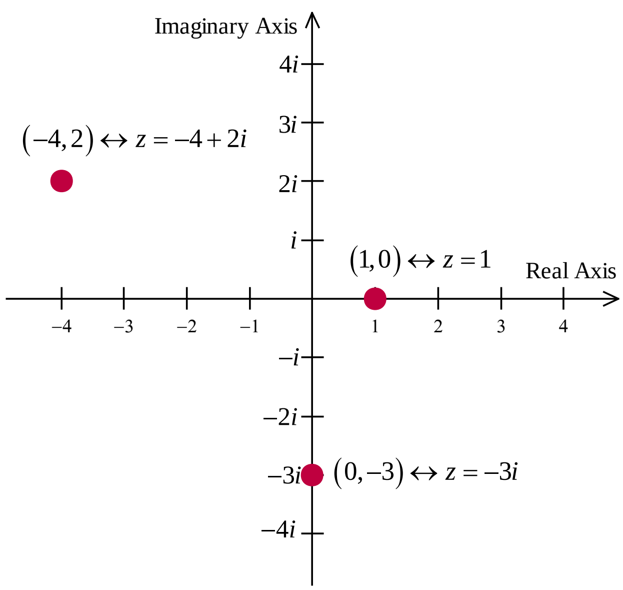
In this section you will:

- **Find the real part, the imaginary part, and the modulus of a complex number.**
- **Graph complex numbers.**
- **Learn the properties of the modulus and the argument of a complex number and be able to apply them.**

Complex Numbers and the Complex Plane

A **complex number** is a number of the form $z = a + bi$ where *a* and *b* are real numbers and *i* is the imaginary unit defined by $i = \sqrt{-1}$. The number *a* is called the **real part** of *z*, denoted $\mathrm{Re}(z)$, while the number *b* is called the **imaginary part** of *z*, denoted $\mathrm{Im}(z)$. If $z = a + bi = c + di$ for real numbers *a*, *b*, *c* and *d*, then $a = c$ and $b = d$, verifying that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are well-defined.[1]

To start off this section, we associate each complex number $z = a + bi$ with the point (a,b) on the coordinate plane. In this case, the *x*-axis is relabeled as the **real axis**, which corresponds to the real number line, and the *y*-axis is relabeled as the **imaginary axis**, which is demarcated in increments of the imaginary unit *i*. The plane determined by these two axes is called the **complex plane**.



The Complex Plane

[1] 'Well-defined' means that no matter how we express *z*, the number Re(*z*) is always the same, and the number Im(*z*) is always the same. In other words, Re and Im are *functions* of complex numbers.

Since the ordered pair (a,b) gives the rectangular coordinates associated with the complex number $z = a + bi$, the expression $z = a + bi$ is called the **rectangular form** of *z*. Of course, we could just as easily associate *z* with a pair of polar coordinates (r,θ). Although it is not as straightforward as the definitions of $\text{Re}(z)$ and $\text{Im}(z)$, we can still give *r* and θ special names in relation to *z*.

The Modulus and Argument of Complex Numbers

Definition. The Modulus and Argument of Complex Numbers: Let $z = a + bi$ be a complex number with $a = \text{Re}(z)$ and $b = \text{Im}(z)$. Let (r,θ) be a polar representation of the point with rectangular coordinates (a,b) where $r \geq 0$.

- The **modulus** of *z*, denoted $|z|$, is defined by $|z| = r$.
- The angle θ is an **argument** of *z*. The set of all arguments of *z* is denoted $\arg(z)$.
- If $z \neq 0$ and $-\pi \leq \theta \leq \pi$, then θ is the **principal argument** of *z*, written $\theta = \text{Arg}(z)$.

Some remarks are in order. We know from **Section 8.1** that every point in the plane has infinitely many polar coordinate representations (r,θ), which means it's worth our time to make sure the quantities 'modulus', 'argument' and 'principal argument' are well-defined.

- Concerning the modulus, if $z = 0$ then the point associated with *z* is the origin. In this case, the only possible *r*-value is $r = 0$. Hence, for $z = 0$, $|z| = 0$ is well-defined.

- If $z \neq 0$, then the point associated with *z* is not the origin, and there are two possibilities for *r*: one positive and one negative. However, we stipulated $r \geq 0$ in our definition so this pins down the value of $|z|$ to one and only one number. Thus, the modulus is well-defined in this case too.[2]

- Even with the requirement $r \geq 0$, there are infinitely many angles θ which can be used in a polar representation of a point (r,θ). If $z \neq 0$ then the point in question is not the origin, so all of these angles θ are coterminal. Since the coterminal angles are exactly 2π radians apart, we are guaranteed that only one of them lies in the interval $(-\pi,\pi]$, and this angle is what we call the principal argument of *z*, $\text{Arg}(z)$.

[2] In case you're wondering, the use of the absolute value notation |*z*| for modulus will be explained shortly.

In fact, the set $\arg(z)$ of all arguments of z can be described using set-builder notation as $\arg(z)=\{\text{Arg}(z)+2\pi k \mid k \text{ is an integer}\}$. Note that since $\arg(z)$ is a set, we will write '$\theta \in \arg(z)$' to mean 'θ is in[3] the set of arguments of z'.

- If $z=0$ then the point in question is the origin, which we know can be represented in polar coordinates as $(0,\theta)$ for any angle θ. In this case, we have $\arg(0)=(-\infty,\infty)$ and since there is no one value of θ which lies in $(-\pi,\pi]$, we leave $\text{Arg}(0)$ undefined.

It is high time for an example.

Example 8.4.1. For each of the following complex numbers find $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$. Plot z in the complex plane.

1. $z=\sqrt{3}-i$ 2. $z=-2+4i$ 3. $z=3i$ 4. $z=-117$

Solution.

1. For $z=\sqrt{3}-i=\sqrt{3}+(-1)i$, we have $\text{Re}(z)=\sqrt{3}$ and $\text{Im}(z)=-1$. To find $|z|$, $\arg(z)$ and $\text{Arg}(z)$, we need to find a polar representation (r,θ) with $r\geq 0$ for the point $P\left(\sqrt{3},-1\right)$ associated with z. We first determine a value for r.

$$r^2=\left(\sqrt{3}\right)^2+\left(-1\right)^2 \quad \textbf{from } r^2=x^2+y^2$$
$$r^2=4$$
$$r=\pm 2$$

We require $r\geq 0$, so we choose $r=2$, and have $|z|=2$.

Next, we find a corresponding angle θ. Since $r>0$ and P lies in Quadrant IV, θ is a Quadrant IV angle. We have

$$\tan(\theta)=\frac{-1}{\sqrt{3}}=-\frac{\sqrt{3}}{3} \qquad \textbf{from } \tan(\theta)=\frac{y}{x}$$
$$\theta=-\frac{\pi}{6}+2\pi k \text{ for integers } k \quad \textbf{since } \theta \textbf{ is a Quadrant IV angle}$$

[3] Recall the symbol being used here, $\in$, is the mathematical symbol which denotes membership in a set.

Thus, $\arg(z) = \left\{-\frac{\pi}{6} + 2\pi k \,\middle|\, k \text{ is an integer}\right\}$. Of these values, only $\theta = -\frac{\pi}{6}$ satisfies the requirement that $-\pi < \theta \le \pi$, hence $\operatorname{Arg}(z) = -\frac{\pi}{6}$.

2. The complex number $z = -2 + 4i$ has $\operatorname{Re}(z) = -2$, $\operatorname{Im}(z) = 4$, and is associated with the point $P(-2,4)$. Our next task is to find a polar representation (r,θ) for *P* where $r \ge 0$. Running through the usual calculations gives $r = 2\sqrt{5}$, so $|z| = 2\sqrt{5}$.

   To find *θ*, we get $\tan(\theta) = -2$. Since $r > 0$ and *P* lies in Quadrant II, we know *θ* is a Quadrant II angle. Thus,

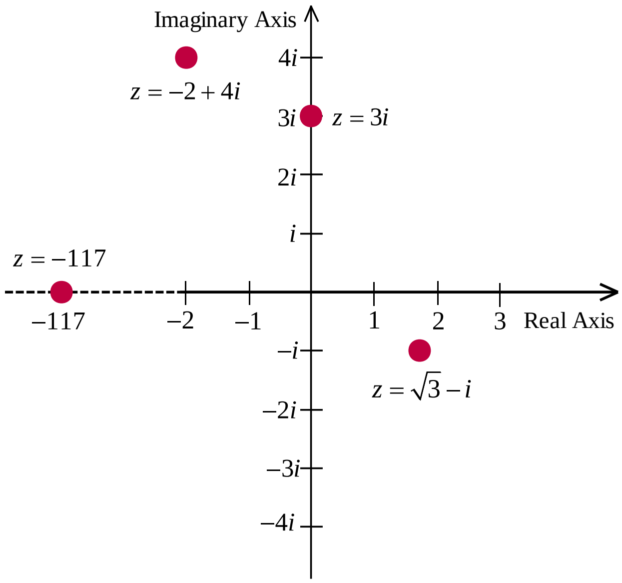
   $$\theta = \pi + \arctan(-2) + 2\pi k \text{ for integers } k \quad \textbf{since } \theta \textbf{ is a Quadrant II angle}$$
   $$\text{or } \theta = \pi - \arctan(2) + 2\pi k \text{ for integers } k \quad \textbf{from odd property of arctangent}$$

   Hence, $\arg(z) = \{\pi - \arctan(2) + 2\pi k \mid k \text{ is an integer}\}$. Only $\theta = \pi - \arctan(2)$ satisfies the requirement $-\pi < \theta \le \pi$, so $\operatorname{Arg}(z) = \pi - \arctan(2)$.

3. We rewrite $z = 3i$ as $z = 0 + 3i$ to find $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) = 3$. The point in the plane which corresponds to *z* is $(0,3)$ and while we could go through the usual calculations to find the required polar form of this point, we can almost 'see' the answer. The point $(0,3)$ lies 3 units away from the origin on the positive *y*-axis. Hence, $r = |z| = 3$ and $\theta = \frac{\pi}{2} + 2\pi k$ for integers *k*. We get $\arg(z) = \left\{\frac{\pi}{2} + 2\pi k \,\middle|\, k \text{ is an integer}\right\}$ and $\operatorname{Arg}(z) = \frac{\pi}{2}$.

4. As in the previous problem, we write $z = -117 = -117 + 0i$, so $\operatorname{Re}(z) = -117$ and $\operatorname{Im}(z) = 0$. The number $z = -117$ corresponds to the point $(-117,0)$, and this is another instance where we can determine the polar form 'by eye'. The point $(-117,0)$ is 117 units away from the origin along the negative *x*-axis. Hence, $r = |z| = 117$ and $\theta = \pi + 2\pi k$ for integers *k*. We have $\arg(z) = \{\pi + 2\pi k \mid k \text{ is an integer}\}$. Only one of these values, $\theta = \pi$, lies in the interval $(-\pi, \pi]$ which means that $\operatorname{Arg}(z) = \pi$.

We plot the four numbers from this example below.



□

Properties of the Modulus and Argument

Now that we've had some practice computing the modulus and argument of some complex numbers, it is time to explore their properties. We have the following theorem.

> **Theorem 8.3. Properties of the Modulus:** Let z and w be complex numbers.
>
> - $|z|$ is the distance from z to 0 in the complex plane
> - $|z|\geq 0$, and $|z|=0$ if and only if $z=0$
> - $|z|=\sqrt{[\text{Re}(z)]^2+[\text{Im}(z)]^2}$
> - **Product Rule:** $|zw|=|z||w|$
> - **Power Rule:** $|z^n|=|z|^n$ for all natural numbers n
> - **Quotient Rule:** $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$, provided $w\neq 0$

To prove the first three properties in **Theorem 8.3**, suppose $z=a+bi$ where a and b are real numbers. To determine $|z|$, we find a polar representation (r,θ) with $r\geq 0$ for the point (a,b). From **Section 8.1**, we know $r^2=a^2+b^2$ so that $r=\pm\sqrt{a^2+b^2}$. Since we require $r\geq 0$, then it must be that $r=\sqrt{a^2+b^2}$, which means $|z|=\sqrt{a^2+b^2}$.

- Using the distance formula, we find the distance from $(0,0)$ to (a,b) is also $\sqrt{a^2+b^2}$, establishing the first property.[4]

- For the second property, note that since $|z|$ is a distance, $|z| \geq 0$. Furthermore, $|z| = 0$ if and only if the distance from *z* to 0 is 0, and the latter happens if and only if $z = 0$, which is what we are asked to show.[5]

- For the third property, we note that since $a = \text{Re}(z)$ and $b = \text{Im}(z)$,

$$\begin{aligned} z &= \sqrt{a^2+b^2} \\ &= \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2} \end{aligned}$$

- To prove the product rule, suppose $z = a+bi$ and $w = c+di$ for real numbers *a*, *b*, *c* and *d*. Then

$$\begin{aligned} zw &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) \quad \textbf{from } i^2 = -1 \\ &= (ac-bd) + (ad+bc)i \end{aligned}$$

Therefore,

$$\begin{aligned} |zw| &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} && \textbf{after expanding} \\ &= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} && \textbf{terms rearranged} \\ &= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} && \textbf{product rule for radicals} \\ &= |z||w| && \textbf{definition of } |z| \textbf{ and } |w| \end{aligned}$$

---

[4] Since the absolute value $|x|$ of a real number *x* can be viewed as the distance from *x* to 0 on the number line, this first property justifies the notation $|z|$ for modulus. We leave it to the reader to show that if *z* is real, then the definition of modulus coincides with absolute value so the notation $|z|$ is unambiguous.

[5] This may be considered by some to be a bit of a cheat, so we work through the underlying Algebra to see this is true. We know $|z| = 0$ if and only if $\sqrt{a^2+b^2} = 0$ if and only if $a^2+b^2 = 0$, which is true if and only if $a = b = 0$. The latter happens if and only if $z = a+bi = 0$. There.

Hence $|zw| = |z||w|$ as required.

- Now that the product rule has been established, we use it and the Principle of Mathematical Induction to prove the power rule. Let $P(n)$ be the statement $|z^n| = |z|^n$. Then $P(1)$ is true since $|z^1| = |z| = |z|^1$. Next, assume $P(k)$ is true. That is, assume $|z^k| = |z|^k$ for some $k \geq 1$. Our job is to show that $P(k+1)$ is true, namely $|z^{k+1}| = |z|^{k+1}$. As is customary with induction proofs, we first try to reduce the problem in such a way as to use the induction hypothesis: $|z^k| = |z|^k$.

$$\begin{aligned} |z^{k+1}| &= |z^k z| && \textbf{property of exponents} \\ &= |z^k||z| && \textbf{product rule of modulus} \\ &= |z|^k |z| && \textbf{induction hypothesis} \\ &= |z|^{k+1} && \textbf{property of exponents} \end{aligned}$$

  Hence, $P(k+1)$ is true, which means $|z^n| = |z|^n$ is true for all natural numbers *n*.

- Like the power rule, the quotient rule can also be established with the help of the product rule. We assume $w \neq 0$, so that $|w| \neq 0$, and get

$$\begin{aligned} \left|\frac{z}{w}\right| &= \left|(z)\left(\frac{1}{w}\right)\right| \\ &= |z|\left|\frac{1}{w}\right| && \textbf{product rule of modulus} \end{aligned}$$

  Hence, the proof really boils down to showing $\left|\frac{1}{w}\right| = \frac{1}{|w|}$. This is left as an exercise.

Next, we characterize the argument of a complex number in terms of its real and imaginary parts.

Theorem 8.4. Properties of the Argument: Let z be a complex number.

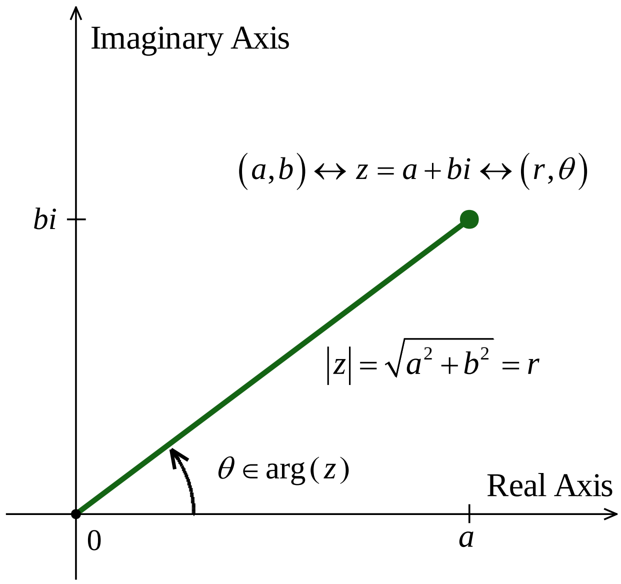
- If $\operatorname{Re}(z) \neq 0$ and $\theta \in \arg(z)$, then $\tan(\theta) = \dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.
- If $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) > 0$, then $\arg(z) = \left\{ \dfrac{\pi}{2} + 2\pi k \,\middle|\, k \text{ is an integer} \right\}$.
- If $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) < 0$, then $\arg(z) = \left\{ -\dfrac{\pi}{2} + 2\pi k \,\middle|\, k \text{ is an integer} \right\}$.
- If $\operatorname{Re}(z) = \operatorname{Im}(z) = 0$, then $z = 0$ and $\arg(z) = (-\infty, \infty)$.

To prove **Theorem 8.4**, suppose $z = a + bi$ for real numbers a and b. By definition, $a = \operatorname{Re}(z)$ and $b = \operatorname{Im}(z)$, so the point associated with z is $(a,b) = (\operatorname{Re}(z), \operatorname{Im}(z))$.

- From **Section 8.1**, we know that if (r,θ) is a polar representation for $(\operatorname{Re}(z), \operatorname{Im}(z))$, then $\tan(\theta) = \dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$, provided $\operatorname{Re}(z) \neq 0$.
- If $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) > 0$, then z lies on the positive imaginary axis. Since we take $r > 0$, we have that θ is coterminal with $\dfrac{\pi}{2}$, and the result follows.
- If $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) < 0$, then z lies on the negative imaginary axis, and a similar argument shows θ is coterminal with $-\dfrac{\pi}{2}$.
- The last property was already discussed following the definition at the beginning of this section.

Polar Form of Complex Numbers

Our next goal is to link the geometry and algebra of the complex numbers. To that end, consider the figure below.



Polar coordinate (r,θ) associated with $z = a+bi$, with $r \geq 0$

We know from **Theorem 8.1** that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Making these substitutions for *a* and *b* gives

$$\begin{aligned} z &= a+bi \\ &= r\cos(\theta) + r\sin(\theta)i \\ &= r\left[\cos(\theta) + i\sin(\theta)\right] \end{aligned}$$

The expression $\cos(\theta)+i\sin(\theta)$ is abbreviated $\operatorname{cis}(\theta)$ so we can write $z = r\operatorname{cis}(\theta)$. Since $r = |z|$ and $\theta \in \arg(z)$, we get

> **Definition. A Polar Form of a Complex Number:** Suppose *z* is a complex number and $\theta \in \arg(z)$. The expression
>
> $$|z|\operatorname{cis}(\theta) = |z|\left[\cos(\theta)+i\sin(\theta)\right]$$
>
> is called a polar form for *z*.

Since there are infinitely many choices for $\theta \in \arg(z)$, there are infinitely many polar forms for *z*, so we used the indefinite article 'a' in the preceding definition. It is time for an example.

Example 8.4.2. Find the rectangular form of the following complex numbers. Find $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$.

1. $z = 4\operatorname{cis}\left(\frac{2\pi}{3}\right)$
2. $z = 2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$
3. $z = 3\operatorname{cis}(0)$
4. $z = \operatorname{cis}\left(\frac{\pi}{2}\right)$

Solution. The key to this problem is to write out $\operatorname{cis}(\theta)$ as $\cos(\theta)+i\sin(\theta)$.

1. By definition,

$$\begin{aligned} z &= 4\text{cis}\left(\frac{2\pi}{3}\right) \\ &= 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \end{aligned}$$

After some simplifying, we get $z = -2 + 2i\sqrt{3}$, so that $\text{Re}(z) = -2$ and $\text{Im}(z) = 2\sqrt{3}$.

2. Expanding, we get

$$\begin{aligned} z &= 2\text{cis}\left(-\frac{3\pi}{4}\right) \\ &= 2\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right] \end{aligned}$$

From this, we find $z = -\sqrt{2} - i\sqrt{2}$, so $\text{Re}(z) = -\sqrt{2} = \text{Im}(z)$.

3. We get

$$\begin{aligned} z &= 3\text{cis}(0) \\ &= 3\left[\cos(0) + i\sin(0)\right] \\ &= 3 \end{aligned}$$

Writing $3 = 3 + 0i$, we get $\text{Re}(z) = 3$ and $\text{Im}(z) = 0$, which makes sense seeing that 3 is a real number.

4. Lastly, we have

$$\begin{aligned} z &= \text{cis}\left(\frac{\pi}{2}\right) \\ &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \\ &= i \end{aligned}$$

Since $i = 0 + 1i$, we get $\text{Re}(z) = 0$ and $\text{Im}(z) = 1$. Since *i* is called the 'imaginary unit', these answers make sense.

□

Example 8.4.3. Use the results from **Example 8.4.1** to find a polar form of the following complex numbers.

1. $z = \sqrt{3} - i$ 2. $z = -2 + 4i$ 3. $z = 3i$ 4. $z = -117$

Solution. To write a polar form of a complex number *z*, we need two pieces of information: the modulus $|z|$ and an argument (not necessarily the principal argument) of *z*. We shamelessly mine our solution to **Example 8.4.1** to find what we need.

1. For $z=\sqrt{3}-i$, $|z|=2$ and $\theta=-\frac{\pi}{6}$, so $z=2\text{cis}\left(-\frac{\pi}{6}\right)$. We can check our answer by converting it back to rectangular form to see that it simplifies to $z=\sqrt{3}-i$.
2. For $z=-2+4i$, $|z|=2\sqrt{5}$ and $\theta=\pi-\arctan(2)$. Hence, $z=2\sqrt{5}\text{cis}(\pi-\arctan(2))$. It is a good exercise to actually show that this polar form reduces to $z=-2+4i$.
3. For $z=3i$, $|z|=3$ and $\theta=\frac{\pi}{2}$. In this case, $z=3\text{cis}\left(\frac{\pi}{2}\right)$. This can be checked geometrically. Head out 3 units from 0 along the positive real axis. Rotating $\frac{\pi}{2}$ radians counter-clockwise lands you exactly 3 units above 0 on the imaginary axis at $z=3i$.
4. Last but not least, for $z=-117$, $|z|=117$ and $\theta=\pi$. We get $z=117\text{cis}(\pi)$. As with the last problem, our answer is easily checked geometrically.

□

8.4 Exercises

In Exercises 1 – 20, find a polar representation for the complex number z and then identify $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$. These exercises should be worked without the aid of a calculator.

1. $z = 9 + 9i$
2. $z = 5 + 5i\sqrt{3}$
3. $z = 6i$
4. $z = -3\sqrt{2} + 3i\sqrt{2}$
5. $z = -6\sqrt{3} + 6i$
6. $z = -2$
7. $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$
8. $z = -3 - 3i$
9. $z = -5i$
10. $z = 2\sqrt{2} - 2i\sqrt{2}$
11. $z = 6$
12. $z = i\sqrt[3]{7}$
13. $z = 3 + 4i$
14. $z = \sqrt{2} + i$
15. $z = -7 + 24i$
16. $z = -2 + 6i$
17. $z = -12 - 5i$
18. $z = -5 - 2i$
19. $z = 4 - 2i$
20. $z = 1 - 3i$

In Exercises 21 – 40, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values. These exercises should be worked without the aid of a calculator.

21. $z = 6\text{cis}(0)$
22. $z = 2\text{cis}\left(\frac{\pi}{6}\right)$
23. $z = 7\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$
24. $z = 3\text{cis}\left(\frac{\pi}{2}\right)$
25. $z = 4\text{cis}\left(\frac{2\pi}{3}\right)$
26. $z = \sqrt{6}\text{cis}\left(\frac{3\pi}{4}\right)$
27. $z = 9\text{cis}(\pi)$
28. $z = 3\text{cis}\left(\frac{4\pi}{3}\right)$
29. $z = 7\text{cis}\left(-\frac{3\pi}{4}\right)$
30. $z = \sqrt{13}\text{cis}\left(\frac{3\pi}{2}\right)$
31. $z = \frac{1}{2}\text{cis}\left(\frac{7\pi}{4}\right)$
32. $z = 12\text{cis}\left(-\frac{\pi}{3}\right)$
33. $z = 8\text{cis}\left(\frac{\pi}{12}\right)$
34. $z = 2\text{cis}\left(\frac{7\pi}{8}\right)$
35. $z = 5\text{cis}\left(\arctan\left(\frac{4}{3}\right)\right)$
36. $z = \sqrt{10}\text{cis}\left(\arctan\left(\frac{1}{3}\right)\right)$
37. $z = 15\text{cis}(\arctan(-2))$
38. $z = \sqrt{3}\text{cis}\left(\arctan\left(-\sqrt{2}\right)\right)$
39. $z = 50\text{cis}\left(\pi - \arctan\left(\frac{7}{24}\right)\right)$
40. $z = \frac{1}{2}\text{cis}\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$

41. Complete the proof of **Theorem 8.4**, Properties of the Modulus, by showing that if $w \neq 0$ then

$$\left|\frac{1}{w}\right| = \frac{1}{|w|}.$$

42. Recall that the complex conjugate of a complex number $z = a + bi$ is denoted $\overline{z}$ and is given by

$$\overline{z} = a - bi.$$

(a) Prove that $\left|\overline{z}\right| = |z|$.

(b) Prove that $|z| = \sqrt{z\overline{z}}$.

(c) Show that $\operatorname{Re}(z) = \dfrac{z+\overline{z}}{2}$ and $\operatorname{Im}(z) = \dfrac{z-\overline{z}}{2i}$.

(d) Show that if $\theta \in \arg(z)$ then $-\theta \in \arg\left(\overline{z}\right)$. Interpret this result geometrically.

(e) Is it always true that $\operatorname{Arg}\left(\overline{z}\right) = -\operatorname{Arg}(z)$?

8.5 Products, Powers, Quotients and Roots of Complex Numbers

Learning Objectives

In this section you will:

- **Find the product, power, quotient and roots of complex number(s).**
- **Learn and apply DeMoivre's Theorem.**

Products, Powers and Quotients of Complex Numbers

The following theorem summarizes the advantages of working with complex numbers in polar form.

Theorem 8.5. Products, Powers and Quotients of Complex Numbers in Polar Form: Suppose z and w are complex numbers with polar forms $z=|z|\operatorname{cis}(\alpha)$ and $w=|w|\operatorname{cis}(\beta)$. Then

- **Product Rule:** $zw=|z||w|\operatorname{cis}(\alpha+\beta)$
- **Power Rule** (DeMoivre's Theorem)**:** $z^n=|z|^n\operatorname{cis}(n\theta)$ for every natural number n
- **Quotient Rule:** $\dfrac{z}{w}=\dfrac{|z|}{|w|}\operatorname{cis}(\alpha-\beta)$, provided $|w|\neq 0$

The proof of **Theorem 8.5** requires a healthy mix of definition, arithmetic and identities.

- We start with the product rule.

$$\begin{aligned} zw &= \left[|z|\operatorname{cis}(\alpha)\right]\left[|w|\operatorname{cis}(\beta)\right] \\ &= |z||w|\left[\cos(\alpha)+i\sin(\alpha)\right]\left[\cos(\beta)+i\sin(\beta)\right] && \textbf{definition of } \operatorname{cis} \end{aligned}$$

  We now focus on the quantities in brackets on the right hand side of the equation.

$$\begin{aligned} &\left[\cos(\alpha)+i\sin(\alpha)\right]\left[\cos(\beta)+i\sin(\beta)\right] \\ &= \cos(\alpha)\cos(\beta)+i\cos(\alpha)\sin(\beta)+i\sin(\alpha)\cos(\beta)+i^2\sin(\alpha)\sin(\beta) \\ &= \cos(\alpha)\cos(\beta)+i^2\sin(\alpha)\sin(\beta)+i\sin(\alpha)\cos(\beta)+i\cos(\alpha)\sin(\beta) && \textbf{rearrange terms} \\ &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)+i\left(\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\right) && \textbf{use } i^2=-1\textbf{; factor out } i \\ &= \cos(\alpha+\beta)+i\sin(\alpha+\beta) && \textbf{sum identities} \\ &= \operatorname{cis}(\alpha+\beta) && \textbf{definition of } \operatorname{cis} \end{aligned}$$

  Putting this together with our earlier work, we get $zw=|z||w|\operatorname{cis}(\alpha+\beta)$.

- Moving right along, we take aim at the power rule, better known as DeMoivre's Theorem. We proceed by induction on n. Let $P(n)$ be the sentence $z^n=|z|^n\operatorname{cis}(n\theta)$. Then $P(1)$ is true, since

$$\begin{aligned} z^1 &= z \\ &= |z|\operatorname{cis}(\theta) \\ &= |z|^1\operatorname{cis}(1\cdot\theta) \end{aligned}$$

We now assume $P(k)$ is true, that is, we assume $z^k = |z|^k \operatorname{cis}(k\theta)$ for some $k \geq 1$. Our goal is to show that $P(k+1)$ is true, or that $z^{k+1} = |z|^{k+1} \operatorname{cis}((k+1)\theta)$. We have

$$\begin{aligned} z^{k+1} &= z^k z && \textbf{property of exponents} \\ &= \left(|z|^k \operatorname{cis}(k\theta)\right)\left(|z| \operatorname{cis}(\theta)\right) && \textbf{induction hypothesis} \\ &= \left(|z|^k |z|\right) \operatorname{cis}(k\theta + \theta) && \textbf{product rule} \\ &= |z|^{k+1} \operatorname{cis}((k+1)\theta) \end{aligned}$$

Hence, assuming $P(k)$ is true, we have that $P(k+1)$ is true, so by the Principle of Mathematical Induction, $z^n = |z|^n \operatorname{cis}(n\theta)$ for all natural numbers *n*.

- The last property in **Theorem 8.5** to prove is the quotient rule. Assuming $|w| \neq 0$, we have

$$\begin{aligned} \frac{z}{w} &= \frac{|z| \operatorname{cis}(\alpha)}{|w| \operatorname{cis}(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)} \end{aligned}$$

Next, we multiply both the numerator and denominator of the right hand side by $(\cos(\beta) - i\sin(\beta))$ to get

$$\begin{aligned} \frac{z}{w} &= \left(\frac{|z|}{|w|}\right) \frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)} \cdot \frac{\cos(\beta) - i\sin(\beta)}{\cos(\beta) - i\sin(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{[\cos(\alpha) + i\sin(\alpha)][\cos(\beta) - i\sin(\beta)]}{[\cos(\beta) + i\sin(\beta)][\cos(\beta) - i\sin(\beta)]} \\ &= \left(\frac{|z|}{|w|}\right) \frac{\cos(\alpha)\cos(\beta) - i\cos(\alpha)\sin(\beta) + i\sin(\alpha)\cos(\beta) - i^2\sin(\alpha)\sin(\beta)}{\cos^2(\beta) - i\cos(\beta)\sin(\beta) + i\sin(\beta)\cos(\beta) - i^2\sin^2(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{[\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)] + i[\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)]}{\cos^2(\beta) - i^2\sin^2(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{\cos(\alpha - \beta) + i\sin(\alpha - \beta)}{\cos^2(\beta) + \sin^2(\beta)} \\ &= \left(\frac{|z|}{|w|}\right) \frac{\operatorname{cis}(\alpha - \beta)}{1} \end{aligned}$$

Finally, we have $\frac{z}{w} = \frac{|z|}{|w|} \operatorname{cis}(\alpha - \beta)$, and we are done.

Example 8.5.1. Let $z = 2\sqrt{3} + 2i$ and $w = -1 + i\sqrt{3}$. Use **Theorem 8.5** to find the following.

1. zw
2. w^5
3. $\dfrac{z}{w}$

Write your final answers in rectangular form.

Solution. In order to use **Theorem 8.5**, we need to write z and w in polar form.

For $z = 2\sqrt{3} + 2i$, we find

$$\begin{aligned} |z| &= \sqrt{\left(2\sqrt{3}\right)^2 + (2)^2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

If $\theta \in \arg(z)$, we know

$$\begin{aligned} \tan(\theta) &= \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \\ &= \frac{2}{2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \end{aligned}$$

Since z lies in Quadrant I, we have $\theta = \dfrac{\pi}{6} + 2\pi k$ for integers k. Hence, $z = 4\operatorname{cis}\left(\dfrac{\pi}{6}\right)$.

For $w = -1 + i\sqrt{3}$, we have

$$\begin{aligned} |w| &= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \\ &= 2 \end{aligned}$$

For an argument θ of w, we have

$$\begin{aligned} \tan(\theta) &= \frac{\sqrt{3}}{-1} \\ &= -\sqrt{3} \end{aligned}$$

Since w lies in Quadrant II, $\theta = \dfrac{2\pi}{3} + 2\pi k$ for integers k and $w = 2\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$.

We can now proceed.

1. We get

$$
\begin{aligned}
zw &= \left(4\operatorname{cis}\left(\frac{\pi}{6}\right)\right)\left(2\operatorname{cis}\left(\frac{2\pi}{3}\right)\right) \\
&= 8\operatorname{cis}\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) \\
&= 8\operatorname{cis}\left(\frac{5\pi}{6}\right) \\
&= 8\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]
\end{aligned}
$$

After simplifying, we get $zw = -4\sqrt{3}+4i$.

2. We use DeMoivre's Theorem which yields

$$
\begin{aligned}
w^5 &= \left[2\operatorname{cis}\left(\frac{2\pi}{3}\right)\right]^5 \\
&= 2^5\operatorname{cis}\left(5\cdot\frac{2\pi}{3}\right) \\
&= 32\operatorname{cis}\left(\frac{10\pi}{3}\right)
\end{aligned}
$$

Since $\frac{10\pi}{3}$ is coterminal with $\frac{4\pi}{3}$, we get

$$
\begin{aligned}
w^5 &= 32\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right] \\
&= -16-16i\sqrt{3}
\end{aligned}
$$

3. Last, but not least, we have

$$
\begin{aligned}
\frac{z}{w} &= \frac{4\operatorname{cis}\left(\frac{\pi}{6}\right)}{2\operatorname{cis}\left(\frac{2\pi}{3}\right)} \\
&= \frac{4}{2}\operatorname{cis}\left(\frac{\pi}{6}-\frac{2\pi}{3}\right) \\
&= 2\operatorname{cis}\left(-\frac{\pi}{2}\right)
\end{aligned}
$$

Since $-\frac{\pi}{2}$ is a quadrantal angle, we can 'see' the rectangular form by moving out 2 units along the positive real axis, then rotating $\frac{\pi}{2}$ radians clockwise to arrive at the point 2 units below 0 on the imaginary axis. The long and short of it is that $\frac{z}{w} = -2i$.

□

Some remarks are in order.

- First, the reader may not be sold on using the polar form of complex numbers to multiply complex numbers – especially if they aren't given in polar form to begin with. Indeed, a lot of work was needed to convert the numbers *z* and *w* in **Example 8.5.1** into polar form, compute their product, and convert back to rectangular form – certainly more work than is required to multiply out $zw = \left(2\sqrt{3}+2i\right)\left(-1+i\sqrt{3}\right)$ the old-fashioned way.

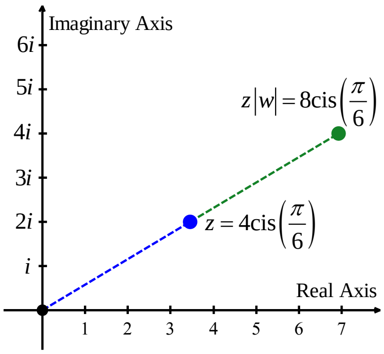
  However, **Theorem 8.5** pays huge dividends when computing powers of complex numbers. Consider how we computed w^5 in **Example 8.5.1** and compare that to accomplishing the same feat by expanding $\left(-1+i\sqrt{3}\right)^5$. With division being tricky in the best of times, we saved ourselves a lot of time and effort using **Theorem 8.5** to find and simplify $\frac{z}{w}$ using their polar forms as opposed to starting with $\frac{2\sqrt{3}+2i}{-1+i\sqrt{3}}$, rationalizing the denominator and so forth.

- There is geometric reason for studying these polar forms. Take the product rule, for instance. If $z = |z|\operatorname{cis}(\alpha)$ and $w = |w|\operatorname{cis}(\beta)$, the formula $zw = |z||w|\operatorname{cis}(\alpha+\beta)$ can be viewed geometrically as a two-step process. The multiplication of $|z|$ by $|w|$ can be interpreted as magnifying[1] the distance $|z|$, from 0 to *z*, by the factor $|w|$. Adding the argument of *w* to the argument of *z* can be interpreted geometrically as a rotation of β radians counter-clockwise.[2]

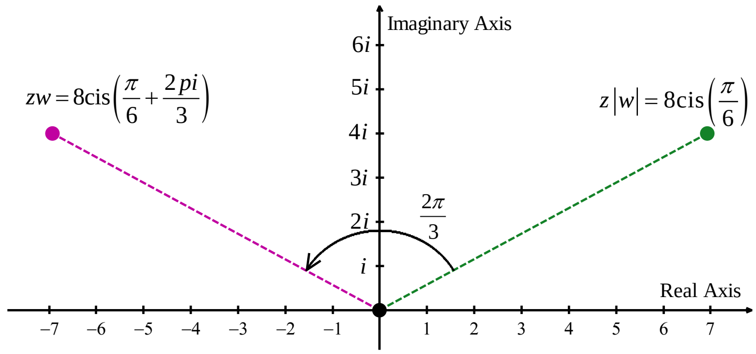
  Focusing on $z = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ from **Example 8.5.1**, we can arrive at the product zw by plotting *z*, doubling its distance from 0 (since $|w| = 2$), and rotating $\frac{2\pi}{3}$ radians counter-clockwise. The sequence of diagrams below attempts to describe this process geometrically.

---

[1] Assuming |w| > 1.
[2] Assuming $\beta > 0$.



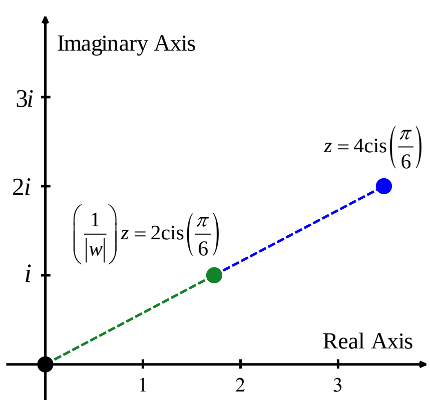
Multiplying z by $|w| = 2$



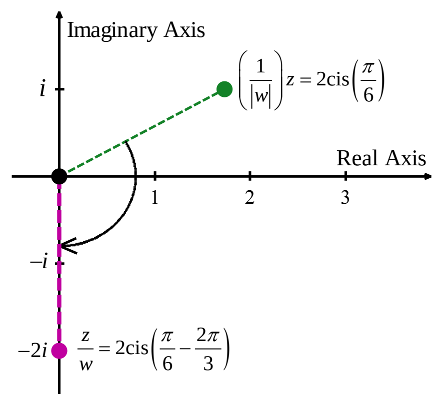
Rotating counter-clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$ radians

We may visualize division similarly. Here, the formula $\frac{z}{w} = \frac{|z|}{|w|}\text{cis}(\alpha - \beta)$ may be interpreted as shrinking[3] the distance from 0 to z by the factor $|w|$, followed by a *clockwise*[4] rotation of β radians.

In the case of $z = 4\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2\text{cis}\left(\frac{2\pi}{3}\right)$ from **Example 8.5.1**, we arrive at $\frac{z}{w}$ by first halving the distance from 0 to z, then rotating clockwise $\frac{2\pi}{3}$ radians, as we visualize below.



Dividing z by $|w| = 2$



Rotating clockwise by $\text{Arg}(w) = \frac{2\pi}{3}$

Roots of Complex Numbers

Our last goal of the section is to reverse DeMoivre's Theorem to extract roots of complex numbers.

[3] Again, assuming $|w| > 1$.
[4] Again, assuming $\beta > 0$.

Definition. Let z and w be complex numbers. If there is a natural number n such that $w^n = z$, then w is an $\boldsymbol{n}^{\text{th}}$ **root** of z.

Here, we do not specify one particular *principal* n^{th} root, hence the use of the indefinite article in defining w as 'an' n^{th} root of z. Using this definition, both 4 and –4 are square roots of 16, while $\sqrt{16}$ means the principal square root of 16, as in $\sqrt{16} = 4$.

Suppose we wish to find the complex third (cube) roots of 8. Algebraically, we are trying to solve $w^3 = 8$. We know that there is only one real solution to this equation, namely $w = \sqrt[3]{8} = 2$, but if we take the time to rewrite this equation as $w^3 - 8 = 0$ and factor, we get $(w-2)\left(w^2 + 2w + 4\right) = 0$. The quadratic factor gives two more cube roots, $w = -1 \pm i\sqrt{3}$, for a total of three cube roots of 8.

Recall from College Algebra that, since the degree of the polynomial $P(w) = w^3 - 8$ is three, there are three complex zeros, counting multiplicity. Since we have found three distinct zeros, we know these are all of the zeros, so there are exactly three distinct cube roots of 8.

Let us now solve this same problem using the machinery developed in this section. To do so, we express $z = 8$ in polar form. Since $z = 8$ lies 8 units away from the origin on the positive real axis, we get $z = 8\text{cis}(0)$. If we let $w = |w|\text{cis}(\alpha)$ be a polar form of w, the equation $w^3 = 8$ becomes

$$\begin{aligned} w^3 &= 8 \\ \left(|w|\text{cis}(\alpha)\right)^3 &= 8\text{cis}(0) \\ |w|^3\text{cis}(3\alpha) &= 8\text{cis}(0) \quad \textbf{DeMoivre's Theorem} \end{aligned}$$

The complex number on the left hand side of the equation corresponds to the point with polar coordinates $\left(|w|^3, 3\alpha\right)$ while the complex number on the right hand side corresponds to the point with polar coordinates $(8,0)$. Since by definition $|w| \geq 0$, so is $|w|^3$, which means $\left(|w|^3, 3\alpha\right)$ and $(8,0)$ are two polar representations corresponding to the same complex number, with both representations having positive r values. Thus, $|w|^3 = 8$ and $3\alpha = 0 + 2\pi k$ for integers k.

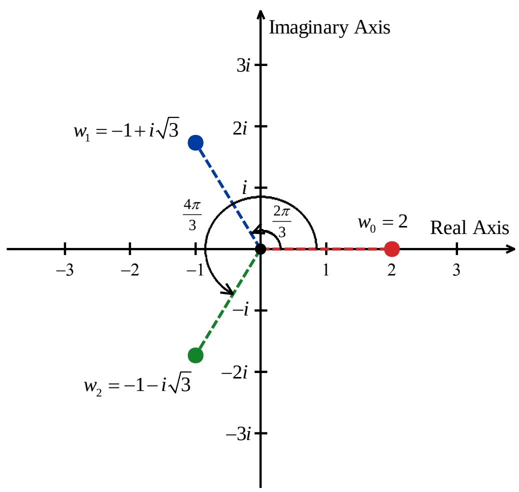
- Since $|w|$ is a real number, we solve $|w|^3 = 8$ by extracting the principal cube root to get $|w| = \sqrt[3]{8} = 2$.

- As for α, we get $\alpha = \frac{2\pi k}{3}$ for integers *k*.

This produces three distinct points with polar coordinates corresponding to $k = 0$, 1, and 2: specifically $(2,0)$, $\left(2,\frac{2\pi}{3}\right)$ and $\left(2,\frac{4\pi}{3}\right)$. The corresponding complex and rectangular forms are listed in following table.

Polar Coordinate	$(2,0)$	$\left(2,\frac{2\pi}{3}\right)$	$\left(2,\frac{4\pi}{3}\right)$
Complex Number	$w_0 = 2\text{cis}(0)$	$w_1 = 2\text{cis}\left(\frac{2\pi}{3}\right)$	$w_2 = 2\text{cis}\left(\frac{4\pi}{3}\right)$
Rectangular Form	$w_0 = 2$	$w_1 = -1 + i\sqrt{3}$	$w_2 = -1 - i\sqrt{3}$

The cube roots of 8 can be visualized geometrically in the complex plane, as follows.



Keeping the geometric picture in mind throughout the remainder of this section will lead to an interesting observation regarding geometric properties of complex roots.

While the process for finding the cube roots of 8 seems a tad more involved than our previous factoring approach, this procedure can be generalized to find, for example, all of the fifth roots of 32. (Try using factoring techniques on that!) If we start with a generic complex number in polar form $z = |z|\text{cis}(\theta)$ and solve $w^n = z$ in the same manner as above, we arrive at the following theorem.

Theorem 8.6. The n^th Roots of a Complex Number: Let $z \neq 0$ be a complex number with polar form $z = r\operatorname{cis}(\theta)$. For each natural number *n*, *z* has *n* distinct *n*^th roots, which we denote by $w_0,\ w_1, \ldots, w_{n-1}$, and they are given by the formula

$$w_k = \sqrt[n]{r}\operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)$$

The proof of **Theorem 8.6** breaks into two parts: first, showing that each w_k is an n^{th} root, and second, showing that the set $\{w_k \mid k = 0, 1, \ldots, (n-1)\}$ consists of *n* different complex numbers.

- To show w_k is an n^{th} root of *z*, we use DeMoivre's Theorem to show $(w_k)^n = z$.

$$\begin{aligned}(w_k)^n &= \left(\sqrt[n]{r}\operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right)^n \\ &= \left(\sqrt[n]{r}\right)^n \operatorname{cis}\left(n \cdot \left[\frac{\theta}{n} + \frac{2\pi}{n}k\right]\right) \quad \textbf{DeMoivre's Theorem} \\ &= r\operatorname{cis}(\theta + 2\pi k)\end{aligned}$$

  Since *k* is a whole number, $\cos(\theta + 2\pi k) = \cos(\theta)$ and $\sin(\theta + 2\pi k) = \sin(\theta)$. Hence, it follows that $\operatorname{cis}(\theta + 2\pi k) = \operatorname{cis}(\theta)$, so $(w_k)^n = r\operatorname{cis}(\theta) = z$, as required.

- To show that the formula in **Theorem 8.6** generates *n* distinct numbers, we assume $n \geq 2$ (or else there is nothing to prove) and note that the modulus of each of the w_k is the same, namely $\sqrt[n]{r}$. Therefore, the only way any two of these polar forms correspond to the same number is if their arguments are coterminal – that is, if the arguments differ by an integer multiple of 2π.

  Suppose *k* and *j* are whole numbers between 0 and $(n-1)$, inclusive, with $k \neq j$. Then $\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) - \left(\frac{\theta}{n} + \frac{2\pi}{n}j\right) = 2\pi\left(\frac{k-j}{n}\right)$. For this to be an integer multiple of 2π, $(k-j)$ must be a multiple of *n*. But because of the restrictions on *k* and *j*, $0 < k - j \leq n - 1$. (Think this through.) Hence, $(k-j)$ is a positive number less than *n*, so it cannot be a multiple of *n*. As a result, w_k and w_j are different complex numbers, and we are done.

From College Algebra, we know there are at most *n* distinct solutions to $w^n = z$, and we have just found all of them. We illustrate **Theorem 8.6** in the next example.

Example 8.5.2. Find the following:

1. both square roots of $z=-2+2i\sqrt{3}$
2. the four fourth roots of $z=-16$
3. the three cube roots of $z=\sqrt{2}+i\sqrt{2}$
4. the five fifth roots of $z=1$

Solution.

1. To find both square roots of $z=-2+2i\sqrt{3}$, we start by writing $z=-2+2i\sqrt{3}=4\text{cis}\left(\frac{2\pi}{3}\right)$. To use **Theorem 8.6**, we identify $r=4$, $\theta=\frac{2\pi}{3}$ and $n=2$. We know that z has two square roots, and in keeping with the notation in the theorem, we'll call them w_0 and w_1. We get

$$\begin{aligned} w_0 &= \sqrt{4}\text{cis}\left(\frac{2\pi/3}{2}+\frac{2\pi}{2}(0)\right) && \textbf{Theorem 8.6 with } k=0 \\ &= 2\text{cis}\left(\frac{\pi}{3}\right) \\ &= 1+i\sqrt{3} && \textbf{rectangular form} \end{aligned}$$

and

$$\begin{aligned} w_1 &= \sqrt{4}\text{cis}\left(\frac{2\pi/3}{2}+\frac{2\pi}{2}(1)\right) && \textbf{Theorem 8.6 with } k=1 \\ &= 2\text{cis}\left(\frac{4\pi}{3}\right) \\ &= 1-i\sqrt{3} && \textbf{rectangular form} \end{aligned}$$

We can check each of these roots by squaring to get $z=-2+2i\sqrt{3}$.

2. To find the four fourth roots of $z=-16$, proceeding as above, we write z as $z=-16=16\text{cis}(\pi)$. With $r=16$, $\theta=\pi$ and $n=4$, we get the four fourth roots of z to be

$$\begin{aligned} w_0 &= \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}(0)\right)=2\text{cis}\left(\frac{\pi}{4}\right) \\ w_1 &= \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}(1)\right)=2\text{cis}\left(\frac{3\pi}{4}\right) \\ w_2 &= \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}(2)\right)=2\text{cis}\left(\frac{5\pi}{4}\right) \\ w_3 &= \sqrt[4]{16}\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{4}(3)\right)=2\text{cis}\left(\frac{7\pi}{4}\right) \end{aligned}$$

Converting these to rectangular form gives $w_0=\sqrt{2}+i\sqrt{2}$, $w_1=-\sqrt{2}+i\sqrt{2}$, $w_2=-\sqrt{2}-i\sqrt{2}$ and $w_3=\sqrt{2}-i\sqrt{2}$.

3. For finding the cube roots of $z=\sqrt{2}+i\sqrt{2}$, we have $z=2\text{cis}\left(\frac{\pi}{4}\right)$. With $r=2$, $\theta=\frac{\pi}{4}$ and $n=3$ the usual computations yield

$$w_0=\sqrt[3]{2}\text{cis}\left(\frac{\pi}{12}\right)$$
$$w_1=\sqrt[3]{2}\text{cis}\left(\frac{9\pi}{12}\right)=\sqrt[3]{2}\text{cis}\left(\frac{3\pi}{4}\right)$$
$$w_2=\sqrt[3]{2}\text{cis}\left(\frac{17\pi}{12}\right)$$

If we were to convert these to rectangular form, we would need to use either sum and difference identities or half-angle identities to evaluate w_0 and w_2. Since we are not explicitly told to do so, we leave this as a good, but messy, exercise.

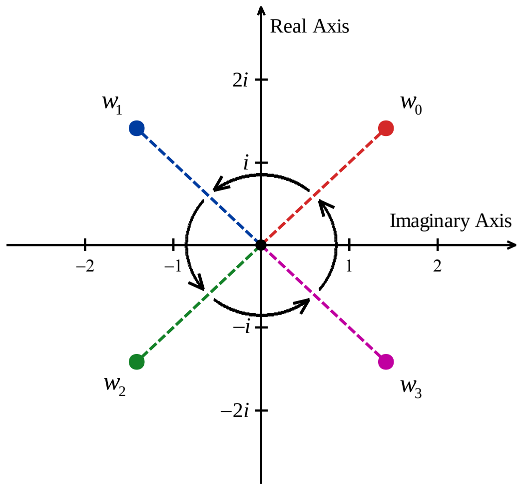
4. To find the five fifth roots of 1, we write $1=1\text{cis}(0)$. We have $r=1$, $\theta=0$ and $n=5$. Since $\sqrt[5]{1}=1$, the roots are

$$w_0=\text{cis}(0)=1$$
$$w_1=\text{cis}\left(\frac{2\pi}{5}\right)$$
$$w_2=\text{cis}\left(\frac{4\pi}{5}\right)$$
$$w_3=\text{cis}\left(\frac{6\pi}{5}\right)$$
$$w_4=\text{cis}\left(\frac{8\pi}{5}\right)$$

The situation here is even graver than in the previous example, since we have not developed any identities to help us determine the cosine or sine of $\frac{2\pi}{5}$. At this stage, we could approximate our answers using a calculator, and we leave this as an exercise.

□

Now that we have done some computations using **Theorem 8.6**, we take a step back to look at things geometrically. Essentially, **Theorem 8.6** says that to find the n^{th} roots of a complex number, we first take the n^{th} root of the modulus and divide the argument by *n*. This gives the first root w_0. Each successive root is found by adding $\frac{2\pi}{n}$ to the argument, which amounts to rotating w_0 by $\frac{2\pi}{n}$ radians. This results in *n* roots, spaced equally around the complex plane. As an example of this, we plot our answers to number 2 in **Example 8.5.2**.



The four fourth roots of $z = -16$ equally spaced $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$ around the plane

We have only glimpsed at the beauty of the complex numbers in this section. The complex plane is without a doubt one of the most important mathematical constructs ever devised. Coupled with Calculus, it is the venue for incredibly important Science and Engineering applications. For now, the following exercises will have to suffice.

8.5 Exercises

In Exercises 1 – 12, use $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ and $w = 3\sqrt{2} - 3i\sqrt{2}$ to compute the quantity. Express your answers in polar form using the principal argument. These exercises should be worked without the aid of a calculator.

1. zw
2. $\frac{z}{w}$
3. $\frac{w}{z}$
4. z^4
5. w^3
6. z^5w^2
7. z^3w^2
8. $\frac{z^2}{w}$
9. $\frac{w}{z^2}$
10. $\frac{z^3}{w^2}$
11. $\frac{w^2}{z^3}$
12. $\left(\frac{w}{z}\right)^6$

In Exercises 13 – 24, use DeMoivre's Theorem to find the indicated power of the given complex number. Express your final answers in rectangular form. These exercises should be worked without the aid of a calculator.

13. $\left(-2+2i\sqrt{3}\right)^3$
14. $\left(-\sqrt{3}-i\right)^3$
15. $\left(-3+3i\right)^4$
16. $\left(\sqrt{3}+i\right)^4$
17. $\left(\frac{5}{2}+\frac{5}{2}i\right)^3$
18. $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^6$
19. $\left(\frac{3}{2}-\frac{3}{2}i\right)^3$
20. $\left(\frac{\sqrt{3}}{3}-\frac{1}{3}i\right)^4$
21. $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^4$
22. $\left(2+2i\right)^5$
23. $\left(\sqrt{3}-i\right)^5$
24. $\left(1-i\right)^8$

In Exercises 25 – 36, find the indicated complex roots. Express your answers in polar form and then convert them into rectangular form. These exercises should be worked without the aid of a calculator.

25. the two square roots of $z = 4i$
26. the two square roots of $z = -25i$
27. the two square roots of $z = 1 + i\sqrt{3}$
28. the two square roots of $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$
29. the three cube roots of $z = 64$
30. the three cube roots of $z = -125$
31. the three cube roots of $z = i$
32. the three cube roots of $z = -8i$

33. the four fourth roots of $z = 16$

34. the four fourth roots of $z = -81$

35. the six sixth roots of $z = 64$

36. the six sixth roots of $z = -729$

37. Use the four complex fourth roots of –4 to show that the factorization of $p(x) = x^4 + 4$ over the real numbers is $p(x) = (x^2 - 2x + 2)(x^2 + 2x + 2)$. You may want to refer to the Complex Factorization Theorem from College Algebra.

38. Use the 12 complex 12$^{\text{th}}$ roots of 4096 to factor $p(x) = x^{12} - 4096$ over the real numbers, into a product of linear and irreducible quadratic factors.

39. Given any natural number $n \geq 2$, the n complex n^{th} roots of the number $z = 1$ are called the ***n***$\boldsymbol{n^{\text{th}}}$ **Roots of Unity**. In the following exercises, assume that n is a fixed, but arbitrary, natural number such that $n \geq 2$.

(a) Show that $w = 1$ is an n^{th} root of unity.

(b) Show that if both w_j and w_k are n^{th} roots of unity then so is their product $w_j w_k$.

(c) Show that if w_j is an n^{th} root of unity then there exists another n^{th} root of unity $w_j{}'$ such that $w_j w_j{}' = 1$. Hint: If $w_j = \text{cis}(\theta)$ let $w_j{}' = \text{cis}(2\pi - \theta)$. You'll need to verify that $w_j{}' = \text{cis}(2\pi - \theta)$ is indeed an n^{th} root of unity.

40. Another way to express the polar form of a complex number is to use the exponential function. For real numbers t, Euler's Formula defines $e^{it} = \cos(t) + i\sin(t)$.

(a) Use **Theorem 8.5** (Products Powers and Quotients of Complex Numbers in Polar Form) to show that $e^{ix}e^{iy} = e^{i(x+y)}$ for all real numbers x and y.

(b) Use **Theorem 8.5** to show that $\left(e^{ix}\right)^n = e^{i(nx)}$ for any real number x and any natural number n.

(c) Use **Theorem 8.5** to show that $\dfrac{e^{ix}}{e^{iy}} = e^{i(x-y)}$ for all real numbers x and y.

(d) If $z = r\text{cis}(\theta)$ is the polar form of z, show that $z = re^{it}$ where $\theta = t$ radians.

(e) Show that $e^{i\pi} + 1 = 0$. (This famous equation relates the five most important constants in all of Mathematics with the three most fundamental operations in Mathematics.)

(f) Show that $\cos(t) = \dfrac{e^{it} + e^{-it}}{2}$ and that $\sin(t) = \dfrac{e^{it} - e^{-it}}{2i}$ for all real numbers *t*.

CHAPTER 9
VECTORS AND PARAMETRIC EQUATIONS

Chapter Outline

9.1 Vector Properties and Operations
9.2 The Unit Vector and Vector Applications
9.3 The Dot Product
9.4 Sketching Curves Described by Parametric Equations
9.5 Finding Parametric Descriptions for Oriented Curves

Introduction

Chapter 9 introduces vectors, their many resulting applications, and parametric equations. We begin in Section 9.1 by learning about the geometric representation of vectors along with vector arithmetic, properties and applications involving bearings. Section 9.2 continues with applications by focusing on component forms of vectors in the solution process. The unit vector is introduced along with operations on vectors in an **i** and **j** format. In this section, vectors are used to model forces. The focus of Section 9.3 is the dot product – operations, properties and applications. In Section 9.4, parametric equations are defined and graphed. Graph behavior is explored through sketching both x and y as a function of t before displaying the resulting parametric graph in the xy-plane. Section 9.5 follows with the elimination of the parameter t in an effort to transform parametric equations to Cartesian equations. Thus, the correlation between Cartesian and parametric equations is established and further efforts are made to transform Cartesian equations into their parametric form.

Throughout Chapter 9, the application of Trigonometry can be seen, providing further evidence of its importance in Mathematics, Engineering, Physics and real-life applications. This chapter is a good stepping off place to conclude the study of Trigonometry before moving on to Calculus.

9.1 Vector Properties and Operations

Learning Objectives

In this section you will:

- **Interpret vectors and vector operations geometrically.**
- **Perform algebraic operations on vectors, including scalar multiplication, addition and determination of inverses.**
- **Determine the component form of a vector.**
- **Find the magnitude and direction of a vector.**

As we have seen numerous times in this book, mathematics can be used to model and solve real-world problems. For many applications, real numbers suffice; that is, real numbers with the appropriate units attached can be used to answer questions like "How close is the nearest Sasquatch nest?" There are other times though, when these kinds of quantities do not suffice. Perhaps it is important to know, for instance, how close the nearest Sasquatch nest is as well as the direction in which it lies. To answer questions like these which involve both a quantitative answer, or *magnitude*, along with a *direction*, we use the mathematical objects called **vectors**[1].

The Geometry of Vectors

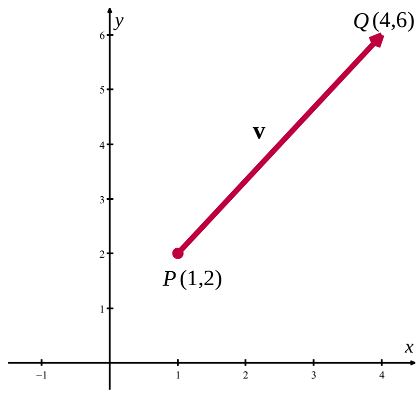
A **vector** is represented geometrically as a directed line segment where the **magnitude** of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrowhead at one endpoint of the segment. A vector has an **initial point**, where it begins, and a **terminal point**, indicated by an arrowhead, where it ends. There are various symbols that distinguish vectors from other quantities:

- Lower case type, boldfaced or with an arrow on top, such as $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$, $\vec{v}$, $\vec{u}$, $\vec{w}$.[2]
- Given an initial point *P* and a terminal point *Q*, a vector can be represented as $\overrightarrow{PQ}$. The arrow on top is what indicates that it is not just a line, but a directed line segment.

Below is a typical vector $\mathbf{v}$ with endpoints $P(1,2)$ and $Q(4,6)$. The point *P* is the initial point, or tail, of $\mathbf{v}$ and the point *Q* is the terminal point, or head, of $\mathbf{v}$. Since we can reconstruct $\mathbf{v}$ completely from *P*

[1] The word 'vector' comes from the Latin *vehere* meaning to convey or to carry.
[2] In this textbook, we will adopt the boldfaced type notation for vectors, without the arrow. In the classroom, your instructor will likely use arrow notation, and arrow notation should be used whenever vectors are written by hand.

and Q, we write $\mathbf{v} = \overrightarrow{PQ}$, where the order of points P (initial point) and Q (terminal point) is important. (Think about this before moving on.)

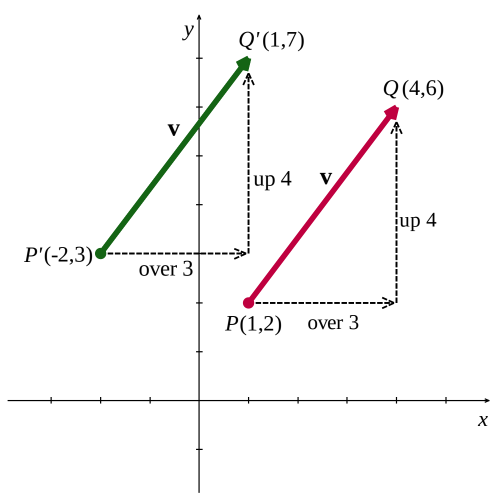


$\mathbf{v} = \overrightarrow{PQ}$

While it is true that P and Q completely determine $\mathbf{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\mathbf{v}$ is considered to be the same vector as $\mathbf{v}$, regardless of its initial point. In the case of our vector $\mathbf{v}$ above, any vector which moves three units to the right and four up[3] from its initial point to arrive at its terminal point is considered the same vector as $\mathbf{v}$.

The Component Form of a Vector

The notation we use to capture this idea is $\mathbf{v} = \langle 3,4 \rangle$, the **component form** of the vector, where the first number, 3, is called the x-component of $\mathbf{v}$ and the second number, 4, is called the y-component of $\mathbf{v}$. If we wanted to reconstruct $\mathbf{v} = \langle 3,4 \rangle$ with initial point $P'(-2,3)$ then we would find the terminal point of $\mathbf{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$, as seen to the right.



$\mathbf{v} = \langle 3,4 \rangle$

[3] If this idea of over and up seems familiar, it should. The slope of the line segment containing **v** is 4/3.

The component form of a vector is what ties these very geometric objects back to Algebra and ultimately Trigonometry. We generalize our example in the following definition.

Definition. Suppose $\mathbf{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\mathbf{v}$ is given by

$$\mathbf{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

Example 9.1.1. Consider the vector whose initial point is $P(2,3)$ and terminal point is $Q(6,4)$. Write $\mathbf{v} = \overrightarrow{PQ}$ in component form.

Solution. Using the definition of component form, we get

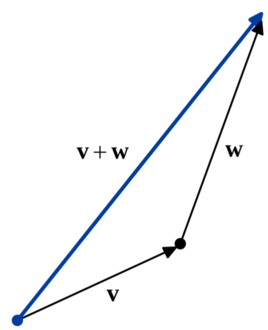
$$\begin{aligned} \mathbf{v} &= \langle 6-2, 4-3 \rangle \\ &= \langle 4,1 \rangle \end{aligned}$$

□

Using the language of components, we have that two vectors are equal if and only if their corresponding components are equal. That is, $\langle v_1, v_2 \rangle = \langle v_1', v_2' \rangle$ if and only if $v_1 = v_1'$ and $v_2 = v_2'$. (Again, think about this before reading on.) We now set about defining operations on vectors.

Vector Addition

Suppose we are given two vectors $\mathbf{v}$ and $\mathbf{w}$. The sum, or **resultant vector** $\mathbf{v}+\mathbf{w}$, is obtained geometrically as follows. First, plot $\mathbf{v}$. Next, plot $\mathbf{w}$ so that its initial point is the terminal point of $\mathbf{v}$. To plot the vector $\mathbf{v}+\mathbf{w}$ we begin at the initial point of $\mathbf{v}$ and end at the terminal point of $\mathbf{w}$. It is helpful to think of the vector $\mathbf{v}+\mathbf{w}$ as the 'net result' of moving along $\mathbf{v}$ then moving along $\mathbf{w}$.

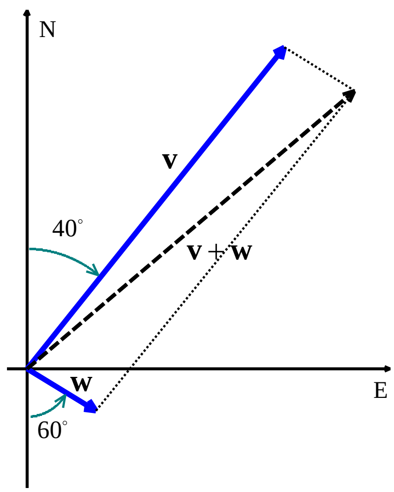


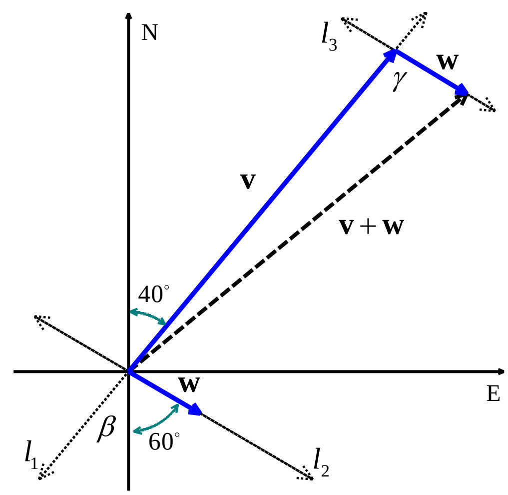
$\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}+\mathbf{w}$

Our next example makes good use of resultant vectors and reviews bearings and the Law of Cosines.[4]

Example 9.1.2. A plane leaves an airport with an airspeed of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution. For both the plane and the wind, we are given their speeds and their directions. Coupling speed (as a magnitude) with direction is the concept of velocity which we've seen a few times before in this textbook. We let $\mathbf{v}$ denote the plane's velocity and $\mathbf{w}$ denote the wind's velocity in the diagram below. The true speed and bearing is found by analyzing the resultant vector, $\mathbf{v}+\mathbf{w}$.

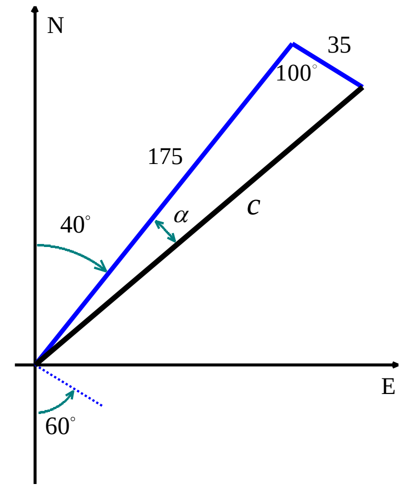




Before proceeding, it will be helpful to determine the angle γ, labeled in the diagram to the right. We extend the vectors $\mathbf{v}$, $\mathbf{w}$ with initial point at origin, and $\mathbf{w}$ with initial point coinciding with the terminal point of $\mathbf{v}$, to give us the three lines l_1, l_2 and l_3, respectively. From vertical angles, we have $\beta = 40^\circ$. With both l_2 and l_3 containing the directed line segment $\mathbf{w}$, we see that they are parallel lines. We then use corresponding angles of parallel lines to conclude that $\gamma = \beta + 60^\circ = 100^\circ$.

[4] If necessary, review the discussion on bearings in the **7.2 Exercises**.

From the vector diagram, we get a triangle, the lengths of whose sides are the magnitude of $\mathbf{v}$, which is 175, the magnitude of $\mathbf{w}$, which is 35, and the magnitude of $\mathbf{v}+\mathbf{w}$, which we'll call c.



The Law of Cosines gives us

$$c^2 = 175^2 + 35^2 - 2(175)(35)\cos(100^\circ)$$
$$c = \sqrt{31850 - 12250\cos(100^\circ)}$$
$$c \approx 184$$

This means the true speed of the plane is approximately 184 miles per hour.

To find the true bearing of the plane we need to determine the angle α. Using the Law of Cosines once more[5], we have

$$35^2 = 175^2 + c^2 - 2(175)(c)\cos(\alpha)$$
$$\cos(\alpha) = \frac{35^2 - 175^2 - c^2}{-2(175)(c)}$$

We use the inverse cosine, along with the value for c from our prior calculation, to find that $\alpha \approx 11^\circ$. Given the geometry of the situation, we add α to the given 40° and find the true bearing of the plane to be approximately N51°E.

□

We next define the addition of vectors component-wise to match the geometry.[6]

Definition. Suppose $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$. The vector $\mathbf{v}+\mathbf{w}$ is defined by

$$\mathbf{v}+\mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

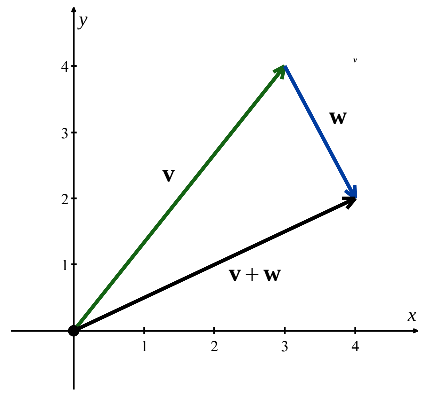
Example 9.1.3. Let $\mathbf{v} = \langle 3,4 \rangle$ and suppose $\mathbf{w} = \overrightarrow{PQ}$ for $P(-3,7)$ and $Q(-2,5)$. Find $\mathbf{v}+\mathbf{w}$ and interpret this sum geometrically.

Solution. Before adding the vectors using the definition, we need to write $\mathbf{w}$ in component form. We get $\mathbf{w} = \langle -2-(-3), 5-7 \rangle = \langle 1,-2 \rangle$, and then

[5] Or, since our given angle, 100°, is obtuse, we could use the Law of Sines without any ambiguity here.
[6] Adding vectors component-wise should look hauntingly familiar. Compare this with matrix addition. In fact, in more advanced courses such as Linear Algebra, vectors are defined as 1 by *n* or *n* by 1 matrices, depending on the situation.

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= \langle 3,4\rangle + \langle 1,-2\rangle \\ &= \langle 3+1, 4+(-2)\rangle \\ &= \langle 4,2\rangle\end{aligned}$$

To visualize this sum, we draw $\mathbf{v}$ with its initial point at $(0,0)$, for convenience, so that its terminal point is $(3,4)$. Next, we graph $\mathbf{w}$ with its initial point at $(3,4)$. Moving one to the right and two down, we find the terminal point of $\mathbf{w}$ to be $(4,2)$. We see that the vector $\mathbf{v}+\mathbf{w}$ has initial point $(0,0)$ and terminal point $(4,2)$ so its component form is $\langle 4,2\rangle$ as required.



□

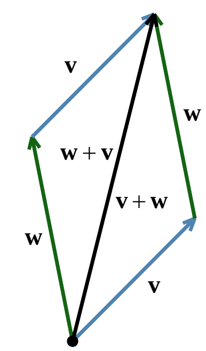
In order for vector addition to enjoy the same kinds of properties as real number addition, it is necessary to extend our definition of vectors to include a **zero vector**, $\mathbf{0} = \langle 0,0\rangle$. Geometrically, $\mathbf{0}$ represents a point, which we can think of as a directed line segment with the same initial and terminal points. The reader may well object to the inclusion of $\mathbf{0}$ since, after all, vectors are supposed to have both a magnitude (length) and a direction. While it seems clear that the magnitude of $\mathbf{0}$ should be 0, it is not clear what its direction is. As we shall see, the direction of $\mathbf{0}$ is in fact undefined, but this minor hiccup in the natural flow of things is worth the benefits we reap by including $\mathbf{0}$ in our discussions. We have the following theorem.

Theorem 9.1. Properties of Vector Addition.

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.
- **Associative Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
- **Identity Property:** The vector $\mathbf{0}$ acts as the additive identity for vector addition. That is, for all vectors $\mathbf{v}$, $\mathbf{v}+\mathbf{0}=\mathbf{0}+\mathbf{v}=\mathbf{v}$.
- **Inverse Property:** Every vector $\mathbf{v}$ has a unique additive inverse, denoted $\mathbf{-v}$. That is, for every vector $\mathbf{v}$, there is a vector $\mathbf{-v}$ so that $\mathbf{v}+(\mathbf{-v})=(\mathbf{-v})+\mathbf{v}=\mathbf{0}$.

The properties in **Theorem 9.1** are easily verified using the definition of vector addition. For the commutative property, we note that if $\mathbf{v}=\langle v_1, v_2\rangle$ and $\mathbf{w}=\langle w_1, w_2\rangle$ then

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= \langle v_1, v_2\rangle + \langle w_1, w_2\rangle \\ &= \langle v_1 + w_1, v_2 + w_2\rangle \quad \textbf{definition of vector addition} \\ &= \langle w_1 + v_1, w_2 + v_2\rangle \quad \textbf{commutative property of real number addition} \\ &= \langle w_1, w_2\rangle + \langle v_1, v_2\rangle \quad \textbf{definition of vector addition} \\ &= \mathbf{w}+\mathbf{v}\end{aligned}$$



Geometrically, we can 'see' the commutative property by realizing that the sums $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{v}$ are the same directed diagonal determined by the parallelogram.

The proofs of the associative and identity properties proceed similarly, and the reader is encouraged to verify them and provide accompanying diagrams.

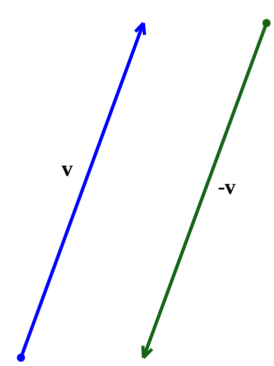
The Additive Inverse

The existence and uniqueness of the additive inverse is yet another property inherited from the real numbers. Given a vector $\mathbf{v} = \langle v_1, v_2\rangle$, suppose we wish to find a vector $\mathbf{w} = \langle w_1, w_2\rangle$ so that $\mathbf{v}+\mathbf{w} = \mathbf{0}$. By the definition of vector addition, we have

$$\mathbf{v}+\mathbf{w} = \mathbf{0}$$
$$\langle v_1 + w_1, v_2 + w_2\rangle = \langle 0, 0\rangle$$

Hence, $v_1 + w_1 = 0$ and $v_2 + w_2 = 0$, from which $w_1 = -v_1$ and $w_2 = -v_2$, with the result that $\mathbf{w} = \langle -v_1, -v_2\rangle$. Hence, $\mathbf{v}$ has an additive inverse, and moreover it is unique and can be obtained by the formula $\mathbf{-v} = \langle -v_1, -v_2\rangle$.

Geometrically, the vectors $\mathbf{v} = \langle v_1, v_2\rangle$ and $\mathbf{-v} = \langle -v_1, -v_2\rangle$ have the same length but opposite directions. As a result, when adding the vectors geometrically, the sum $\mathbf{v}+(\mathbf{-v})$ results in starting at the initial point of $\mathbf{v}$ and ending back at the initial point of $\mathbf{v}$. Or, in other words, the net result of moving $\mathbf{v}$ then $\mathbf{-v}$ is not moving at all.[7]

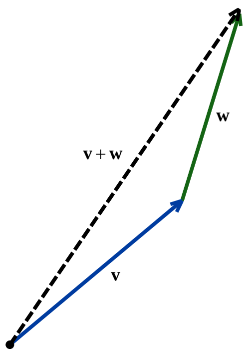


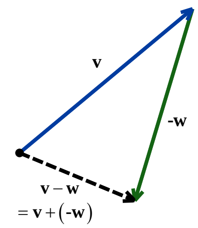
[7] An interesting property of a vector and its inverse is that the two vectors are 'parallel'. In fact, we say two non-zero vectors are **parallel** when they have the same or opposite directions. That is, **v** is parallel to **w** if **v**=*k***w** for some real, non-zero, number *k*.

Using the additive inverse of a vector, we can define **vector subtraction**, or the difference of two vectors, as $\mathbf{v}-\mathbf{w}=\mathbf{v}+(\text{-}\mathbf{w})$. If $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$ then

$$\begin{aligned}\mathbf{v}-\mathbf{w}&=\mathbf{v}+(\text{-}\mathbf{w})\\&=\langle v_1,v_2\rangle+\langle -w_1,-w_2\rangle\\&=\langle v_1+(-w_1),v_2+(-w_2)\rangle\\&=\langle v_1-w_1,v_2-w_2\rangle\end{aligned}$$

In other words, like vector addition, vector subtraction works component-wise. Below, we observe a geometrical interpretation of vector addition and vector subtraction.

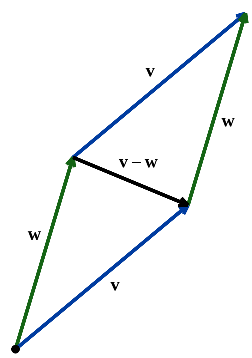




To interpret the vector $\mathbf{v}-\mathbf{w}$ geometrically, we note

$$\begin{aligned}\mathbf{w}+(\mathbf{v}-\mathbf{w})&=\mathbf{w}+\big(\mathbf{v}+(\text{-}\mathbf{w})\big) && \text{definition of vector subtraction}\\&=\mathbf{w}+\big((\text{-}\mathbf{w})+\mathbf{v}\big) && \text{commutativity of vector addition}\\&=\big(\mathbf{w}+(\text{-}\mathbf{w})\big)+\mathbf{v} && \text{associativity of vector addition}\\&=\mathbf{0}+\mathbf{v} && \text{definition of additive inverse}\\&=\mathbf{v} && \text{definition of additive identity}\end{aligned}$$

This means that the net result of moving along $\mathbf{w}$, then moving along $\mathbf{v}-\mathbf{w}$, is just $\mathbf{v}$ itself. From the diagram, we see that $\mathbf{v}-\mathbf{w}$ may be interpreted as the vector whose initial point is the terminal point of $\mathbf{w}$ and whose terminal point is the terminal point of $\mathbf{v}$. It is also worth mentioning that in the parallelogram determined by the vectors $\mathbf{v}$ and $\mathbf{w}$, the vector $\mathbf{v}-\mathbf{w}$ is one of the diagonals, the other being $\mathbf{v}+\mathbf{w}$.



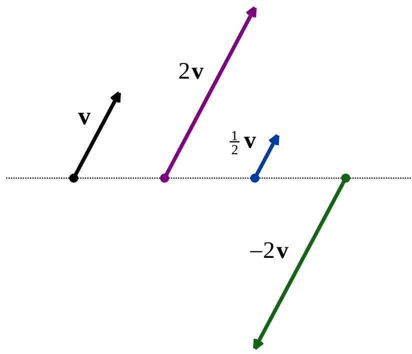
Scalar Multiplication

Next, we discuss *scalar* multiplication, which is taking a real number times a vector. We define scalar multiplication for vectors in the same way we defined it for matrices.

Definition. If k is a real number and $\mathbf{v} = \langle v_1, v_2 \rangle$, we define $k\mathbf{v}$ by

$$k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$$

Scalar multiplication of vectors by k can be understood geometrically as scaling the vector (if $k > 0$) or scaling the vector and reversing its direction (if $k < 0$) as demonstrated to the right.



Note that, by definition,

$$\begin{aligned}(-1)\,\mathbf{v} &= (-1)\langle v_1, v_2 \rangle \\ &= \langle (-1)v_1, (-1)v_2 \rangle \\ &= \langle -v_1, -v_2 \rangle \\ &= \mathbf{-v}\end{aligned}$$

This and other properties of scalar multiplication are summarized below.

Theorem 9.2. Properties of Scalar Multiplication.

- **Associative Property:** For every vector $\mathbf{v}$ and scalars k and r, $(kr)\mathbf{v} = k(r\mathbf{v})$.
- **Identity Property:** For all vectors $\mathbf{v}$, $1\mathbf{v} = \mathbf{v}$.
- **Additive Inverse Property:** For all vectors $\mathbf{v}$, $\mathbf{-v} = (-1)\mathbf{v}$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every vector $\mathbf{v}$ and scalars k and r, $(k + r)\mathbf{v} = k\mathbf{v} + r\mathbf{v}$.
- **Distributive Property of Scalar Multiplication over Vector Addition:** For all vectors $\mathbf{v}$ and $\mathbf{w}$ and scalars k, $k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w}$.
- **Zero Product Property:** If $\mathbf{v}$ is a vector and k is a scalar, then $k\mathbf{v} = \mathbf{0}$ if and only if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

The proof of **Theorem 9.2**, like the proof of **Theorem 9.1**, ultimately boils down to the definition of scalar multiplication and properties of real numbers. For example, to prove the associative property, we let $\mathbf{v} = \langle v_1, v_2 \rangle$. If k and r are scalars, then

$$
\begin{aligned}
(kr)\mathbf{v} &= (kr)\langle v_1, v_2\rangle \\
&= \langle (kr)v_1, (kr)v_2\rangle && \textbf{definition of scalar multiplication} \\
&= \langle k(rv_1), k(rv_2)\rangle && \textbf{associative property of real number multiplication} \\
&= k\langle rv_1, rv_2\rangle && \textbf{definition of scalar multiplication} \\
&= k\left(r\langle v_1, v_2\rangle\right) && \textbf{definition of scalar multiplication} \\
&= k(r\mathbf{v})
\end{aligned}
$$

The remaining properties are proved similarly and are left as exercises.

Our next example demonstrates how **Theorem 9.2** allows us to do the same kind of algebraic manipulations with vectors as we do with variables, multiplication and division of vectors notwithstanding.

Example 9.1.4. Solve $5\mathbf{v} - 2\left(\mathbf{v} + \langle 1, -2\rangle\right) = \mathbf{0}$ for $\mathbf{v}$.

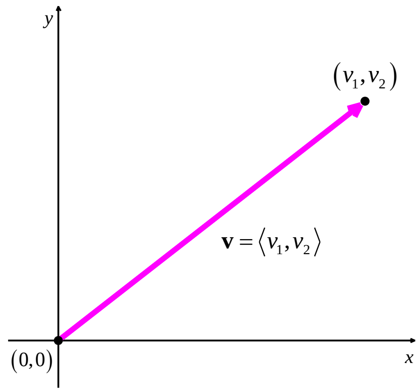
Solution.

$$
\begin{aligned}
5\mathbf{v} - 2\left(\mathbf{v} + \langle 1, -2\rangle\right) &= \mathbf{0} \\
5\mathbf{v} + (-2)\left(\mathbf{v} + \langle 1, -2\rangle\right) &= \mathbf{0} \\
5\mathbf{v} + (-2)\mathbf{v} + (-2)\langle 1, -2\rangle &= \mathbf{0} && \textbf{distributive property over vector addition} \\
3\mathbf{v} + (-2)\langle 1, -2\rangle &= \mathbf{0} && \textbf{distributive property over scalar addition} \\
3\mathbf{v} + \langle -2, 4\rangle &= \mathbf{0} && \textbf{definition of scalar multiplication} \\
3\mathbf{v} + \langle -2, 4\rangle + \langle 2, -4\rangle &= \mathbf{0} + \langle 2, -4\rangle \\
3\mathbf{v} + \langle 0, 0\rangle &= \mathbf{0} + \langle 2, -4\rangle && \textbf{definition of vector addition} \\
3\mathbf{v} &= \langle 2, -4\rangle && \textbf{property of additive identity} \\
\frac{1}{3}(3\mathbf{v}) &= \frac{1}{3}\langle 2, -4\rangle \\
1\mathbf{v} &= \left\langle \frac{2}{3}, -\frac{4}{3}\right\rangle && \textbf{associative property, scalar multiplication} \\
\mathbf{v} &= \left\langle \frac{2}{3}, -\frac{4}{3}\right\rangle && \textbf{property of multiplicative identity}
\end{aligned}
$$

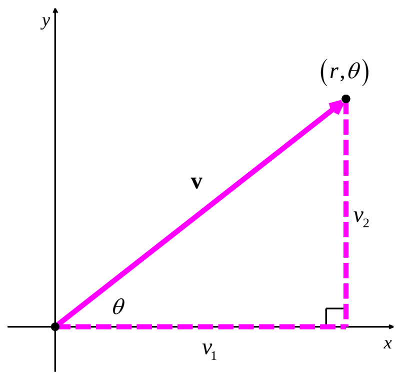
□

Vectors in Standard Position

A vector whose initial point is $(0,0)$ is said to be in **standard position**. If $\mathbf{v} = \langle v_1, v_2\rangle$ is plotted in standard position, then its terminal point is necessarily (v_1, v_2). (Once more, think about this before reading on.)



$\mathbf{v} = \langle v_1, v_2 \rangle$ in standard position



Plotting a vector in standard position enables us to more easily quantify the concepts of magnitude and direction of the vector. We can convert the point (v_1, v_2) in rectangular coordinates to a pair (r, θ) in polar coordinates where $r \geq 0$.

The magnitude of $\mathbf{v}$, which we said earlier was the length of the directed line segment, is denoted $\|\mathbf{v}\|$. We can see from the right triangle corresponding to the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ that $\|\mathbf{v}\| = r$, for $r \geq 0$, and we use the Pythagorean Theorem to find $r = \sqrt{v_1^2 + v_2^2}$. Using right triangle trigonometry, we also find that

$$v_1 = r\cos(\theta) = \|\mathbf{v}\|\cos(\theta)$$
$$v_2 = r\sin(\theta) = \|\mathbf{v}\|\sin(\theta)$$

From the definition of scalar multiplication and vector equality, we get

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= \langle \|\mathbf{v}\|\cos(\theta), \|\mathbf{v}\|\sin(\theta) \rangle \\ &= \|\mathbf{v}\| \langle \cos(\theta), \sin(\theta) \rangle \end{aligned}$$

This motivates the following definition.

Definition. Suppose $\mathbf{v}$ is a vector with component form $\mathbf{v} = \langle v_1, v_2 \rangle$. Let (r, θ) be a polar representation of the point with rectangular coordinates (v_1, v_2) with $r \geq 0$.

- The **magnitude** of $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is given by $\|\mathbf{v}\| = r = \sqrt{v_1^2 + v_2^2}$.
- If $\mathbf{v} \neq \mathbf{0}$, the **direction angle** of $\mathbf{v}$ is given by θ.

Taken together, we get $\mathbf{v} = \langle \|\mathbf{v}\|\cos(\theta), \|\mathbf{v}\|\sin(\theta) \rangle$.

A few remarks are in order.

- We note that if $\mathbf{v} \neq \mathbf{0}$ then even though there are infinitely many angles θ which satisfy the preceding definition, the stipulation $r > 0$ means that all of the angles are coterminal. Hence, if θ and θ' both satisfy the conditions of the definition, then $\cos(\theta) = \cos(\theta')$ and $\sin(\theta) = \sin(\theta')$, and as such, $\langle \cos(\theta), \sin(\theta) \rangle = \langle \cos(\theta'), \sin(\theta') \rangle$, making $\mathbf{v}$ well-defined.
- If $\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \langle 0, 0 \rangle$, and we know from **Section 8.1** that $(0, \theta)$ is a polar representation for the origin for any angle θ. For this reason, the direction of $\mathbf{0}$ is undefined.

The following theorem summarizes the important facts about the magnitude and direction of a vector.

Theorem 9.3. Properties of Magnitude and Direction: Suppose $\mathbf{v}$ is a vector.

- Then $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- For all scalars k, $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.
- If $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle$

The proof of the first property in **Theorem 9.3** is a direct consequence of the definition of $\|\mathbf{v}\|$. If $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$ which is by definition greater than or equal to 0. Moreover, $\sqrt{v_1^2 + v_2^2} = 0$ if and only if $v_1^2 + v_2^2 = 0$, if and only if $v_1 = v_2 = 0$. Hence, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \langle 0, 0 \rangle = \mathbf{0}$, as required.

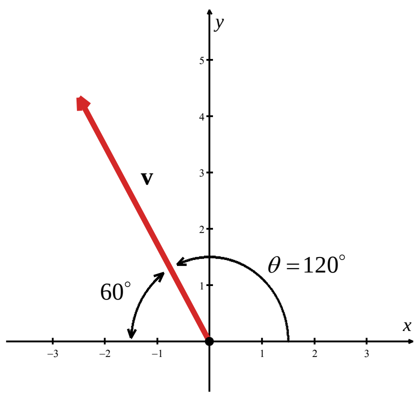
The second property is a result of the definition of magnitude and scalar multiplication along with a property of radicals. If $\mathbf{v} = \langle v_1, v_2 \rangle$ and k is a scalar then

$$
\begin{aligned}
\|k\mathbf{v}\| &= \|k\langle v_1, v_2 \rangle\| \\
&= \|\langle kv_1, kv_2 \rangle\| && \textbf{definition of scalar multiplication} \\
&= \sqrt{(kv_1)^2 + (kv_2)^2} && \textbf{definition of magnitude} \\
&= \sqrt{k^2 v_1^2 + k^2 v_2^2} \\
&= \sqrt{k^2 (v_1^2 + v_2^2)} \\
&= \sqrt{k^2}\sqrt{v_1^2 + v_2^2} && \textbf{product rule for radicals} \\
&= |k|\sqrt{v_1^2 + v_2^2} && \textbf{since } \sqrt{k^2} = |k| \\
&= |k|\|\mathbf{v}\| && \textbf{definition of magnitude}
\end{aligned}
$$

The equation $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle$ in **Theorem 9.3** is a consequence of the component definition for vectors, $\mathbf{v} = \langle \|\mathbf{v}\|\cos(\theta), \|\mathbf{v}\|\sin(\theta) \rangle$, and was worked out prior to that definition. In words, the equation $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle$ says that any given vector is the product of its magnitude and direction, an important concept to keep in mind when studying and using vectors.

Example 9.1.5. Find the component form of the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 5$ so that when $\mathbf{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle[8] with the negative *x*-axis.

Solution. We are told that $\|\mathbf{v}\| = 5$ and are given information about its direction, so we can use the formula $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle$ to get the component form of $\mathbf{v}$. To determine θ, we are told that $\mathbf{v}$ lies in Quadrant II and makes a 60° angle with the negative *x*-axis, so a polar form of the terminal point of $\mathbf{v}$, when plotted in standard position, is $(5, 120^\circ)$. (See the diagram below.)



Thus,

$$\begin{aligned} \mathbf{v} &= \|\mathbf{v}\|\left\langle \cos(120^\circ), \sin(120^\circ) \right\rangle \\ &= 5\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\ &= \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle \end{aligned}$$

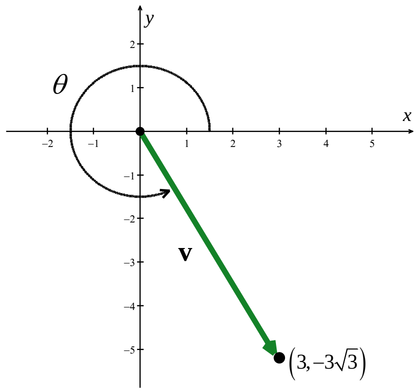
□

Example 9.1.6. For $\mathbf{v} = \langle 3, -3\sqrt{3} \rangle$, find $\|\mathbf{v}\|$ and θ, $0 \le \theta < 2\pi$ so that $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta) \rangle$.

[8] Due to the utility of vectors in real-world applications, we will usually use degree measure for the angle when giving the vector's direction.

Solution. For $\mathbf{v} = \langle 3, -3\sqrt{3} \rangle$, we get $\|\mathbf{v}\| = \sqrt{(3)^2 + \left(-3\sqrt{3}\right)^2} = 6$.

We can find the θ we're after by converting the point with rectangular coordinates $\left(3, -3\sqrt{3}\right)$ to polar form (r, θ), where $r = \|\mathbf{v}\| > 0$. From **Section 8.1**, we have



$$\begin{aligned}\tan(\theta) &= \frac{y}{x} \\ &= \frac{-3\sqrt{3}}{3} \\ &= -\sqrt{3}\end{aligned}$$

Since $\left(3, -3\sqrt{3}\right)$ is a point in Quadrant IV, θ is a Quadrant IV angle. Hence, we pick $\theta = \frac{5\pi}{3}$.

We may check our answer by verifying that $\mathbf{v} = \langle 3, -3\sqrt{3} \rangle = 6\left\langle \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right\rangle$.

□

Example 9.1.7. For the vectors $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle 1, -2 \rangle$, find the following:

1. $\|\mathbf{v}\| - 2\|\mathbf{w}\|$
2. $\|\mathbf{v} - 2\mathbf{w}\|$

Solution.

1. For $\mathbf{v} = \langle 3, 4 \rangle$, we have $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$. The magnitude of $\mathbf{w} = \langle 1, -2 \rangle$ is $\|\mathbf{w}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Hence, $\|\mathbf{v}\| - 2\|\mathbf{w}\| = 5 - 2\sqrt{2}$.

2. In the expression $\|\mathbf{v} - 2\mathbf{w}\|$, notice that the arithmetic on the vectors comes first, then the magnitude. Hence, our first step is to find the component form of the vector $\mathbf{v} - 2\mathbf{w}$.

$$\begin{aligned}\mathbf{v} - 2\mathbf{w} &= \langle 3, 4 \rangle - 2\langle 1, -2 \rangle \\ &= \langle 3, 4 \rangle + \langle -2, 4 \rangle \\ &= \langle 1, 8 \rangle\end{aligned}$$

Then

$$\begin{aligned}\|\mathbf{v} - 2\mathbf{w}\| &= \|\langle 1, 8 \rangle\| \\ &= \sqrt{1^2 + 8^2} \\ &= \sqrt{65}\end{aligned}$$

□

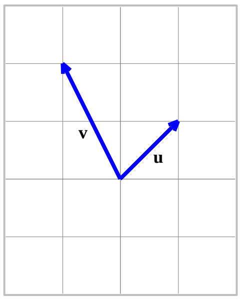
9.1 Exercises

In Exercises 1 – 3, sketch $\mathbf{v}$, $3\mathbf{v}$ and $\frac{1}{2}\mathbf{v}$.

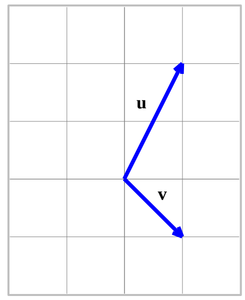
1. $\mathbf{v} = \langle 2, -1 \rangle$
2. $\mathbf{v} = \langle -1, 4 \rangle$
3. $\mathbf{v} = \langle -3, -2 \rangle$

In Exercises 4 – 6, sketch $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$ and $2\mathbf{u}$.

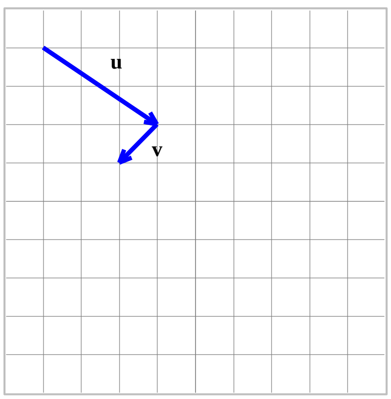
4.



5.



6.



In Exercises 7 – 8, use the given pair of vectors to compute $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$ and $2\mathbf{u}-3\mathbf{v}$.

7. $\mathbf{u} = \langle 2, -3 \rangle$, $\mathbf{v} = \langle 1, 5 \rangle$
8. $\mathbf{u} = \langle -3, 4 \rangle$ $\mathbf{v} = \langle -2, 1 \rangle$

In Exercises 9 – 16, use the given pair of vectors to find the following quantities. State whether the result is a vector or a scalar.

- $\mathbf{v}+\mathbf{w}$
- $\mathbf{w}-2\mathbf{v}$
- $\|\mathbf{v}+\mathbf{w}\|$
- $\|\mathbf{v}\|+\|\mathbf{w}\|$
- $\|\mathbf{v}\|\mathbf{w}-\|\mathbf{w}\|\mathbf{v}$
- $\|\mathbf{w}\|\mathbf{v}$

Finally, verify that the vectors satisfy the Parallelogram Law: $\|\mathbf{v}\|^2+\|\mathbf{w}\|^2=\frac{1}{2}\left[\|\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{v}-\mathbf{w}\|^2\right]$.

9. $\mathbf{v} = \langle 12, -5 \rangle$, $\mathbf{w} = \langle 3, 4 \rangle$
10. $\mathbf{v} = \langle -7, 24 \rangle$, $\mathbf{w} = \langle -5, -12 \rangle$
11. $\mathbf{v} = \langle 2, -1 \rangle$, $\mathbf{w} = \langle -2, 4 \rangle$
12. $\mathbf{v} = \langle 10, 4 \rangle$, $\mathbf{w} = \langle -2, 5 \rangle$

13. $\mathbf{v} = \left\langle -\sqrt{3}, 1 \right\rangle$, $\mathbf{w} = \left\langle 2\sqrt{3}, 2 \right\rangle$

14. $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $\mathbf{w} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$

15. $\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$, $\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

16. $\mathbf{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, $\mathbf{w} = \left\langle -1, -\sqrt{3} \right\rangle$

17. Given a vector with initial point $(5, 2)$ and terminal point $(-1, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b \rangle$.

18. Given a vector with initial point $(-4, 2)$ and terminal point $(3, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b \rangle$.

19. Given a vector with initial point $(7, -1)$ and terminal point $(-1, -7)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b \rangle$.

In Exercises 20 – 34, find the component form of the vector $\mathbf{v}$ using the information given about its magnitude and direction. Give exact values.

20. $\|\mathbf{v}\| = 6$; when drawn in standard position $\mathbf{v}$ lies in Quadrant I and makes a 60° angle with the positive *x*-axis.

21. $\|\mathbf{v}\| = 3$; when drawn in standard position $\mathbf{v}$ lies in Quadrant I and makes a 45° angle with the positive *x*-axis.

22. $\|\mathbf{v}\| = \frac{2}{3}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant I and makes a 60° angle with the positive *x*-axis.

23. $\|\mathbf{v}\| = 12$; when drawn in standard position $\mathbf{v}$ lies along the positive *y*-axis.

24. $\|\mathbf{v}\| = 4$; when drawn in standard position $\mathbf{v}$ lies in Quadrant II and makes a 30° angle with the negative *x*-axis.

25. $\|\mathbf{v}\| = 2\sqrt{3}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant II and makes a 30° angle with the positive *y*-axis.

26. $\|\mathbf{v}\| = \frac{7}{2}$; when drawn in standard position $\mathbf{v}$ lies along the negative *x*-axis.

27. $\|\mathbf{v}\| = 5\sqrt{6}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant III and makes a 45° angle with the negative *x*-axis.

28. $\|\mathbf{v}\| = 6.25$; when drawn in standard position $\mathbf{v}$ lies along the negative *y*-axis.

29. $\|\mathbf{v}\| = 4\sqrt{3}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant IV and makes a 30° angle with the positive *x*-axis.

30. $\|\mathbf{v}\| = 5\sqrt{2}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant IV and makes a 45° angle with the negative *y*-axis.

31. $\|\mathbf{v}\| = 2\sqrt{5}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant I and makes an angle measuring $\arctan(2)$ with the positive *x*-axis.

32. $\|\mathbf{v}\| = \sqrt{10}$; when drawn in standard position $\mathbf{v}$ lies in Quadrant II and makes an angle measuring $\arctan(3)$ with the negative *x*-axis.

33. $\|\mathbf{v}\| = 5$; when drawn in standard position $\mathbf{v}$ lies in Quadrant III and makes an angle measuring $\arctan\left(\frac{4}{3}\right)$ with the negative *x*-axis.

34. $\|\mathbf{v}\| = 26$; when drawn in standard position $\mathbf{v}$ lies in Quadrant IV and makes an angle measuring $\arctan\left(\frac{5}{12}\right)$ with the positive *x*-axis.

In Exercises 35 – 40, approximate the component form of the vector $\mathbf{v}$ using the information given about its magnitude and direction. Round your approximations to two decimal places.

35. $\|\mathbf{v}\| = 392$; when drawn in standard position $\mathbf{v}$ makes a 117° angle with the positive *x*-axis.

36. $\|\mathbf{v}\| = 63.92$; when drawn in standard position $\mathbf{v}$ makes a 78.3° angle with the positive *x*-axis.

37. $\|\mathbf{v}\| = 5280$; when drawn in standard position $\mathbf{v}$ makes a 12° angle with the positive *x*-axis.

38. $\|\mathbf{v}\| = 450$; when drawn in standard position $\mathbf{v}$ makes a 210.75° angle with the positive *x*-axis.

39. $\|\mathbf{v}\| = 168.7$; when drawn in standard position $\mathbf{v}$ makes a 252° angle with the positive *x*-axis.

40. $\|\mathbf{v}\| = 26$; when drawn in standard position $\mathbf{v}$ makes a 304.5° angle with the positive *x*-axis.

In Exercises 41 – 58, for the given vector $\mathbf{v}$, find the magnitude $\|\mathbf{v}\|$ and an angle θ with $0^\circ \le \theta < 360^\circ$ so that $\mathbf{v} = \|\mathbf{v}\|\langle \cos(\theta), \sin(\theta)\rangle$ (See definition of magnitude and direction.) Round approximations to two decimal places.

41. $\mathbf{v} = \langle 1, \sqrt{3}\rangle$
42. $\mathbf{v} = \langle 5, 5\rangle$
43. $\mathbf{v} = \langle -2\sqrt{3}, 2\rangle$
44. $\mathbf{v} = \langle -\sqrt{2}, \sqrt{2}\rangle$
45. $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$
46. $\mathbf{v} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$
47. $\mathbf{v} = \langle 6, 0\rangle$
48. $\mathbf{v} = \langle -2.5, 0\rangle$
49. $\mathbf{v} = \langle 0, \sqrt{7}\rangle$
50. $\mathbf{v} = \langle 3, 4\rangle$
51. $\mathbf{v} = \langle 12, 5\rangle$
52. $\mathbf{v} = \langle -4, 3\rangle$
53. $\mathbf{v} = \langle -7, 24\rangle$
54. $\mathbf{v} = \langle -2, -1\rangle$
55. $\mathbf{v} = \langle -2, -6\rangle$
56. $\mathbf{v} = \langle 123.4, -77.05\rangle$
57. $\mathbf{v} = \langle 965.15, 831.6\rangle$
58. $\mathbf{v} = \langle -114.1, 42.3\rangle$

59. A small boat leaves the dock at Camp DuNuthin and heads across the Nessie River at 17 miles per hour (that is, with respect to the water) at a bearing of S68°W. The river is flowing due east at 8 miles per hour. What is the boat's true speed and heading? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

60. The HMS Sasquatch leaves port with bearing S20°E maintaining a speed of 42 miles per hour (that is, with respect to the water). If the ocean current is 5 miles per hour with a bearing of N60°E, find the HMS Sasquatch's true speed and bearing. Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

61. The goal of this exercise is to use vectors to describe non-vertical lines in the plane. To that end, consider the line $y = 2x - 4$. Let $\mathbf{v_0} = \langle 0, -4\rangle$ and let $\mathbf{s} = \langle 1, 2\rangle$. Let *t* be any real number. Show that the vector defined by $\mathbf{v} = \mathbf{v_0} + t\mathbf{s}$, when drawn in standard position, has its terminal point on the line $y = 2x - 4$. (Hint: Show that $\mathbf{v_0} + t\mathbf{s} = \langle t, 2t - 4\rangle$ for any real number *t*.)

Now consider the non-vertical line $y = mx + b$. Repeat the previous analysis with $\mathbf{v_0} = \langle 0, b\rangle$ and let $\mathbf{s} = \langle 1, m\rangle$. Thus, any non-vertical line can be thought of as a collection of terminal points of the vector sum of $\langle 0, b \rangle$ (the position vector of the *y*-intercept) and a scalar multiple of the slope vector $\mathbf{s} = \langle 1, m \rangle$.

62. Prove the associative and identity properties of vector addition in **Theorem 9.1**.

63. Prove the properties of scalar multiplication in **Theorem 9.2**.

9.2 The Unit Vector and Vector Applications

Learning Objectives

In this section you will:

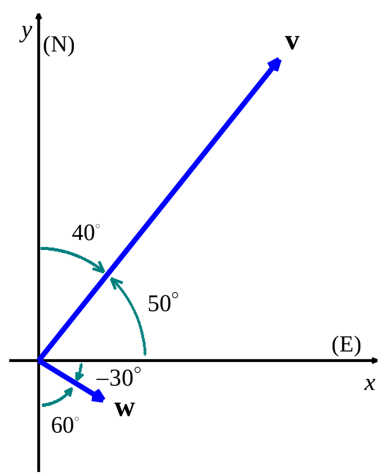
- **Use vectors in component form to solve applications.**
- **Find the unit vector in a given direction.**
- **Perform operations on vectors in terms of i and j.**
- **Use vectors to model forces.**

Using Vectors in Component Form to Solve Applications

We continue our discussion of component forms of vectors from **Section 9.1** and resume the process of **resolving** vectors into their components. This next example revisits **Example 9.1.2**, making use of component forms and vector algebra to solve this problem.

Example 9.2.1. A plane leaves an airport with an airspeed of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution. We proceed as we did in **Example 9.1.2** and let $\mathbf{v}$ denote the plane's velocity and $\mathbf{w}$ denote the wind's velocity, and set about determining $\mathbf{v}+\mathbf{w}$. If we regard the airport as being at the origin, the positive *y*-axis as acting as due north and the positive *x*-axis acting as due east, we see that the vectors $\mathbf{v}$ and $\mathbf{w}$ are in standard position and their directions correspond to the angles 50° and -30°, respectively.



Hence, the component forms:

$$\mathbf{v} = 175\langle \cos(50^\circ), \sin(50^\circ)\rangle \qquad \mathbf{w} = 35\langle \cos(-30^\circ), \sin(-30^\circ)\rangle$$
$$= \langle 175\cos(50^\circ), 175\sin(50^\circ)\rangle \qquad = \langle 35\cos(-30^\circ), 35\sin(-30^\circ)\rangle$$

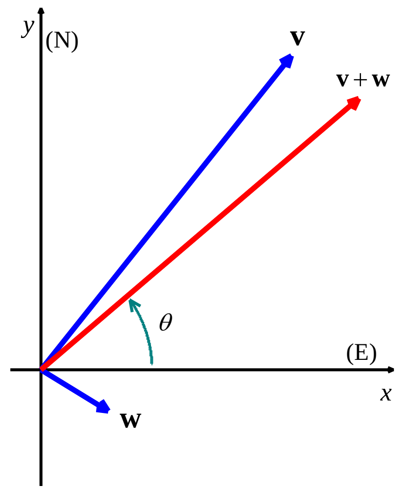
Since we have no convenient way to express the exact values of cosine and sine of 50°, we leave both vectors in terms of cosines and sines. Adding corresponding components, we find the resultant vector:

$$\mathbf{v}+\mathbf{w} = \langle 175\cos(50^\circ)+35\cos(-30^\circ), 175\sin(50^\circ)+35\sin(-30^\circ)\rangle$$

To find the true speed of the plane, we compute the magnitude of the resultant vector:

$$\|\mathbf{v}+\mathbf{w}\| = \sqrt{\left(175\cos(50^\circ)+35\cos(-30^\circ)\right)^2+\left(175\sin(50^\circ)+35\sin(-30^\circ)\right)^2}$$
$$\approx 184$$

Hence, the true speed of the plane is approximately 184 miles per hour. To find the true bearing, we need to find the angle θ which corresponds to the polar form (r,θ), $r>0$, of the point $(x,y)=\left(175\cos(50^\circ)+35\cos(-30^\circ),175\sin(50^\circ)+35\sin(-30^\circ)\right)$. Since both of these coordinates are positive[1], we know θ is a Quadrant I angle, as depicted below.



Furthermore,

$$\tan(\theta) = \frac{y}{x}$$
$$= \frac{175\sin(50^\circ)+35\sin(-30^\circ)}{175\cos(50^\circ)+35\cos(-30^\circ)}$$

Using the arctangent function, we get $\theta \approx 39^\circ$. Since, for the purposes of bearing, we need the angle between $\mathbf{v}+\mathbf{w}$ and the positive *y*-axis, we take the complement of θ and find the true bearing of the plane to be approximately N51°E.

□

The Unit Vector

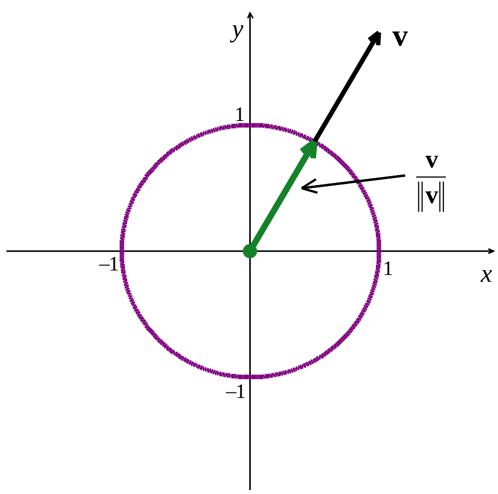
In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**.

[1] Yes, a calculator approximation is the quickest way to see this, but you can also use good old-fashioned inequalities and the fact that $45^\circ \le 50^\circ \le 60^\circ$.

Definition. Let $\mathbf{v}$ be a vector. If $\|\mathbf{v}\|=1$, we say that $\mathbf{v}$ is a **unit vector**.

Any nonzero vector divided by its magnitude is a unit vector. If $\mathbf{v}$ is a nonzero vector, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a 'unit vector in the **direction** of $\mathbf{v}$'. Noting that magnitude is always a scalar, and that dividing by a scalar is the same as multiplying by its reciprocal, a unit vector for any nonzero vector $\mathbf{v}$ can be found through multiplication by $\frac{1}{\|\mathbf{v}\|}$. The process of multiplying a nonzero vector by the reciprocal of its magnitude is called '**normalizing** the vector'. We leave it as an exercise to show that $\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$ is a unit vector for any nonzero vector $\mathbf{v}$.

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. (You should take time to show this.) As a result, we visualize normalizing a nonzero vector $\mathbf{v}$ as shrinking[2] its terminal point, when plotted in standard position, back to the Unit Circle.



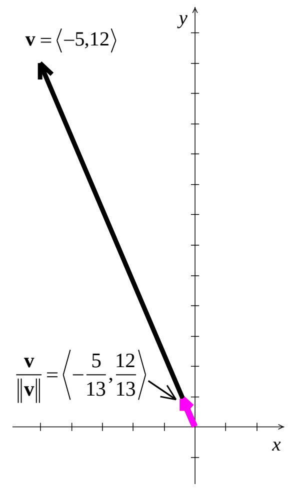
Example 9.2.2. Find a unit vector in the same direction as $\mathbf{v}=\langle -5,12\rangle$.

Solution. We begin by finding the magnitude.

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-5)^2+(12)^2}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

[2] …if the magnitude of **v** is greater than 1…

Next, we divide $\mathbf{v}=\langle -5,12\rangle$ by $\|\mathbf{v}\|=13$.



$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|}&=\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\\&=\frac{1}{13}\langle -5,12\rangle\\&=\left\langle -\frac{5}{13},\frac{12}{13}\right\rangle\end{aligned}$$

We can check that $\frac{\mathbf{v}}{\|\mathbf{v}\|}=\left\langle -\frac{5}{13},\frac{12}{13}\right\rangle$ is indeed a unit vector by verifying that its magnitude is 1. Try it!

□

Note that since a unit vector has length 1, multiplying a unit vector by the magnitude of a vector $\mathbf{v}$ results in the vector $\mathbf{v}$ itself, provided the unit vector has the same direction as $\mathbf{v}$. (Try this with the unit vector we found in **Example 9.2.2**.) As a rule of thumb, for any nonzero vector $\mathbf{v}$,

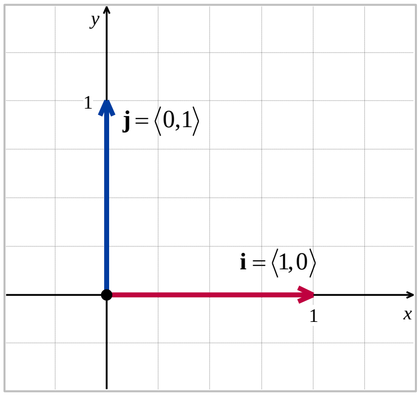
$$(\text{magnitude of }\mathbf{v})\times(\text{unit vector in the direction of }\mathbf{v})=\mathbf{v}$$

The Principal Unit Vectors

Of all of the unit vectors, two deserve special mention.

Definition. The Principal Unit Vectors:

- The vector $\mathbf{i}$ is defined by $\mathbf{i}=\langle 1,0\rangle$.
- The vector $\mathbf{j}$ is defined by $\mathbf{j}=\langle 0,1\rangle$.



We can think of the vector $\mathbf{i}$ as representing the positive *x*-direction while $\mathbf{j}$ represents the positive *y*-direction. We have the following 'decomposition' theorem.[3]

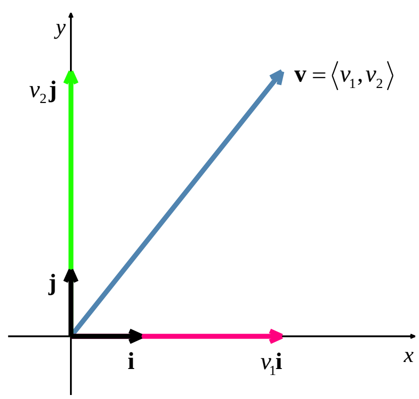
Theorem 9.4. Principal Vector Decomposition Theorem:

Let $\mathbf{v}$ be a vector with component form $\mathbf{v} = \langle v_1, v_2 \rangle$. Then $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.

The proof of **Theorem 9.4** is straightforward. Since $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$\begin{aligned} v_1\mathbf{i} + v_2\mathbf{j} &= v_1\langle 1,0\rangle + v_2\langle 0,1\rangle && \textbf{definition of i and j} \\ &= \langle v_1, 0\rangle + \langle 0, v_2\rangle && \textbf{scalar multiplication} \\ &= \langle v_1, v_2\rangle && \textbf{vector addition} \\ &= \mathbf{v} \end{aligned}$$

Geometrically, the situation looks like this:



$\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$

In **Section 9.1**, we found the component form of a vector $\overrightarrow{PQ}$ with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$ to be $\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle$. It follows from **Theorem 9.4** that $\overrightarrow{PQ}$ may also be written in terms of $\mathbf{i}$ and $\mathbf{j}$ as $\overrightarrow{PQ} = (x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j}$.

Example 9.2.3. Given a vector $\mathbf{v}$ with initial point $P(2,-6)$ and terminal point $Q(-6,6)$, write the vector in terms of $\mathbf{i}$ and $\mathbf{j}$.

[3] We will see a generalization of **Theorem 9.4** in **Section 9.3**. Stay tuned!

Solution.

$$\begin{aligned}\mathbf{v} &= (-6-2)\,\mathbf{i} + (6-(-6))\,\mathbf{j} \\ &= -8\mathbf{i} + 12\mathbf{j}\end{aligned}$$

□

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of $\mathbf{i}$ and $\mathbf{j}$, we carry out addition, subtraction and scalar multiplication by performing operations on corresponding components.

Operations on Vectors Written in terms of $\mathbf{i}$ and $\mathbf{j}$: Given $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$, then

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j}$
- $\mathbf{v} - \mathbf{w} = (v_1 - w_1)\mathbf{i} + (v_2 - w_2)\mathbf{j}$
- $k\mathbf{v} = (kv_1)\mathbf{i} + (kv_2)\mathbf{j}$ for any scalar k

These results can be verified using definitions of addition, subtraction and scalar multiplication from **Section 9.1** along with **Theorem 9.4**. Their verification is left to the student.

Example 9.2.4. Use vectors $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + \mathbf{j}$ to find $3\mathbf{v} + \mathbf{w}$.

Solution.

$$\begin{aligned}3\mathbf{v} + \mathbf{w} &= 3(4\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\ &= 3(4\mathbf{i} + (-2)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\ &= (12\mathbf{i} + (-6)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) && \textbf{scalar multiplication} \\ &= (12 + (-3))\mathbf{i} + (-6 + 1)\mathbf{j} && \textbf{vector addition} \\ &= 9\mathbf{i} - 5\mathbf{j}\end{aligned}$$

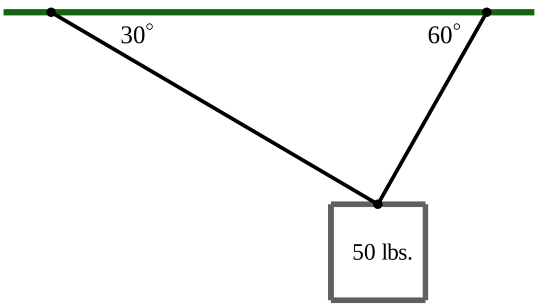
□

Using Vectors to Model Forces

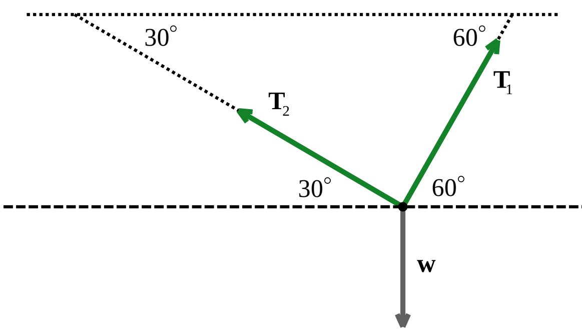
We conclude this section with a classic example which demonstrates how vectors are used to model forces. A **force** is defined as a ‘push’ or a ‘pull’. The intensity of the push or pull is the magnitude of the force, and is measured in Newtons (N) in the SI system or pounds (lbs.) in the English system. The following example should be studied in great detail.

Example 9.2.5. A 50 pound speaker is suspended from the ceiling by two support braces. If one of them makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each of the supports?

Solution. We first represent the problem schematically.

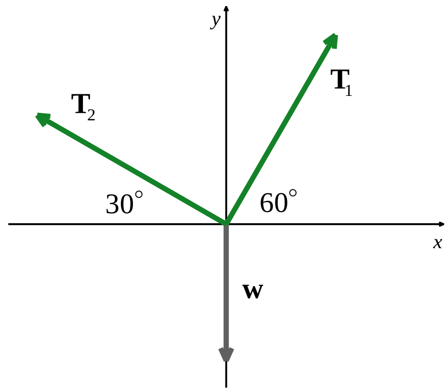


We have three forces acting on the speaker: the weight of the speaker, which we'll call $\mathbf{w}$, pulling the speaker directly downward, and the forces on the support rods, which we'll call $\mathbf{T}_1$ and $\mathbf{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively. We provide the corresponding vector diagram below.



Note that we have used alternate interior angles to determine the added angle measures in the above diagram. We are looking for the tensions on the supports, which are the magnitudes $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$. In order for the speaker to remain stationary[4], we require $\mathbf{w}+\mathbf{T}_1+\mathbf{T}_2=\mathbf{0}$. Viewing the common initial point of these vectors as the origin and the dashed line as the *x*-axis, we find component representations for the three vectors involved.

[4] This is the criteria for 'static equilibrium'.



- We can model the weight of the speaker as a vector pointing directly downward with a magnitude of 50 pounds. That is, $\|\mathbf{w}\| = 50$. Since the vector $\mathbf{w}$ is directed strictly downward, $-\mathbf{j} = \langle 0, -1 \rangle$ is a unit vector in the direction of $\mathbf{w}$. Hence,

$$\begin{aligned} \mathbf{w} &= 50\langle 0, -1 \rangle \\ &= \langle 0, -50 \rangle \end{aligned}$$

- For the force in the first support, applying **Theorem 9.3**, we get

$$\begin{aligned} \mathbf{T}_1 &= \|\mathbf{T}_1\| \langle \cos(60^\circ), \sin(60^\circ) \rangle \\ &= \|\mathbf{T}_1\| \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\ &= \left\langle \frac{\|\mathbf{T}_1\|}{2}, \frac{\|\mathbf{T}_1\|\sqrt{3}}{2} \right\rangle \end{aligned}$$

- For the second support, we note that the angle 30° is measured from the negative *x*-axis, so the angle needed to write $\mathbf{T}_2$ in component form is 150°. Hence,

$$\begin{aligned} \mathbf{T}_2 &= \|\mathbf{T}_2\| \langle \cos(150^\circ), \sin(150^\circ) \rangle \\ &= \|\mathbf{T}_2\| \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \\ &= \left\langle -\frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_2\|}{2} \right\rangle \end{aligned}$$

The requirement $\mathbf{w} + \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}$ gives us

$$\langle 0,-50\rangle + \left\langle \frac{\|\mathbf{T}_1\|}{2}, \frac{\|\mathbf{T}_1\|\sqrt{3}}{2} \right\rangle + \left\langle -\frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_2\|}{2} \right\rangle = \langle 0,0\rangle$$

$$\left\langle \frac{\|\mathbf{T}_1\|}{2} - \frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_1\|\sqrt{3}}{2} + \frac{\|\mathbf{T}_2\|}{2} - 50 \right\rangle = \langle 0,0\rangle$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$.

$$\begin{cases} (\text{E1}) & \dfrac{\|\mathbf{T}_1\|}{2} - \dfrac{\|\mathbf{T}_2\|\sqrt{3}}{2} = 0 \\ (\text{E2}) & \dfrac{\|\mathbf{T}_1\|\sqrt{3}}{2} + \dfrac{\|\mathbf{T}_2\|}{2} - 50 = 0 \end{cases}$$

From (E1) we get $\|\mathbf{T}_1\| = \|\mathbf{T}_2\|\sqrt{3}$. Substituting into (E2) gives

$$\frac{\left(\|\mathbf{T}_2\|\sqrt{3}\right)\sqrt{3}}{2} + \frac{\|\mathbf{T}_2\|}{2} - 50 = 0$$

$$\frac{3\|\mathbf{T}_2\| + \|\mathbf{T}_2\|}{2} = 50$$

$$\|\mathbf{T}_2\| = 25$$

Hence, $\|\mathbf{T}_2\| = 25$ pounds and $\|\mathbf{T}_1\| = \|\mathbf{T}_2\|\sqrt{3} = 25\sqrt{3}$ pounds.

□

9.2 Exercises

1. Given initial point $P_1 = (-3,1)$ and terminal point $P_2 = (5,2)$, write the vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

2. Given initial point $P_1 = (6,0)$ and terminal point $P_2 = (-1,-3)$, write the vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

In Exercises 3 – 4, use the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - \mathbf{j}$ to find the following.

3. $\mathbf{u} + (\mathbf{v} - \mathbf{w})$

4. $4\mathbf{v} + 2\mathbf{u}$

5. Let $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j}$. Find a vector that is half the length and points in the same direction as $\mathbf{v}$.

6. Let $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$. Find a vector that is twice the length and points in the opposite direction as $\mathbf{v}$.

In Exercises 7 – 10, find a unit vector in the same direction as the given vector.

7. $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$

8. $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$

9. $\mathbf{c} = 10\mathbf{i} - \mathbf{j}$

10. $\mathbf{d} = -\frac{1}{3}\mathbf{i} + \frac{5}{2}\mathbf{j}$

In Exercises 11 – 12, use the given pair of vectors to find the following quantities. State whether the result is a vector or scalar.

- $\mathbf{v} + \mathbf{w}$
- $\mathbf{w} - 2\mathbf{v}$
- $\|\mathbf{v} + \mathbf{w}\|$
- $\|\mathbf{v}\| + \|\mathbf{w}\|$
- $\|\mathbf{v}\|\mathbf{w} - \|\mathbf{w}\|\mathbf{v}$
- $\|\mathbf{w}\|\mathbf{v}$

11. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{w} = -2\mathbf{j}$

12. $\mathbf{v} = \frac{1}{2}(\mathbf{i} + \mathbf{j})$, $\mathbf{w} = \frac{1}{2}(\mathbf{i} - \mathbf{j})$

In Exercises 13 – 15, for the given vector $\mathbf{v}$, find the magnitude $\|\mathbf{v}\|$ and an angle θ with $0^\circ \le \theta < 360^\circ$ so that $\mathbf{v} = \|\mathbf{v}\|\langle\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\rangle$. Round approximations to two decimal places.

13. $\mathbf{v} = -10\mathbf{j}$

14. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

15. $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$

16. Let $\mathbf{v} = \langle v_1, v_2 \rangle$ be any non-zero vector. Show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ has length 1.

17. A woman leaves her home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

18. If the captain of the HMS Sasquatch wishes to reach Chupacabra Cove, an island 100 miles away at a bearing of S20°E from port, in three hours, what speed and heading should she set to take into

account an ocean current of 5 miles per hour? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

HINT: If $\mathbf{v}$ denotes the velocity of the HMS Sasquatch and $\mathbf{w}$ denotes the velocity of the current, what does $\mathbf{v}+\mathbf{w}$ need to be to reach Chupacabra Cove in three hours?

19. In calm air, a plane flying from the Pedimaxus International Airport can reach Cliffs of Insanity Point in two hours by following a bearing of N8.2°E at 96 miles an hour. (The distance between the airport and the cliffs is 192 miles.) If the wind is blowing from the southeast at 25 miles per hour, what speed and bearing should the pilot take so that she makes the trip in two hours along the original heading? Round the speed to the nearest hundredth of a mile per hour and your angle to the nearest tenth of a degree.

20. The SS Bigfoot leaves Yeti Bay on a course of N37°W at a speed of 50 miles per hour. After traveling half an hour, the captain determines he is 30 miles from the bay and his bearing back to the bay is S40°E. What is the speed and bearing of the ocean current? Round the speed to the nearest mile per hour and express the heading as a bearing, rounded to the nearest tenth of a degree.

21. A 600 pound Sasquatch statue is suspended by two cables from a gymnasium ceiling. If each cable makes a 60° angle with the ceiling, find the tension on each cable. Round your answer to the nearest pound.

22. Two cables are to support an object hanging from a ceiling. If the cables are each to make a 42° angle with the ceiling, and each cable is rated to withstand a maximum tension of 100 pounds, what is the heaviest object that can be supported? Round your answer down to the nearest pound.

23. A 300 pound metal star is hanging on two cables which are attached to the ceiling. The left hand cable makes a 72° angle with the ceiling while the right hand cable makes an 18° angle with the ceiling. What is the tension on each of the cables? Round your answers to three decimal places.

24. Two drunken college students have filled an empty beer keg with rocks and tied ropes to it in order to drag it down the street in the middle of the night. The stronger of the two students pulls with a force of 100 pounds at a heading of N77°E and the other pulls at a heading of S68°E. What force should the weaker student apply to his rope so that the keg of rocks heads due east? What resultant force is applied to the keg? Round your answer to the nearest pound.

25. Emboldened by the success of their late night keg pull in **Exercise 24** above, our intrepid young scholars have decided to pay homage to the chariot race scene from the movie 'Ben-Hur' by tying three ropes to a couch, loading the couch with all but one of their friends and pulling it due west down the street. The first rope points N80°W, the second points due west and the third points S80°W. The force applied to the first rope is 100 pounds, the force applied to the second rope is 40 pounds and the force applied (by the non-riding friend) to the third rope is 160 pounds. They need the resultant force to be at least 300 pounds; otherwise the couch won't move. Does it move? If so, is it heading due west?

9.3 The Dot Product

Learning Objectives

In this section you will:

- **Find the dot product of two vectors.**
- **Learn properties of the dot product.**
- **Determine the angle between two vectors.**
- **Determine whether or not two vectors are orthogonal.**
- **Solve applications of the dot product.**

Thus far in Chapter 9, we have learned how to add and subtract vectors and how to multiply vectors by scalars. In this section, we define a product of vectors.

Definition and Algebraic Properties of the Dot Product

We begin with the following definition.

Definition. Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors whose component forms are $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$. The **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v}\cdot\mathbf{w}=\langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle=v_1w_1+v_2w_2$$

Example 9.3.1. Find the dot product of $\mathbf{v}=\langle 3,4\rangle$ and $\mathbf{w}=\langle 1,-2\rangle$.

Solution. We have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w}&=\langle 3,4\rangle\cdot\langle 1,-2\rangle\\&=(3)(1)+(4)(-2)\\&=-5\end{aligned}$$

□

Note that the dot product takes two vectors and produces a scalar. For that reason, the quantity $\mathbf{v}\cdot\mathbf{w}$ is often called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$. The dot product enjoys the following properties.

Theorem 9.5. Properties of the Dot Product:

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$.
- **Distributive Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$.
- **Scalar Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, and scalars k, $(k\mathbf{v})\cdot\mathbf{w}=k(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(k\mathbf{w})$.
- **Relation to Magnitude:** For all vectors $\mathbf{v}$, $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$.

Like most of the theorems involving vectors, the proof of **Theorem 9.5** amounts to using the definition of the dot product and properties of real number arithmetic. To show the commutative property, for instance, we let $\mathbf{v}=\langle v_1,v_2\rangle$ and $\mathbf{w}=\langle w_1,w_2\rangle$. Then

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} &= \langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle \\
&= v_1w_1+v_2w_2 && \textbf{definition of dot product} \\
&= w_1v_1+w_2v_2 && \textbf{commutativity of real number multiplication} \\
&= \langle w_1,w_2\rangle\cdot\langle v_1,v_2\rangle && \textbf{definition of dot product} \\
&= \mathbf{w}\cdot\mathbf{v}
\end{aligned}$$

The distributive property is proved similarly and is left as an exercise.

For the scalar property, assume that $\mathbf{v}=\langle v_1,v_2\rangle$, $\mathbf{w}=\langle w_1,w_2\rangle$ and k is a scalar. Then

$$\begin{aligned}
(k\mathbf{v})\cdot\mathbf{w} &= \left(k\langle v_1,v_2\rangle\right)\cdot\langle w_1,w_2\rangle \\
&= \langle kv_1,kv_2\rangle\cdot\langle w_1,w_2\rangle && \textbf{definition of scalar multiplication} \\
&= (kv_1)(w_1)+(kv_2)(w_2) && \textbf{definition of dot product} \\
&= k(v_1w_1)+k(v_2w_2) && \textbf{associativity of real number multiplication} \\
&= k(v_1w_1+v_2w_2) && \textbf{distributive law for real numbers} \\
&= k\left(\langle v_1,v_2\rangle\cdot\langle w_1,w_2\rangle\right) && \textbf{definition of dot product} \\
&= k(\mathbf{v}\cdot\mathbf{w})
\end{aligned}$$

We leave the proof of $k(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(k\mathbf{w})$ as an exercise.

For the last property, we note that if $\mathbf{v}=\langle v_1,v_2\rangle$, then

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{v} &= \langle v_1,v_2\rangle\cdot\langle v_1,v_2\rangle \\
&= v_1^2+v_2^2 \\
&= \|\mathbf{v}\|^2 && \textbf{definition of magnitude}
\end{aligned}$$

The following example puts **Theorem 9.5** to good use.

Example 9.3.2. Prove the identity: $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$.

Solution. We begin by using **Theorem 9.5** to rewrite $\|\mathbf{v}-\mathbf{w}\|^2$ in terms of the dot product.

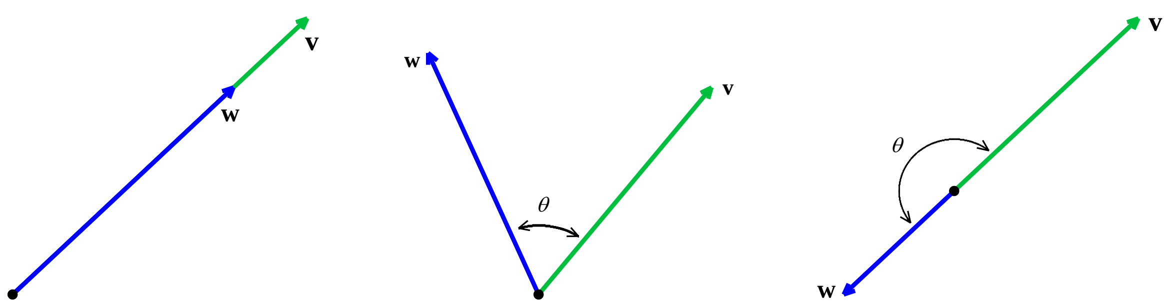
$$\begin{aligned}
\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) && \textbf{relation to magnitude property}\\
&= [\mathbf{v}+(\text{-}\mathbf{w})]\cdot\mathbf{v} + [\mathbf{v}+(\text{-}\mathbf{w})]\cdot(\text{-}\mathbf{w}) && \textbf{distributive property}\\
&= \mathbf{v}\cdot[\mathbf{v}+(\text{-}\mathbf{w})] + (\text{-}\mathbf{w})\cdot[\mathbf{v}+(\text{-}\mathbf{w})] && \textbf{commutative property}\\
&= \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot(\text{-}\mathbf{w}) + (\text{-}\mathbf{w})\cdot\mathbf{v} + (\text{-}\mathbf{w})\cdot(\text{-}\mathbf{w}) && \textbf{distributive property}\\
&= \mathbf{v}\cdot\mathbf{v} + (-1)(\mathbf{v}\cdot\mathbf{w}) + (-1)(\mathbf{v}\cdot\mathbf{w}) + (-1)(-1)(\mathbf{w}\cdot\mathbf{w}) && \textbf{scalar \& commutative properties}\\
&= \mathbf{v}\cdot\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{w}) + \mathbf{w}\cdot\mathbf{w}\\
&= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 && \textbf{relation to magnitude property}
\end{aligned}$$

Hence, $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$ as required. □

If we take a step back from the pedantry in **Example 9.3.2**, we see that the bulk of the work is needed to show that $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{w}) + \mathbf{w}\cdot\mathbf{w}$. If this looks familiar, it should. Since the dot product enjoys many of the same properties as real numbers, the machinations required to expand $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})$ for vectors $\mathbf{v}$ and $\mathbf{w}$ match those required to expand $(v-w)(v-w)$ for real numbers v and w, and hence we get similar looking results. The identity verified in **Example 9.3.2** plays a large role in the development of the geometric properties of the dot product, which we now explore.

Geometric Properties of the Dot Product

Suppose $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors. If we draw $\mathbf{v}$ and $\mathbf{w}$ with the same initial point, we define the **angle between** $\mathbf{v}$ and $\mathbf{w}$ to be the angle θ determined by the rays containing the vectors $\mathbf{v}$ and $\mathbf{w}$, as illustrated below. We choose to define $0 \le \theta \le \pi$.



$\theta = 0$ $\quad$ $0 < \theta < \pi$ $\quad$ $\theta = \pi$

The following theorem gives us some insight into the geometric role the dot product plays.

Theorem 9.6. Geometric Interpretation of the Dot Product: If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors then $\mathbf{v}\cdot\mathbf{w}=\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

We prove **Theorem 9.6** in cases.

Case 1: If $\theta=0$, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction. Thus, the unit vector in the direction of $\mathbf{v}$ is also the unit vector in the direction of $\mathbf{w}$, and we have

$$\frac{\mathbf{w}}{\|\mathbf{w}\|}=\frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \textbf{unit vectors in directions of } \mathbf{w} \textbf{ and } \mathbf{v} \textbf{ are the same}$$

$$\mathbf{w}=\|\mathbf{w}\|\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \quad \textbf{scalar multiplication}$$

$$\mathbf{w}=\frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v} \quad \textbf{associative property of scalar multiplication}$$

$$\mathbf{w}=k\mathbf{v} \quad \textbf{letting } k=\frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}$$

Thus, we have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w}&=\mathbf{v}\cdot(k\mathbf{v})\\&=k(\mathbf{v}\cdot\mathbf{v}) && \textbf{scalar property of dot product}\\&=k\|\mathbf{v}\|^2 && \textbf{relation to magnitude property}\\&=k\|\mathbf{v}\|\|\mathbf{v}\|\end{aligned}$$

We note that $k>0$, from which we get $k=|k|$. It follows, from **Theorem 9.3**, that $k\|\mathbf{v}\|=|k|\|\mathbf{v}\|=\|k\mathbf{v}\|$. Hence,

$$\begin{aligned}k\|\mathbf{v}\|\|\mathbf{v}\|&=\|\mathbf{v}\|\left(k\|\mathbf{v}\|\right) && \textbf{scalar property of dot product}\\&=\|\mathbf{v}\|\|k\mathbf{v}\|\\&=\|\mathbf{v}\|\|\mathbf{w}\|\end{aligned}$$

We get

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w}&=k\|\mathbf{v}\|\|\mathbf{v}\|\\&=\|\mathbf{v}\|\|\mathbf{w}\|\\&=\|\mathbf{v}\|\|\mathbf{w}\|\cos(0) && \textbf{since } \cos(0)=1\end{aligned}$$

This proves the formula holds for $\theta=0$.

Case 2: If $\theta = \pi$, we repeat the argument with the difference being that $\frac{\mathbf{w}}{\|\mathbf{w}\|} = -\frac{\mathbf{v}}{\|\mathbf{v}\|}$, so that $k = -\frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}$.

Thus, $\mathbf{w} = k\mathbf{v}$ where $k < 0$. It follows that, $|k| = -k$, resulting in

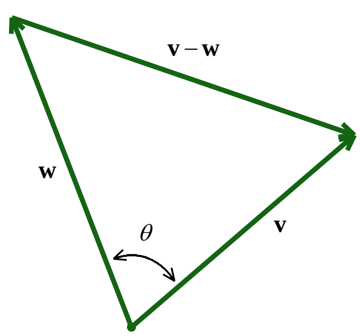
$$\begin{aligned} k\|\mathbf{v}\| &= -|k|\|\mathbf{v}\| \\ &= -\|k\mathbf{v}\| \\ &= -\|\mathbf{w}\| \end{aligned}$$

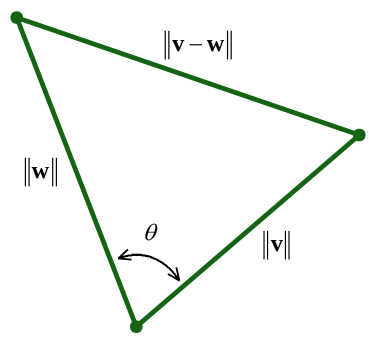
We have

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} &= -\|\mathbf{v}\|\|\mathbf{w}\| \\ &= \|\mathbf{v}\|\|\mathbf{w}\|\cos(\pi) \quad \textbf{since } \cos(\pi) = -1 \end{aligned}$$

Thus, the formula holds for $\theta = \pi$.

Case 3: Next, if $0 < \theta < \pi$, the vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ determine a triangle with side lengths $\|\mathbf{v}\|$, $\|\mathbf{w}\|$ and $\|\mathbf{v}-\mathbf{w}\|$, respectively, as seen below.





The Law of Cosines yields $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$. From **Example 9.3.2**, we know $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$. Equating these two expressions for $\|\mathbf{v}-\mathbf{w}\|^2$ gives

$$\begin{aligned} \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \\ -2\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= -2(\mathbf{v}\cdot\mathbf{w}) \\ \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta) &= \mathbf{v}\cdot\mathbf{w} \end{aligned}$$

Thus, $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, as required.

Determining the Angle Between Two Vectors

An immediate consequence of **Theorem 9.6** is the following.

Theorem 9.7. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

We obtain the formula in **Theorem 9.7** by solving the equation given in **Theorem 9.6** for θ. Since $\mathbf{v}$ and $\mathbf{w}$ are nonzero, so are $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$. Hence, we may divide both sides of $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$ by $\|\mathbf{v}\|\|\mathbf{w}\|$ to get

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Since $0 \le \theta \le \pi$ by definition, the values of θ exactly match the range of the arccosine function. Hence,

$$\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

We are long overdue for an example.

Example 9.3.3. Find the angle between the following pairs of vectors.

1. $\mathbf{v} = \langle 3, -3\sqrt{3}\rangle$ and $\mathbf{w} = \langle -\sqrt{3}, 1\rangle$
2. $\mathbf{v} = \langle 2, 2\rangle$ and $\mathbf{w} = \langle 5, -5\rangle$
3. $\mathbf{v} = \langle 3, -4\rangle$ and $\mathbf{w} = \langle 2, 1\rangle$

Solution. We use the formula $\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$ from **Theorem 9.7** in each case below.

1. For $\mathbf{v} = \langle 3, -3\sqrt{3}\rangle$ and $\mathbf{w} = \langle -\sqrt{3}, 1\rangle$, we have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \langle 3, -3\sqrt{3}\rangle \cdot \langle -\sqrt{3}, 1\rangle \\ &= -3\sqrt{3} - 3\sqrt{3} \\ &= -6\sqrt{3}\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{3^2 + \left(-3\sqrt{3}\right)^2} \\ &= \sqrt{36} \\ &= 6\end{aligned}$$

$$\begin{aligned}\|\mathbf{w}\| &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

Then

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)\\ &= \arccos\left(\frac{-6\sqrt{3}}{12}\right)\\ &= \arccos\left(-\frac{\sqrt{3}}{2}\right)\\ &= \frac{5\pi}{6}\end{aligned}$$

2. We have $\mathbf{v} = \langle 2,2\rangle$ and $\mathbf{w} = \langle 5,-5\rangle$, so that

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \langle 2,2\rangle\cdot\langle 5,-5\rangle\\ &= 10-10\\ &= 0\end{aligned}$$

It follows that

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)\\ &= \arccos\left(\frac{0}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)\\ &= \arccos(0) \qquad \mathbf{v}\neq\mathbf{0}\text{ and }\mathbf{w}\neq\mathbf{0}\Rightarrow\|\mathbf{v}\|\neq 0\text{ and }\|\mathbf{w}\|\neq 0\\ &= \frac{\pi}{2}\end{aligned}$$

3. We find, for $\mathbf{v} = \langle 3,-4\rangle$ and $\mathbf{w} = \langle 2,1\rangle$,

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \langle 3,-4\rangle\cdot\langle 2,1\rangle\\ &= 6-4\\ &= 2\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{3^2+(-4)^2}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

$$\begin{aligned}\|\mathbf{w}\| &= \sqrt{2^2+1^2}\\ &= \sqrt{5}\end{aligned}$$

Then

$$\begin{aligned}\arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) &= \arccos\left(\frac{2}{5\sqrt{5}}\right)\\ &= \arccos\left(\frac{2\sqrt{5}}{25}\right)\end{aligned}$$

Since $\frac{2\sqrt{5}}{25}$ isn't the cosine of one of the common angles, we leave our answer as

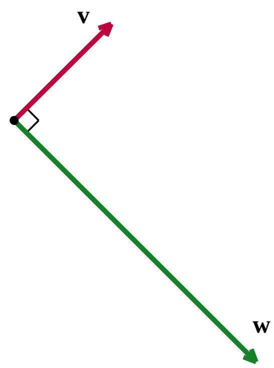
$\theta = \arccos\left(\frac{2\sqrt{5}}{25}\right)$.

□

Orthogonal Vectors

The vectors $\mathbf{v} = \langle 2,2 \rangle$ and $\mathbf{w} = \langle 5,-5 \rangle$ are called **orthogonal**, and we write $\mathbf{v} \perp \mathbf{w}$, because the angle between them is $\frac{\pi}{2}$ radians, or 90°. Geometrically, when orthogonal vectors are sketched with the same initial point, the lines containing the vectors are perpendicular.

In the illustration to the right, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, and we write $\mathbf{v} \perp \mathbf{w}$.



We state the relationship between orthogonal vectors and their dot product in the following theorem.

> **Theorem 9.8. The Dot Product Detects Orthogonality:** Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. Then $\mathbf{v} \perp \mathbf{w}$ if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

To prove **Theorem 9.8**, we first assume $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors with $\mathbf{v} \perp \mathbf{w}$. By definition, the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{2}$. By **Theorem 9.6**, $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\left(\frac{\pi}{2}\right) = 0$.

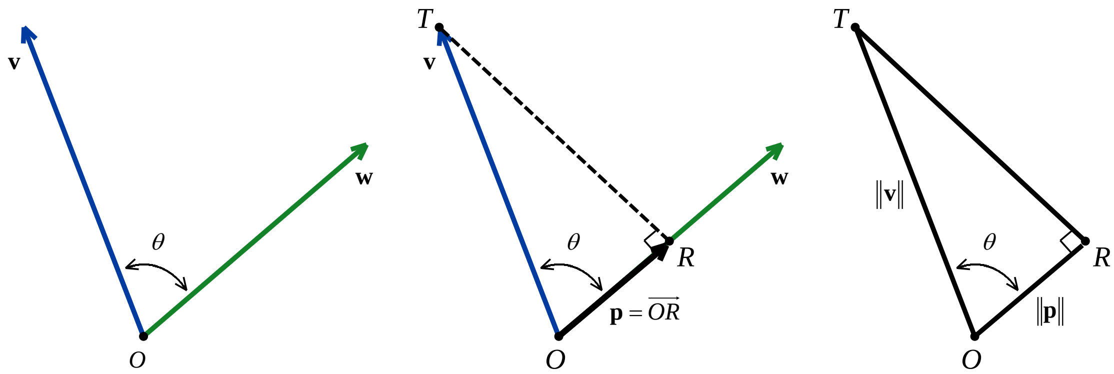
Conversely, if $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors and $\mathbf{v} \cdot \mathbf{w} = 0$, then **Theorem 9.7** gives

$$\begin{aligned} \theta &= \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos\left(\frac{0}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \\ &= \arccos(0) \\ &= \frac{\pi}{2} \qquad \text{thus verifying } \mathbf{v} \perp \mathbf{w} \end{aligned}$$

While **Theorem 9.8** certainly gives us some insight into what the dot product means geometrically, there is more to the story of the dot product.

Orthogonal Projection

Consider the two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\mathbf{v}$ and $\mathbf{w}$, which we'll denote θ, is acute. We wish to develop a formula for the vector $\mathbf{p}$, indicated below, which is called the **orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$**. The vector $\mathbf{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\mathbf{v}$ to the vector $\mathbf{w}$ and call the point of intersection R. The vector $\mathbf{p}$ is then defined as $\mathbf{p} = \overrightarrow{OR}$.



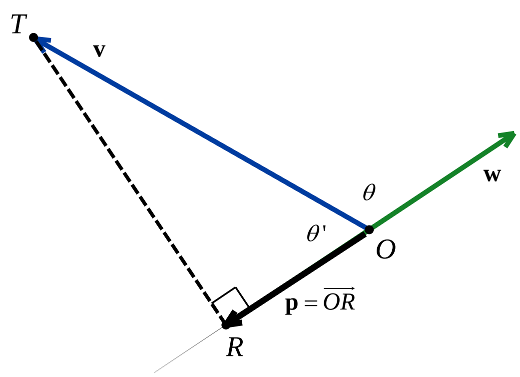
Like any vector, $\mathbf{p}$ is determined by its magnitude $\|\mathbf{p}\|$ and its direction. To determine the magnitude, we observe that

$$\begin{aligned}
\cos(\theta) &= \frac{\|\mathbf{p}\|}{\|\mathbf{v}\|} \\
\|\mathbf{p}\| &= \|\mathbf{v}\|\cos(\theta) \\
\|\mathbf{p}\| &= \|\mathbf{v}\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \quad \text{from } \mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta), \text{ Theorem 9.6} \\
\|\mathbf{p}\| &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}
\end{aligned}$$

We determine the direction of $\mathbf{p}$ by finding the unit vector in the direction of $\mathbf{p}$, which is the same as the unit vector in the direction of $\mathbf{w}$; that is, $\frac{\mathbf{w}}{\|\mathbf{w}\|}$. It follows that

$$\mathbf{p} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}\right)\left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) \quad \textbf{magnitude of } \mathbf{p} \textbf{ times unit vector in direction of } \mathbf{p}$$
$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|\|\mathbf{w}\|}\right)\mathbf{w}$$
$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

Thus, we have a formula for the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$ for an acute angle θ. Suppose next that the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is obtuse, and consider the diagram below.



In this case we see that $\theta + \theta' = \pi$, from which

$$\cos(\theta') = \cos(\pi - \theta)$$
$$= \cos(\pi)\cos(\theta) + \sin(\pi)\sin(\theta) \quad \textbf{difference identity for cosine}$$
$$= (-1)\cos(\theta) + (0)\sin(\theta)$$
$$= -\cos(\theta)$$

Thus, from $\cos(\theta') = \frac{\|\mathbf{p}\|}{\|\mathbf{v}\|}$, we have

$$\|\mathbf{p}\| = \|\mathbf{v}\|\cos(\theta')$$
$$= -\|\mathbf{v}\|\cos(\theta) \quad \textbf{since } \cos(\theta') = -\cos(\theta)$$
$$= -\|\mathbf{v}\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \quad \textbf{from Theorem 9.6: } \mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$$
$$= -\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$$

The unit vector in the direction of $\mathbf{p}$ is the unit vector in the direction of $-\mathbf{w}$, which is $\frac{-\mathbf{w}}{\|-\mathbf{w}\|} = \frac{-\mathbf{w}}{\|\mathbf{w}\|}$. We note that reversing the direction of $\mathbf{w}$ does not affect the magnitude. Finally,

$$\mathbf{p} = \left(-\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}\right)\left(\frac{-\mathbf{w}}{\|\mathbf{w}\|}\right) \quad \text{magnitude times direction}$$
$$= -\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|\|\mathbf{w}\|}\right)(-\mathbf{w})$$
$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

This formula for orthogonal projection when θ is obtuse matches the formula for an acute angle θ. If the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{2}$, then $\mathbf{p} = \mathbf{0}$ [1]. Since $\mathbf{v} \perp \mathbf{w}$ in this case, $\mathbf{v}\cdot\mathbf{w} = 0$. It follows that

$$\mathbf{p} = \mathbf{0}$$
$$= 0\mathbf{w}$$
$$= \left(\frac{0}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$
$$= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

Finally, we have the following theorem.

Theorem 9.9. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then the **orthogonal projection of** $\mathbf{v}$ onto $\mathbf{w}$, denoted $\text{proj}_{\mathbf{w}}(\mathbf{v})$, is given by

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

It is time for an example.

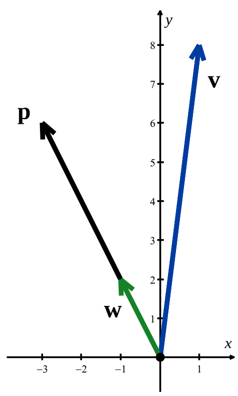
Example 9.3.4. Let $\mathbf{v} = \langle 1,8\rangle$ and $\mathbf{w} = \langle -1,2\rangle$. Find $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v})$, and plot $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$ in standard position.

Solution. We find

[1] In this case, the point *R* coincides with the point *O*, so that $\mathbf{p} = \overrightarrow{OR} = \overrightarrow{OO} = \mathbf{0}$.

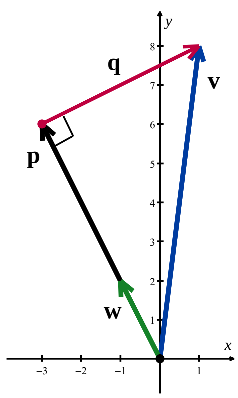
$$\begin{aligned}\text{proj}_{\mathbf{w}}(\mathbf{v}) &= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} \\ &= \left(\frac{\langle 1,8\rangle\cdot\langle -1,2\rangle}{\|\langle -1,2\rangle\|^2}\right)\langle -1,2\rangle \\ &= \left(\frac{(1)(-1)+(8)(2)}{\left(\sqrt{(-1)^2+(2)^2}\right)^2}\right)\langle -1,2\rangle \\ &= \left(\frac{-1+16}{\left(\sqrt{5}\right)^2}\right)\langle -1,2\rangle \\ &= 3\langle -1,2\rangle\end{aligned}$$

Hence, $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v}) = \langle -3,6\rangle$. We plot $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$ below.



□

Suppose we wanted to verify that our answer $\mathbf{p}$ in **Example 9.3.4** is indeed the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$. We first note that, since $\mathbf{p} = 3\mathbf{w}$, $\mathbf{p}$ is a scalar multiple of $\mathbf{w}$ and so it has the correct direction. It remains to check the orthogonality condition. Consider the vector $\mathbf{q}$ whose initial point is the terminal point of $\mathbf{p}$ and whose terminal point is the terminal point of $\mathbf{v}$.



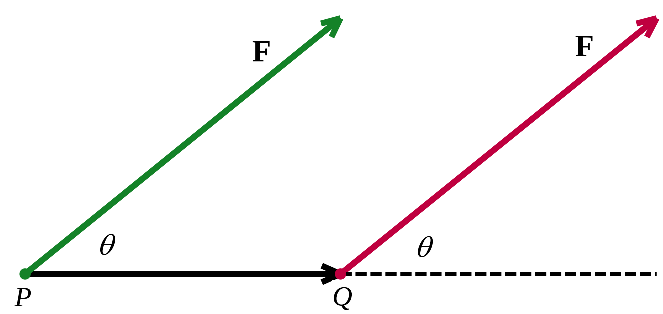
From the definition of vector arithmetic, $\mathbf{p}+\mathbf{q}=\mathbf{v}$, so that $\mathbf{q}=\mathbf{v}-\mathbf{p}$. In the case of **Example 9.3.4**, $\mathbf{v}=\langle 1,8\rangle$ and $\mathbf{p}=\langle -3,6\rangle$, so $\mathbf{q}=\langle 1,8\rangle-\langle -3,6\rangle=\langle 4,2\rangle$. Then

$$\begin{aligned}\mathbf{q}\cdot\mathbf{v}&=\langle 4,2\rangle\cdot\langle -1,2\rangle\\&=(-4)+4\\&=0\end{aligned}$$

This shows $\mathbf{q}\perp\mathbf{w}$ as required.

Work

We close this section with an application of the dot product. In Physics, if a constant force $\mathbf{F}$ moves an object a distance *d*, then the **work**, *W*, done by the force is given by the magnitude of the force times the amount of displacement, or $W=\|\mathbf{F}\|d$, where the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done. Consider the scenario below where the constant force $\mathbf{F}$ is applied to move an object from the point *P* to the point *Q*.



To determine the work *W* done in this scenario, we find that the magnitude of the force $\mathbf{F}$ in the direction of $\overrightarrow{PQ}$ is $\|\mathbf{F}\|\cos(\theta)$, where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$. The distance the object travels is $\|\overrightarrow{PQ}\|$. Since work is the magnitude of the force $\mathbf{F}$ in the direction of $\overrightarrow{PQ}$ times the distance traveled from *P* to *Q*, we get

$$\begin{aligned}W&=\|\mathbf{F}\|\cos(\theta)\|\overrightarrow{PQ}\|\\&=\|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos(\theta)\\&=\mathbf{F}\cdot\overrightarrow{PQ} && \textbf{from Theorem 9.6}\end{aligned}$$

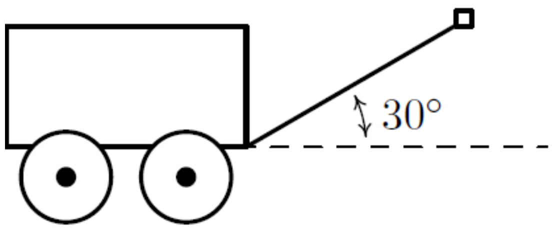
We have proved the following.

Theorem 9.10. Work as a Dot Product: Suppose a constant force $\mathbf{F}$ is applied to move an object along the vector $\overrightarrow{PQ}$, from P to Q. The work W done by $\mathbf{F}$ is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos(\theta),$$

where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$.

Example 9.3.5. Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume Taylor exerts the force of 10 pounds at a 30° angle for the duration of the 50 feet.



Solution. There are two ways to attack this problem.

- One way is to find the vectors $\mathbf{F}$ and $\overrightarrow{PQ}$ mentioned in **Theorem 9.10** and compute $W = \mathbf{F} \cdot \overrightarrow{PQ}$. To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above. Since the force applied is a constant 10 pounds, we have $\|\mathbf{F}\| = 10$. Since it is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis, **Theorem 9.3** gives us

$$\begin{aligned}\mathbf{F} &= 10\left\langle \cos\left(30^\circ\right), \sin\left(30^\circ\right)\right\rangle \\ &= 10\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle 5\sqrt{3}, 5\right\rangle\end{aligned}$$

Since the wagon is being pulled along 50 feet in the positive direction, the displacement vector is

$$\begin{aligned}\overrightarrow{PQ} &= 50i \\ &= 50\langle 1, 0\rangle \\ &= \langle 50, 0\rangle\end{aligned}$$

We get

$$\begin{aligned} W &= \mathbf{F} \cdot \overrightarrow{PQ} \\ &= \left\langle 5\sqrt{3}, 5 \right\rangle \cdot \left\langle 50, 0 \right\rangle \\ &= 250\sqrt{3} \end{aligned}$$

Since force is measured in pounds and distance is measured in feet, we get $W = 250\sqrt{3}$ foot-pounds.

- Alternately, we can use the formulation $W = \|\mathbf{F}\| \left\| \overrightarrow{PQ} \right\| \cos(\theta)$ to get

$$\begin{aligned} W &= (10 \text{ pounds})(50 \text{ feet})\cos\left(30^\circ\right) \\ &= (500)\left(\frac{\sqrt{3}}{2}\right) \text{ foot-pounds} \\ &= 250\sqrt{3} \text{ foot-pounds of work} \end{aligned}$$

□

9.3 Exercises

1. Given $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$, calculate $\mathbf{u} \cdot \mathbf{v}$.

2. Given $\mathbf{u} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 5\mathbf{j}$, calculate $\mathbf{u} \cdot \mathbf{v}$.

3. Given $\mathbf{u} = \langle -2, 4 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$, calculate $\mathbf{u} \cdot \mathbf{v}$.

4. Given $\mathbf{u} = \langle -1, 6 \rangle$ and $\mathbf{v} = \langle 6, -1 \rangle$, calculate $\mathbf{u} \cdot \mathbf{v}$.

In Exercises 5 – 24, use the give pair of vectors, $\mathbf{v}$ and $\mathbf{w}$, to find the following quantities.

- $\mathbf{v} \cdot \mathbf{w}$
- $\text{proj}_{\mathbf{w}}(\mathbf{v})$
- the angle θ (in degrees) between $\mathbf{v}$ and $\mathbf{w}$
- $\mathbf{q} = \mathbf{v} - \text{proj}_{\mathbf{w}}(\mathbf{v})$ (Show that $\mathbf{q} \cdot \mathbf{w} = 0$.)

5. $\mathbf{v} = \langle -2, -7 \rangle$ and $\mathbf{w} = \langle 5, -9 \rangle$

6. $\mathbf{v} = \langle -6, -5 \rangle$ and $\mathbf{w} = \langle 10, -12 \rangle$

7. $\mathbf{v} = \langle 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 1, -\sqrt{3} \rangle$

8. $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle -6, -8 \rangle$

9. $\mathbf{v} = \langle -2, 1 \rangle$ and $\mathbf{w} = \langle 3, 6 \rangle$

10. $\mathbf{v} = \langle -3\sqrt{3}, 3 \rangle$ and $\mathbf{w} = \langle -\sqrt{3}, -1 \rangle$

11. $\mathbf{v} = \langle 1, 17 \rangle$ and $\mathbf{w} = \langle -1, 0 \rangle$

12. $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle 5, 12 \rangle$

13. $\mathbf{v} = \langle -4, -2 \rangle$ and $\mathbf{w} = \langle 1, -5 \rangle$

14. $\mathbf{v} = \langle -5, 6 \rangle$ and $\mathbf{w} = \langle 4, -7 \rangle$

15. $\mathbf{v} = \langle -8, 3 \rangle$ and $\mathbf{w} = \langle 2, 6 \rangle$

16. $\mathbf{v} = \langle 34, -91 \rangle$ and $\mathbf{w} = \langle 0, 1 \rangle$

17. $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 4\mathbf{j}$

18. $\mathbf{v} = -24\mathbf{i} + 7\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i}$

19. $\mathbf{v} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$

20. $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$

21. $\mathbf{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

22. $\mathbf{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\mathbf{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$

23. $\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\mathbf{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

24. $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\mathbf{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

25. A force of 1500 pounds is required to tow a trailer. Find the work done towing the trailer 300 feet along a flat stretch of road Assume the force is applied in the direction of the motion.

26. Find the work done lifting a 10 pound book 3 feet straight up into the air. Assume the force of gravity is acting straight downwards.

27. Suppose Taylor fills her wagon with rocks and must exert a force of 13 pounds to pull her wagon across the yard. If she maintains a 15° angle between the handle of the wagon and the horizontal, compute how much work Taylor does pulling her wagon 25 feet. Round your answer to two decimal places.

28. In **Exercise 24** in **Section 9.2**, two drunken college students have filled an empty beer keg with rocks which they drag down the street by pulling on two attached ropes. The stronger of the two students pulls with a force of 100 pounds on a rope which makes a 13° angle with the direction of motion. (In this case, the keg was being pulled due east and the student's heading was N77°E.) Find the work done by this student if the keg is dragged 42 feet.

29. Find the work done pushing a 200 pound barrel 10 feet up a 12.5° incline. Ignore all forces acting on the barrel except gravity, which acts downwards. Round your answer to two decimal places.

    HINT: Since you are working to overcome gravity only, the force being applied acts directly upwards. This means that the angle between the applied force in this case and the motion of the object is *not* the 12.5° of the incline!

30. Prove the distributive property of the dot product in **Theorem 9.5**.

31. Finish the proof of the scalar property of the dot product in **Theorem 9.5**.

32. Use the identity in **Example 9.3.2**, $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2$, to prove the Parallelogram Law:

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \frac{1}{2}\left[\|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2\right]$$

33. We know that $|x+y| \le |x| + |y|$ for all real numbers x and y by the Triangle Inequality from College Algebra. We can now establish a Triangle Inequality for vectors. In this exercise, we prove that $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ for all pairs of vectors $\mathbf{u}$ and $\mathbf{v}$.

    a) (Step 1) Show that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$.

b) (Step 2) Show that $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$. This is the celebrated Cauchy-Schwarz Inequality. (Hint: To show this inequality, start with the fact that $|\mathbf{u}\cdot\mathbf{v}| \le \left| \ \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta) \ \right|$ and use the fact that $|\cos(\theta)| \le 1$ for all θ.)

c) (Step 3) Show that

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2\end{aligned}$$

d) (Step 4) Use Step 3 to show that $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ for all pairs of vectors $\mathbf{u}$ and $\mathbf{v}$.

e) As an added bonus, we can now show that the Triangle Inequality $|z+w| \le |z| + |w|$ holds for all complex numbers *z* and *w* as well. Identify the complex number $z = a+bi$ with the vector $\mathbf{u} = \langle a,b\rangle$ and identify the complex number $w = c+di$ with the vector $\mathbf{v} = \langle c,d\rangle$ and just follow your nose!

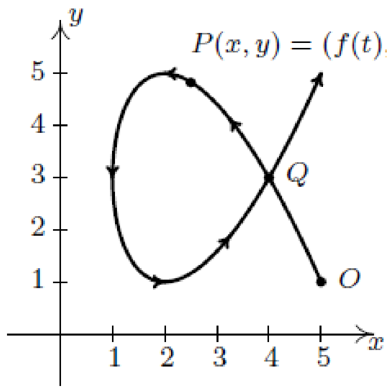
9.4 Sketching Curves Described by Parametric Equations

Learning Objectives

In this section you will:

- **Graph plane curves described by parametric equations.**
- **Analyze behavior in the graphs of parametric equations.**

As we have seen, most recently in **Section 8.3**, there are scores of interesting curves which, when plotted in the *xy*-plane, neither represent *y* as a function of *x* nor *x* as a function of *y*. In this section, we present a new concept which allows us to use functions to study these kinds of curves. To motivate the idea, we imagine a bug crawling across a table top starting at the point *O* and tracing out a curve *C* in the plane, as shown below.



The curve *C* does not represent *y* as a function of *x* because it fails the Vertical Line Test and it does not represent *x* as a function of *y* because it fails the Horizontal Line Test. However, since the bug can be in only one place $P(x, y)$ at any given time *t*, we can define the *x*-coordinate of *P* as a function of *t* and the *y*-coordinate of *P* as a (usually but not necessarily) different function of *t*.

Curves Described by Parametric Equations

The functions describing the curve *C*, traditionally, use $f(t)$ to represent *x* and $g(t)$ to represent *y*. The independent variable *t* in this case is called a **parameter** and the system of equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

is called a **system of parametric equations** or a **parametrization** of the curve *C*.[1]

[1] Note the use of the indefinite article 'a'. As we shall see, there are infinitely many different parametric representations for any given curve.

The parametrization of C endows it with an **orientation** and the arrows on C indicate motion in the direction of increasing values of t. In this case, our bug starts at the point O, travels upwards to the left, then loops back around to cross its path[2] at the point Q and finally heads off into the first quadrant.

It is important to note that the curve itself is a set of points and as such is devoid of any orientation. The parametrization determines the orientation and as we shall see, different parametrizations can determine different orientations. If all of this seems hauntingly familiar, it should. By definition, the system of equations

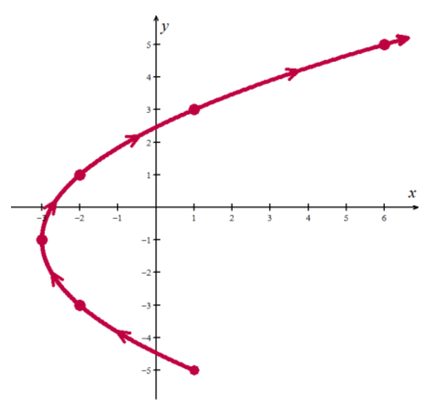
$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

parametrizes the Unit Circle, giving it a counter-clockwise orientation. It is time for an example.

Example 9.4.1. Sketch the curve described by $\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$ for $t \geq -2$.

Solution. We follow the same procedure here as we have time and time again when asked to graph anything new. We choose friendly values of t, plot the corresponding points and connect the results in a pleasing fashion. Since we are told $t \geq -2$, we start there and as we plot successive points, we draw an arrow to indicate the direction of the path for increasing values of t.

t	$x(t) = t^2 - 3$	$y(t) = 2t - 1$	$(x(t), y(t))$
-2	1	-5	$(1,-5)$
-1	-2	-3	$(-2,-3)$
0	-3	-1	$(-3,-1)$
1	-2	1	$(-2,1)$
2	1	3	$(1,3)$
3	6	5	$(6,5)$



□

The curve sketched out in **Example 9.4.1** certainly looks like a parabola and the presence of the t^2 term in the equation $x = t^2 - 3$ reinforces this hunch. In **Section 9.5**, we will use the technique of substitution to eliminate the parameter t and get an equation involving just x and y. As we will see, the resulting Cartesian equation $(y+1)^2 = 4(x+3)$ describes a parabola with vertex $(-3,1)$.

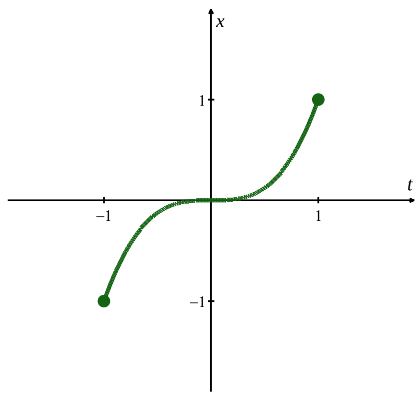
---

[2] Here, the bug reaches the point Q at two different times. While this does not contradict our claim that $f(t)$ and $g(t)$ are functions of t, it shows that neither f nor g can be one-to-one. (Think about this before reading on,)

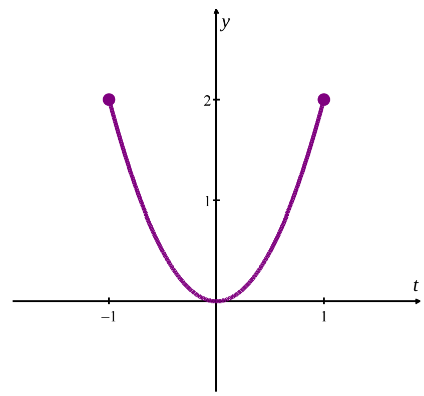
Graphing Parametric Equations

Example 9.4.2. Sketch the curve described by the parametric equations $\begin{cases} x = t^3 \\ y = 2t^2 \end{cases}$ for $-1 \le t \le 1$.

Solution. To get a feel for the curve described by the system, we first sketch the graphs of each equation, $x = t^3$ and $y = 2t^2$, over the interval $[-1,1]$.



$x = t^3$, $-1 \le t \le 1$

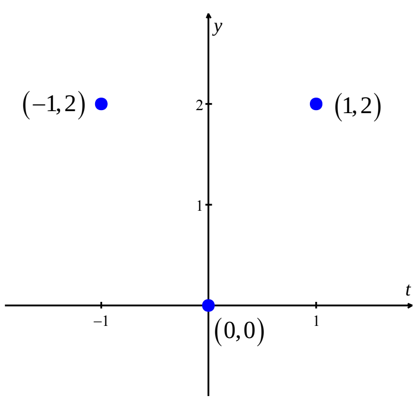


$y = 2t^2$, $-1 \le t \le 1$

We note that as t takes on values on the interval $[-1,1]$, $x = t^3$ ranges between -1 and 1, and $y = 2t^2$ ranges between 0 and 2. This means that all of the action is happening on a portion of the plane, namely $\{(x, y) \mid -1 \le x \le 1,\ 0 \le y \le 2\}$. More generally, we see that $x = t^3$ is increasing over the entire interval $[-1,1]$ whereas $y = 2t^2$ is decreasing over the interval $[-1,0]$ and then increasing over $[0,1]$.

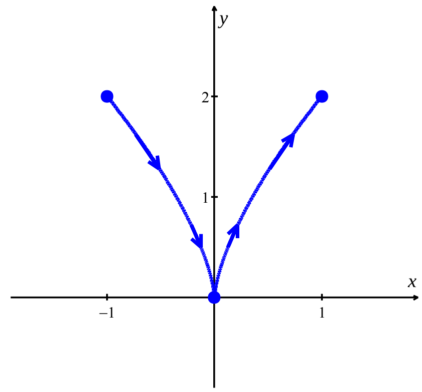
Next, we plot a few points to get a sense of the position and orientation of the curve.

t	$x(t) = t^3$	$y(t) = 2t^2$	$(x(t), y(t))$
-1	-1	2	$(-1,2)$
0	0	0	$(0,0)$
1	1	2	$(1,2)$



To trace out the path described by the parametric equations:

- We start at $(-1,2)$, where $t=-1$, then move to the right (since *x* is increasing) and down (since *y* is decreasing) to $(0,0)$.
- We continue to move to the right (since *x* is still increasing) but now move upwards (since *y* is now increasing) until we reach $(1,2)$, where $t=1$.

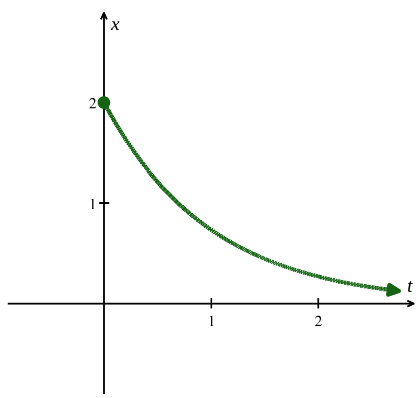


$\begin{cases} x = t^3 \\ y = 2t^2 \end{cases}$ for $-1 \le t \le 1$

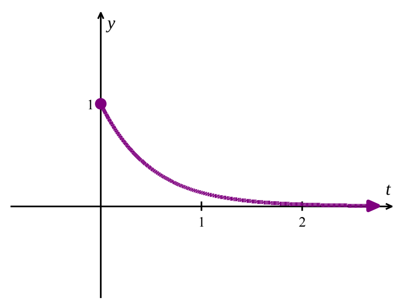
□

Example 9.4.3. Sketch the curve described by the parametric equations $\begin{cases} x = 2e^{-t} \\ y = e^{-2t} \end{cases}$ for $t \ge 0$.

Solution. We proceed as in the previous example and graph $x = 2e^{-t}$ and $y = e^{-2t}$ over the interval $[0,\infty)$.



$x = 2e^{-t}$, $t \ge 0$



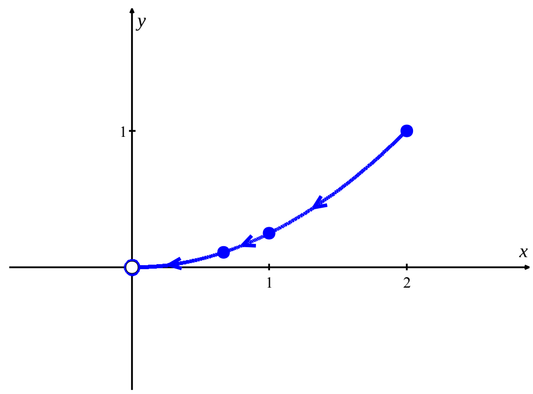
$y = e^{-2t}$, $t \ge 0$

We find that the range of *x* in this case is $(0,2]$ and the range of *y* is $(0,1]$. Since *t* is ranging over the unbounded interval $[0,\infty)$, we take the time to analyze the behavior of both *x* and *y*.

- As $t \to \infty$, $x = 2e^{-t} \to 0^+$ and $y = e^{-2t} \to 0^+$ as well. This means the graph of the resulting function approaches the point $(0,0)$.
- Since both $x = 2e^{-t}$ and $y = e^{-2t}$ are always decreasing for $t \ge 0$, we know that our final graph will start at $(2,1)$, where $t = 0$, and move consistently to the left (since *x* is decreasing) and down (since *y* is decreasing) to approach the origin.

Next, we plug in some friendly values of *t* to get a sense of the orientation of the curve. Since *t* lies in the exponent here, friendly values of *t* involve natural logarithms.

t	$x(t)=2e^{-t}$	$y(t)=e^{-2t}$	$(x(t),y(t))$
$\ln(1)=0$	2	1	$(2,1)$
$\ln(2)$	1	$\frac{1}{4}$	$\left(1,\frac{1}{4}\right)$
$\ln(3)$	$\frac{2}{3}$	$\frac{1}{9}$	$\left(\frac{2}{3},\frac{1}{9}\right)$

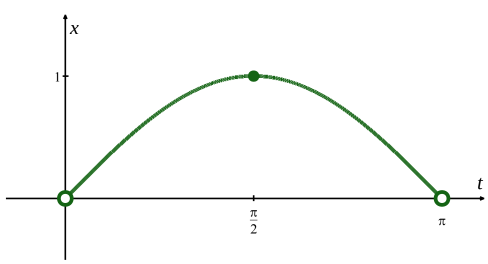


$$\begin{cases} x=2e^{-t} \\ y=e^{-2t} \end{cases} \text{ for } t\ge 0$$

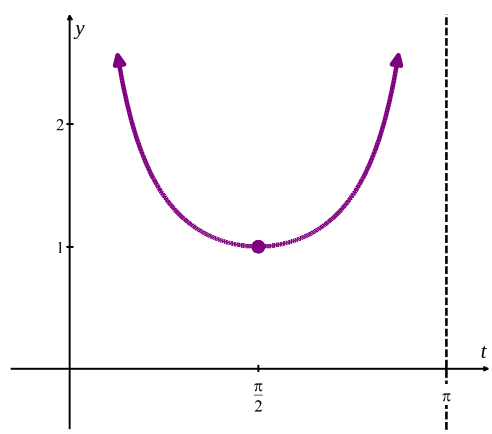
□

Example 9.4.4. Sketch the curve described by the parametric equations $\begin{cases} x=\sin(t) \\ y=\csc(t) \end{cases}$ for $0<t<\pi$.

Solution. We start by graphing $x=\sin(t)$ and $y=\csc(t)$ over the interval $(0,\pi)$.



$x=\sin(t),\ 0<t<\pi$



$y=\csc(t),\ 0<t<\pi$

We find that the range of *x* is $(0,1]$ while the range of *y* is $[1,\infty)$. Before moving on, we take a closer look at these two graphs.

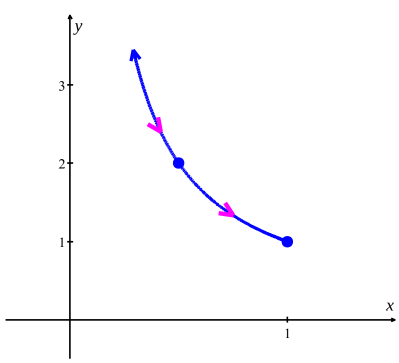
- Since $t=0$ and $t=\pi$ aren't included in the domain for *t*, we analyze the behavior of the system as *t* approaches each of these values. We find that as $t\to 0^+$, and when $t\to\pi^-$, we get $x=\sin(t)\to 0^+$ and $y=\csc(t)\to\infty$. Piecing this information together, we get that for *t* near 0, and for *t* near π, we have points with very small positive *x*-values, but very large *y*-values.

- As t ranges through the interval $\left(0, \frac{\pi}{2}\right]$, $x = \sin(t)$ is increasing and $y = \csc(t)$ is decreasing. This means that we are moving to the right and downwards.
- Once $t = \frac{\pi}{2}$, the orientation reverses, and we start to head to the left, since $x = \sin(t)$ is now decreasing, and up, since $y = \csc(t)$ is now increasing.

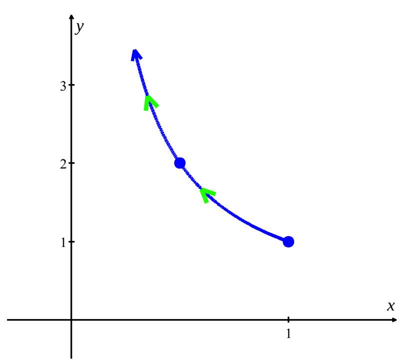
We plot a few points before sketching the curve.

t	$x(t) = \sin(t)$	$y(t) = \csc(t)$	$(x(t), y(t))$
$\frac{\pi}{6}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\csc\left(\frac{\pi}{6}\right) = 2$	$\left(\frac{1}{2}, 2\right)$
$\frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\csc\left(\frac{\pi}{2}\right) = 1$	$(1,1)$
$\frac{5\pi}{6}$	$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$	$\csc\left(\frac{5\pi}{6}\right) = 2$	$\left(\frac{1}{2}, 2\right)$

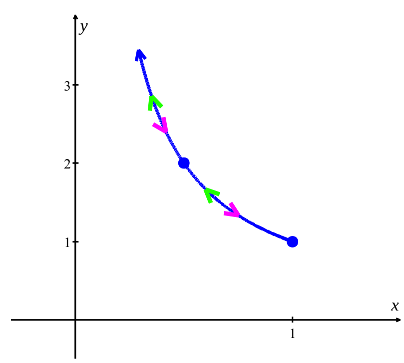
We combine the above information to first graph the system of equations on the interval $\left(0, \frac{\pi}{2}\right]$, followed by the interval $\left[\frac{\pi}{2}, \pi\right)$, and finally on the combined interval $(0, \pi)$.



$0 < t \leq \frac{\pi}{2}$



$\frac{\pi}{2} \leq t < \pi$



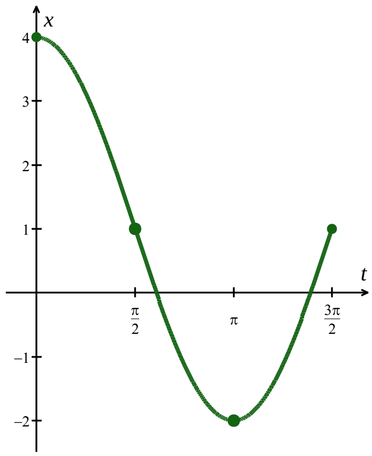
$\begin{cases} x = \sin(t) \\ y = \csc(t) \end{cases}$ for $0 < t < \pi$

□

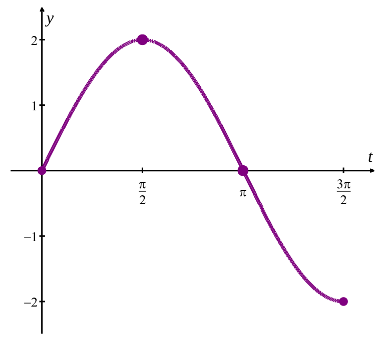
We see that the parametrization given above traces out this portion of the curve twice as t runs through the interval $(0, \pi)$.

Example 9.4.5. Sketch the curve described by the parametric equations $\begin{cases} x = 1 + 3\cos(t) \\ y = 2\sin(t) \end{cases}$ for $0 \le t \le \frac{3\pi}{2}$.

Solution. Proceeding as above, we set about graphing the system of parametric equations by first graphing $x = 1 + 3\cos(t)$ and $y = 2\sin(t)$ on the interval $\left[0, \frac{3\pi}{2}\right]$.



$x = 1 + 3\cos(t)$, $0 \le t \le \frac{3\pi}{2}$



$y = 2\sin(t)$, $0 \le t \le \frac{3\pi}{2}$

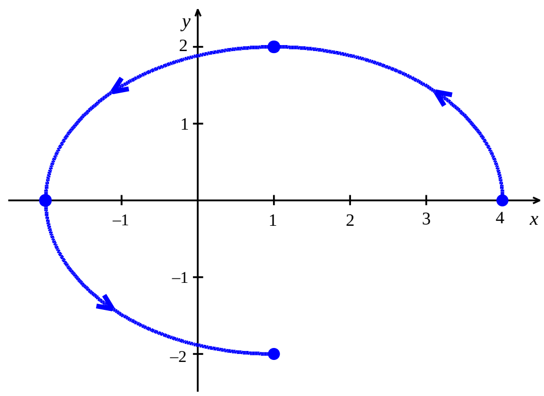
We see that x ranges from -2 to 4 and y ranges from -2 to 2. The direction of the curve is as follows.

- As t ranges from 0 to $\frac{\pi}{2}$, x is decreasing while y is increasing, resulting in a movement left and upwards.
- For $\frac{\pi}{2} \le t \le \pi$, x is decreasing as is y, so the motion is still right to left but is now downwards.
- On the interval $\left[\pi, \frac{3\pi}{2}\right]$, x begins to increase while y continues to decrease. Hence, the motion becomes left to right but continues downwards.

Plugging in the values $t = 0$, $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ gives the following (x, y) coordinates.

t	$x(t)=1+3\cos(t)$	$y(t)=2\sin(t)$	$(x(t),y(t))$
0	$1+3\cos(0)=4$	$2\sin(0)=0$	$(4,0)$
$\frac{\pi}{2}$	$1+3\cos\left(\frac{\pi}{2}\right)=1$	$2\sin\left(\frac{\pi}{2}\right)=2$	$(1,2)$
π	$1+3\cos(\pi)=-2$	$2\sin(\pi)=0$	$(-2,0)$
$\frac{3\pi}{2}$	$1+3\cos\left(\frac{3\pi}{2}\right)=1$	$2\sin\left(\frac{3\pi}{2}\right)=-2$	$(1,-2)$

We put all of our information together to get the following graph.



$$\begin{cases} x=1+3\cos(t) \\ y=2\sin(t) \end{cases} \text{ for } 0\le t\le \frac{3\pi}{2}$$

□

If this graph looks suspiciously like that of an ellipse, we will find in the next section that this is indeed the case. The next section is devoted to converting between parametric and Cartesian equations.

9.4 Exercises

In Exercises 1 – 6, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

1. $\begin{cases} x(t) = t \\ y(t) = t^2 - 1 \end{cases}$ for $-3 \le t \le 3$

t	x	y
–3		
–2		
–1		
0		
1		
2		
3		

2. $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$ for $-3 \le t \le 2$

t	x	y
–3		
–2		
–1		
0		
1		
2		

3. $\begin{cases} x(t) = 2 + t \\ y(t) = 3 - 2t \end{cases}$ for $-2 \le t \le 3$

t	x	y
–2		
–1		
0		
1		
2		
3		

4. $\begin{cases} x(t) = -2 - 2t \\ y(t) = 3 + t \end{cases}$ for $-3 \le t \le 1$

t	x	y
–3		
–2		
–1		
0		
1		

5. $\begin{cases} x(t) = t^3 \\ y(t) = t + 2 \end{cases}$ for $-2 \le t \le 2$

t	x	y
–2		
–1		
0		
1		
2		

6. $\begin{cases} x(t) = t^2 \\ y(t) = t + 3 \end{cases}$ for $-2 \le t \le 2$

t	x	y
–2		
–1		
0		
1		
2		

In Exercises 7 – 30, plot the set of parametric equations by hand. Be sure to indicate the orientation imparted on the curve by the parametrization.

7. $\begin{cases} x = 4t - 3 \\ y = 6t - 2 \end{cases}$ for $0 \le t \le 1$

8. $\begin{cases} x = 4t - 1 \\ y = 3 - 4t \end{cases}$ for $0 \le t \le 1$

9. $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ for $-1 \le t \le 2$

10. $\begin{cases} x = t^2 \\ y = 3t \end{cases}$ for $0 \le t \le 5$

11. $\begin{cases} x = t - 1 \\ y = 3 + 2t - t^2 \end{cases}$ for $0 \le t \le 3$

12. $\begin{cases} x = t^2 + 2t + 1 \\ y = t + 1 \end{cases}$ for $t \le 1$

13. $\begin{cases} x = \frac{1}{9}\left(18 - t^2\right) \\ y = \frac{1}{3}t \end{cases}$ for $t \ge -3$

14. $\begin{cases} x = t \\ y = t^3 \end{cases}$ for $-\infty < t < \infty$

15. $\begin{cases} x = t^3 \\ y = t \end{cases}$ for $-\infty < t < \infty$

16. $\begin{cases} x = t \\ y = \sqrt{25 - t^2} \end{cases}$ for $0 \le t \le 5$

17. $\begin{cases} x = -t \\ y = \sqrt{t} \end{cases}$ for $t \ge 0$

18. $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

19. $\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ for $0 \le t \le \pi$

20. $\begin{cases} x = -2\cos(t) \\ y = 6\sin(t) \end{cases}$ for $0 \le t \le \pi$

21. $\begin{cases} x = -1 + 3\cos(t) \\ y = 4\sin(t) \end{cases}$ for $0 \le t \le 2\pi$

22. $\begin{cases} x = 3\cos(t) \\ y = 2\sin(t) + 1 \end{cases}$ for $\frac{\pi}{2} \le t \le 2\pi$

23. $\begin{cases} x = 2\cos(t) \\ y = \sec(t) \end{cases}$ for $0 \le t < \frac{\pi}{2}$

24. $\begin{cases} x = 2\tan(t) \\ y = \cot(t) \end{cases}$ for $0 < t < \frac{\pi}{2}$

25. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

26. $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$

27. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

28. $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$

29. $\begin{cases} x = \cos(t) \\ y = t \end{cases}$ for $0 \le t \le \pi$

30. $\begin{cases} x = \sin(t) \\ y = t \end{cases}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

In Exercises 31 – 34, plot the set of parametric equations with the help of a graphing utility. Be sure to indicate the orientation imparted on the curve by the parametrization.

31. $\begin{cases} x = t^3 - 3t \\ y = t^2 - 4 \end{cases}$ for $-2 \le t \le 2$

32. $\begin{cases} x = 4\cos^3(t) \\ y = 4\sin^3(t) \end{cases}$ for $0 \le t \le 2\pi$

33. $\begin{cases} x = e^t + e^{-t} \\ y = e^t - e^{-t} \end{cases}$ for $-2 \le t \le 2$

34. $\begin{cases} x = \cos(3t) \\ y = \sin(4t) \end{cases}$ for $0 \le t \le 2\pi$

In Exercises 35 – 38, use a graphing utility to view the graph of each of the four sets of parametric equations. Although they look unusual and beautiful, they are so common they have names, as indicated in each exercise.

35. An epicycloid: $\begin{cases} x = 14\cos(t) - \cos(14t) \\ y = 14\sin(t) + \sin(14t) \end{cases}$ for $0 \le t \le 2\pi$

36. A hypocycloid: $\begin{cases} x = 6\sin(t) + 2\sin(6t) \\ y = 6\cos(t) - 2\cos(6t) \end{cases}$ for $0 \le t \le 2\pi$

37. A hypotrochoid: $\begin{cases} x = 2\sin(t) + 5\cos(6t) \\ y = 5\cos(t) - 2\sin(6t) \end{cases}$ for $0 \le t \le 2\pi$

38. A rose: $\begin{cases} x = 5\sin(2t)\sin(t) \\ y = 5\sin(2t)\cos(t) \end{cases}$ for $0 \le t \le 2\pi$

9.5 Finding Parametric Descriptions for Oriented Curves

Learning Objectives

In this section you will:

- **Eliminate the parameter in a pair of parametric equations.**
- **Parametrize curves given in Cartesian coordinates.**
- **Reverse orientation and shift starting point of a curve described by parametric equations.**

Now that we have had some good practice sketching the graphs of parametric equations, we turn to the problem of eliminating the parameter in a pair of parametric equations to get an equation involving just x and y.

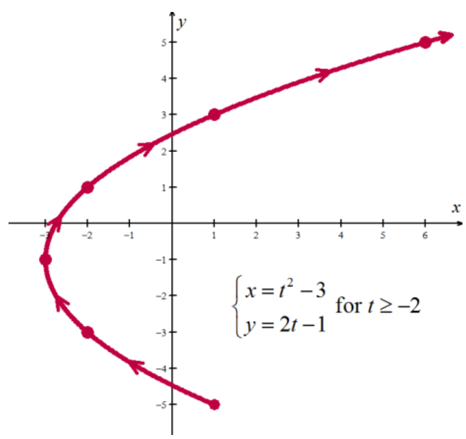
Eliminating the Parameter in Parametric Equations

Recall that several curves in the examples from **Section 9.4** resembled graphs of functions we've seen before. We revisit these examples, eliminating the parameter to determine a Cartesian equation.

Example 9.5.1. Eliminate the parameter t in the system of equations $\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$ from **Example 9.4.1** to determine a Cartesian equation. Recall $t \geq -2$ in this example.

Solution. We use the technique of substitution to eliminate the parameter in the system of equations. The first step is to solve $y = 2t - 1$ for t.

$$\begin{aligned} y &= 2t - 1 \\ y + 1 &= 2t \\ t &= \frac{y+1}{2} \end{aligned}$$



Substituting this result into the equation $x = t^2 - 3$ yields

$$\begin{aligned} x &= \left(\frac{y+1}{2}\right)^2 - 3 \\ x + 3 &= \frac{(y+1)^2}{4} \\ (y+1)^2 &= 4(x+3) \end{aligned}$$

We see that the graph of this equation is a parabola with vertex $(-3, -1)$ which opens to the right.

Technically speaking, the equation $(y+1)^2 = 4(x+3)$ describes the entire parabola, while the parametric equations describe only a portion of the parabola. In this case, we can remedy the situation by restricting the bounds on *y*. Since the portion of the parabola we want is exactly the part where $y \geq -5$, the equation $(y+1)^2 = 4(x+3)$ coupled with the restriction $y \geq -5$ describes the same curve as the given parametric equations. The one piece of information we can never recover after eliminating the parameter is the orientation of the curve.

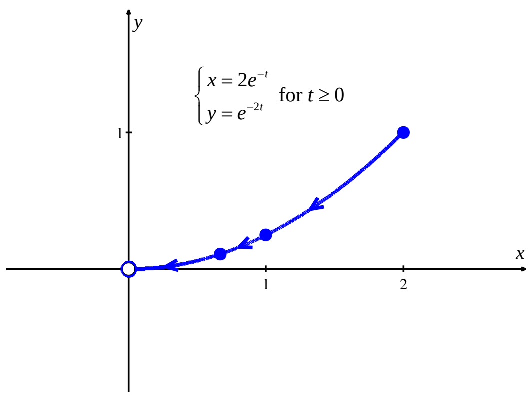
□

Example 9.5.2. Eliminate the parameter *t* in the system of equations $\begin{cases} x = 2e^{-t} \\ y = e^{-2t} \end{cases}$ from **Example 9.4.3**, where the restriction on *t* was $t \geq 0$.

Solution. To eliminate the parameter, one way to proceed is to solve $x = 2e^{-t}$ for *t* to get $t = -\ln\left(\frac{x}{2}\right)$.

Substituting this for *t* in $y = e^{-2t}$ gives

$$\begin{aligned} y &= e^{-2\left(-\ln\left(\frac{x}{2}\right)\right)} \\ &= e^{2\ln\left(\frac{x}{2}\right)} \\ &= e^{\ln\left(\frac{x}{2}\right)^2} \\ &= \left(\frac{x}{2}\right)^2 \\ &= \frac{x^2}{4} \end{aligned}$$



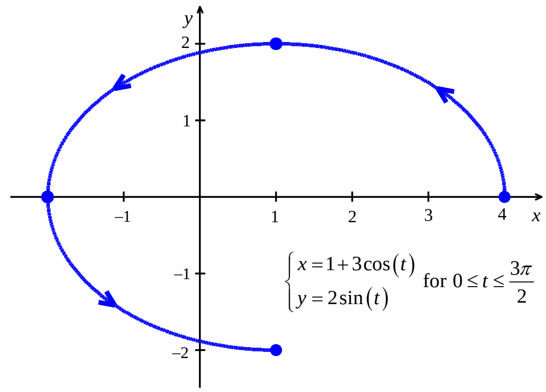
The parametrized curve is the portion of the parabola $y = \frac{x^2}{4}$ which starts at the point $(2,1)$ and heads toward, but never reaches, $(0,0)$.

□

Example 9.5.3. Eliminate the parameter *t* in the system of equations $\begin{cases} x = 1 + 3\cos(t) \\ y = 2\sin(t) \end{cases}$ from **Example 9.4.5**, where the restriction on *t* was $0 \leq t \leq \frac{3\pi}{2}$.

Solution. To eliminate the parameter here, we note that the trigonometric functions involved, namely $\cos(t)$ and $\sin(t)$, are related by the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$. Hence, we solve

$x = 1 + 3\cos(t)$ for $\cos(t)$ to get $\cos(t) = \frac{x-1}{3}$ and we solve $y = 2\sin(t)$ for $\sin(t)$ to get $\sin(t) = \frac{y}{2}$. Substituting these expressions into $\cos^2(t) + \sin^2(t) = 1$ gives



$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$
$$\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$$

This is the equation of an ellipse centered at $(1,0)$ with vertices at $(-2,0)$ and $(4,0)$, and with a minor axis of length 4. The parametric equations trace out three-quarters of this ellipse in a counter-clockwise direction.

□

We next turn to the problem of finding parametric representations of curves.

Parametrizing Curves

We start with the following.

Parametrizations of Common Curves

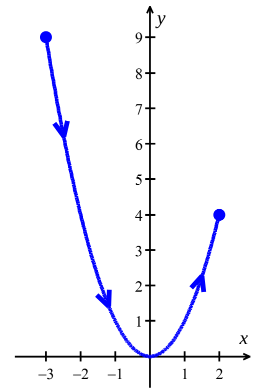
- To parametrize $y = f(x)$ as *x* runs through some interval *I*, let $x = t$, $y = f(t)$, and let *t* run through *I*.
- To parametrize $x = g(y)$ as *y* runs through the interval *I*, let $x = g(t)$, $y = t$, and let *t* run through *I*.
- To parametrize a directed line segment with initial point (x_0, y_0) and terminal point (x_1, y_1), let $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$ for $0 \le t \le 1$.
- To parametrize $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a > 0$ and $b > 0$, let $x = h + a\cos(t)$ and $y = k + b\sin(t)$ for $0 \le t < 2\pi$. (This will impart a counter-clockwise orientation.)

The reader is encouraged to verify the above formulas by eliminating the parameter and, when indicated, checking the orientation. We put these formulas to good use in the following examples.

Example 9.5.4. Find a parametrization for the curve $y = x^2$ from $x = -3$ to $x = 2$.

Solution. Since $y = x^2$ is written in the form $y = f(x)$, we let $x = t$ and $y = f(t) = t^2$. Since $x = t$, the bounds on t match precisely the bounds on x so we get

$$\begin{cases} x = t \\ y = t^2 \end{cases} \text{ for } -3 \le t \le 2$$

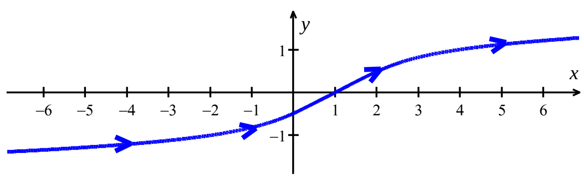


□

Example 9.5.5. Find a parametrization for the curve $x = y^5 + 2y + 1$.

Solution. While we could attempt to solve this equation for y, we don't need to. We can parametrize $x = f(y) = y^5 + 2y + 1$ by setting $y = t$ so that $x = t^5 + 2t + 1$. Since $y = t$ and there are no bounds placed on y, it follows that there are no bounds placed on t. Our final answer is

$$\begin{cases} x = t^5 + 2t + 1 \\ y = t \end{cases} \text{ for } -\infty < t < \infty$$



□

Example 9.5.6. Find a parametrization for the line segment which starts at $(2,-3)$ and ends at $(1,5)$.

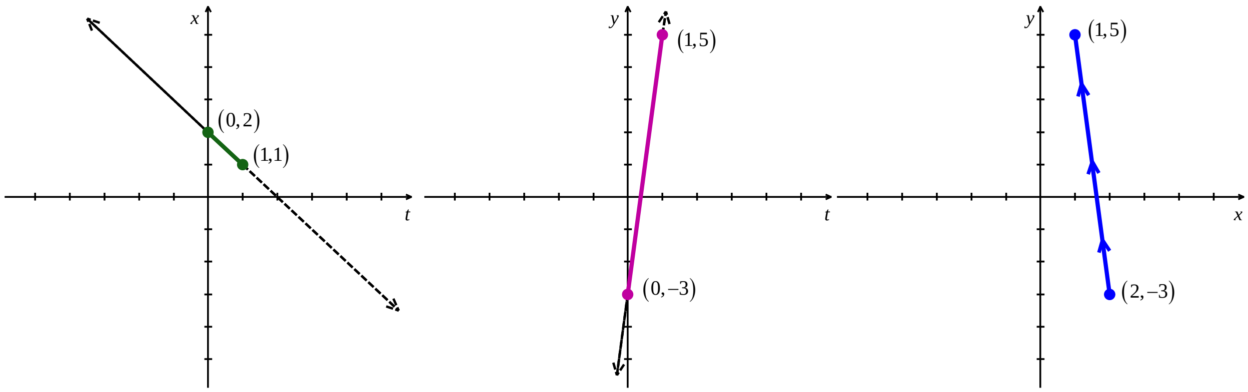
Solution. We make use of the formulas $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$ for $0 \le t \le 1$. These formulas can be summarized as

$$\text{starting point} + (\text{displacement})t$$

- To find the equation for x, we have that the line segment starts at $x = 2$ and ends at $x = 1$. This means that the displacement in the x-direction is $1 - 2 = -1$. Hence, the equation for x is $x = 2 + (-1)t$, or $x = 2 - t$.
- For y, we note that the line segment starts at $y = -3$ and ends at $y = 5$. Thus, the displacement in the y-direction is $5 - (-3) = 8$, so we get $y = -3 + 8t$.

Our final answer is

$$\begin{cases} x = 2 - t \\ y = -3 + 8t \end{cases} \text{ for } 0 \le t \le 1$$



$x = 2 - t$ line segment $0 \le t \le 1$

$y = -3 + 8t$ line segment $0 \le t \le 1$

$\begin{cases} x = 2 - t \\ y = -3 + 8t \end{cases}$ for $0 \le t \le 1$

□

Example 9.5.7. Find a parametrization for the circle $x^2 + 2x + y^2 - 4y = 4$.

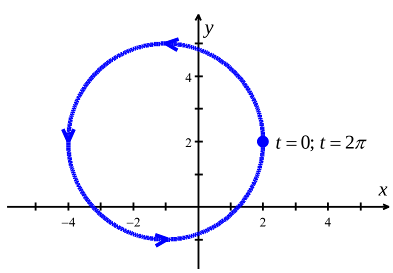
Solution. In order to use the formulas $x = h + a\cos(t)$ and $y = k + b\sin(t)$ to parametrize the circle $x^2 + 2x + y^2 - 4y = 4$, we first need to put it into the correct form, $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

$$\begin{gathered} x^2 + 2x + y^2 - 4y = 4 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4 \\ (x+1)^2 + (y-2)^2 = 9 \\ \frac{(x+1)^2}{9} + \frac{(y-2)^2}{9} = 1 \end{gathered}$$

The formulas $x = h + a\cos(t)$ and $y = k + b\sin(t)$ can be a challenge to memorize, but they come from the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$. By writing the equation $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{9} = 1$ as $\left(\dfrac{x+1}{3}\right)^2 + \left(\dfrac{y-2}{3}\right)^2 = 1$, we identify $\cos(t) = \dfrac{x+1}{3}$ and $\sin(t) = \dfrac{y-2}{3}$.

Rearranging these last two equations, we get $x = -1 + 3\cos(t)$ and $y = 2 + 3\sin(t)$. In order to complete one revolution around the circle, we let t range through the interval $[0, 2\pi)$. Our final answer is

$$\begin{cases} x = -1 + 3\cos(t) \\ y = 2 + 3\sin(t) \end{cases} \text{ for } 0 \le t < 2\pi$$



□

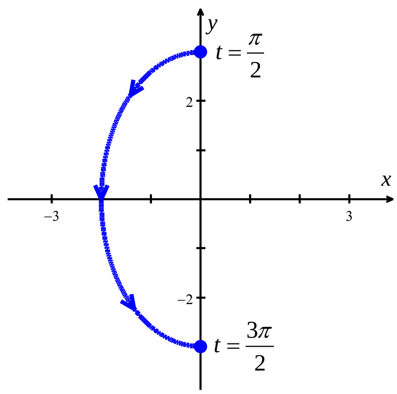
Example 9.5.8. Find a parametrization for the left half of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution. In the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$, we can either use the formulas or think back to the Pythagorean identity $\cos^2(t)+\sin^2(t)=1$, along with $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$, to get

$$\frac{x}{2}=\cos(t) \qquad\qquad \frac{y}{3}=\sin(t)$$
$$x=2\cos(t) \qquad\qquad y=3\sin(t)$$

The normal range on the parameter in this case is $0\le t<2\pi$, but since we are interested in only the left half of the ellipse, we restrict t to the values which correspond to Quadrant II and Quadrant III angles, namely $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$. Our final answer is

$$\begin{cases} x=2\cos(t) \\ y=3\sin(t) \end{cases} \text{ for } \frac{\pi}{2}\le t\le\frac{3\pi}{2}$$



□

In the last two examples we avoided the formulas by instead associating the circle and ellipse equations with the Pythagorean identity $\cos^2(t)+\sin^2(t)=1$. By getting a feel for the mechanics behind each of the previous five examples, reliance on formulas can be minimized. We note that the formulas provided prior to these examples offer only one of literally infinitely many ways to parametrize the common curves listed there.

Adjusting Parametric Equations

At times, the formulas that define a parametric curve need to be altered to suit the situation. Two easy ways to alter parametrizations are given below.

Adjusting Parametric Equations

- **Reversing Orientation:** Replacing every occurrence of t with $-t$ in a parametric description for a curve (including any inequalities which describe the bounds on t) reverses the orientation of the curve.
- **Shift of Parameter:** Replacing every occurrence of t with $t-c$ in a parametric description for a curve (including any inequalities which describe the bounds on t) shifts the start of the parameter t ahead by c units.

We demonstrate these techniques in the following example.

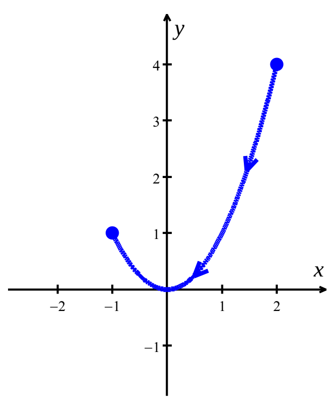
Example 9.5.9. Find a parametrization for the following curves.

1. The curve which starts at $(2,4)$ and follows the parabola $y=x^2$ to end at $(-1,1)$. Shift the parameter so the path starts at $t=0$.
2. The two part path which starts at $(0,0)$, travels along a line to $(3,4)$, then travels along a line to $(5,0)$.
3. The Unit Circle, oriented clockwise, with $t=0$ corresponding to $(0,-1)$.

Solution.

1. We can parametrize $y=x^2$ from $x=-1$ to $x=2$ as $\begin{cases} x=t \\ y=t^2 \end{cases}$ for $-1\le t\le 2$. This parametrization, however, starts at $(-1,1)$ and ends at $(2,4)$. Hence, we need to reverse the orientation. To do so, we replace every occurrence of t with $-t$ to get $x=-t$ and $y=(-t)^2$ for $-1\le -t\le 2$. After simplifying, we have

$$\begin{cases} x=-t \\ y=t^2 \end{cases} \text{ for } -2\le t\le 1$$



We would next like t to begin at $t=0$ instead of $t=-2$. The problem here is that the parametrization we have starts 2 units too soon, so we need to introduce a time delay of 2. Replacing every occurrence of t with $t-2$ gives $x=-(t-2)$ and $y=(t-2)^2$ for $-2\le t-2\le 1$. Simplifying yields

$$\begin{cases} x=2-t \\ y=t^2-4t+4 \end{cases} \text{ for } 0\le t\le 3$$

2. When parameterizing line segments, we think: starting point + $(\text{displacement})t$. For the first part of the path, which starts at $(0,0)$ and travels along a line to $(3,4)$, we get

$$\begin{cases} x = 3t \\ y = 4t \end{cases} \text{ for } 0 \le t \le 1$$

For the second part we get

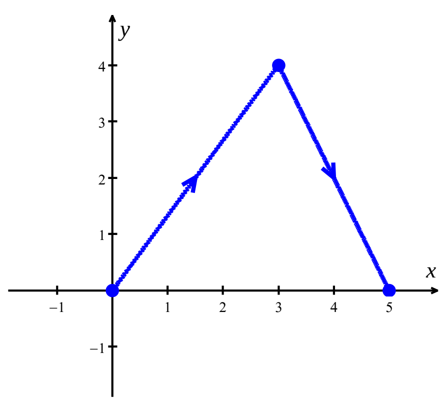
$$\begin{cases} x = 3+2t \\ y = 4-4t \end{cases} \text{ for } 0 \le t \le 1$$

Since the first parametrization leaves off at $t = 1$, we shift the parameter in the second part so it starts at $t = 1$. Our current description of the second part starts at $t = 0$, so we introduce a time delay of 1 unit to the second set of parametric equations. Replacing t with $t-1$ in the second set of parametric equations gives $x = 3+2(t-1)$ and $y = 4-4(t-1)$ for $0 \le t-1 \le 1$. Simplifying yields

$$\begin{cases} x = 1+2t \\ y = 8-4t \end{cases} \text{ for } 1 \le t \le 2$$

Hence, we may parametrize the path as

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \text{ for } 0 \le t \le 2, \text{ where } f(t) = \begin{cases} 3t \text{ for } 0 \le t \le 1 \\ 1+2t \text{ for } 1 \le t \le 2 \end{cases} \text{ and } g(t) = \begin{cases} 4t \text{ for } 0 \le t \le 1 \\ 8-4t \text{ for } 1 \le t \le 2 \end{cases}$$



3. To parametrize the Unit Circle with a clockwise orientation and 'starting point' of $(0,-1)$ corresponding to $t = 0$, we first note that a counter-clockwise orientation is given by

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \text{ for } 0 \le t < 2\pi$$

The first order of business is to reverse the orientation. Replacing t with $-t$ gives $x=\cos(-t)$ and $y=\sin(-t)$ for $0\leq -t<2\pi$, which simplifies[1] to

$$\begin{cases} x=\cos(t) \\ y=-\sin(t) \end{cases} \text{ for } -2\pi<t\leq 0$$

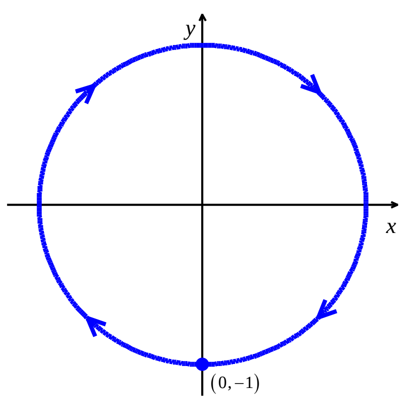
This parametrization gives a clockwise orientation, but $t=0$ still corresponds to the point $(1,0)$; the point $(0,-1)$ is reached when $t=-\frac{3\pi}{2}$. Our strategy is to first get the parametrization to start at the point $(0,-1)$ and then shift the parameter accordingly so the start coincides with $t=0$.

- We know that any interval of length 2π will parametrize the entire circle, so we keep the equations $x=\cos(t)$ and $y=-\sin(t)$, but start the parameter t at $-\frac{3\pi}{2}$, and find the upper bound by adding 2π so that

$$\begin{cases} x=\cos(t) \\ y=-\sin(t) \end{cases} \text{ for } -\frac{3\pi}{2}\leq t<\frac{\pi}{2}$$

  The reader can verify that the Unit Circle is traced out clockwise starting at the point $(0,-1)$.

- To shift the parameter so that the start coincides with $t=0$, we introduce a time delay of $\frac{3\pi}{2}$ units by replacing each occurrence of t with $t-\frac{3\pi}{2}$. We get $x=\cos\left(t-\frac{3\pi}{2}\right)$ and $y=-\sin\left(t-\frac{3\pi}{2}\right)$ for $-\frac{3\pi}{2}\leq t-\frac{3\pi}{2}<\frac{\pi}{2}$. This simplifies[2] to



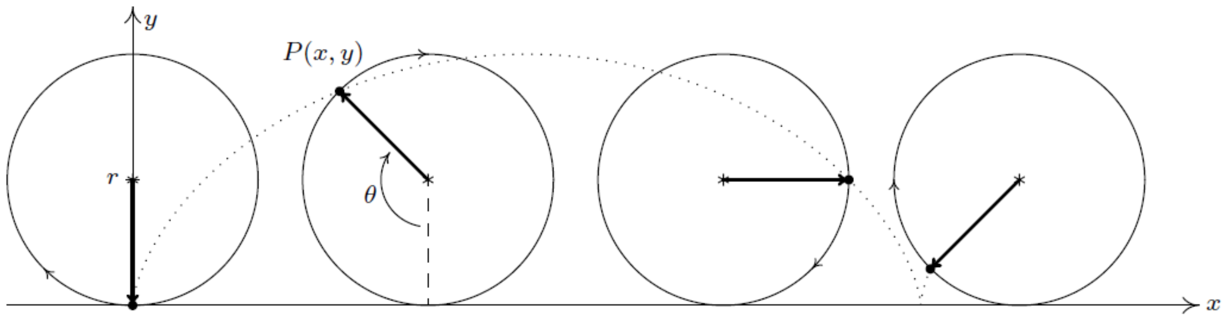
$$\begin{cases} x=-\sin(t) \\ y=-\cos(t) \end{cases} \text{ for } 0\leq t<2\pi$$

□

We put our answer to **Example 9.5.9** number **3** to good use to derive the equation of a cycloid. Suppose a circle of radius r rolls along the positive x-axis at a constant velocity v as pictured below. Let θ be the

[1] courtesy of the even/odd identities
[2] courtesy of the sum/difference formulas

angle in radians which measures the amount of clockwise rotation experienced by the radius highlighted in the figure.



Our goal is to find parametric equations for the coordinates of the point $P(x, y)$ in terms of θ. From our work in **Example 9.5.9** number 3, we know that clockwise motion along the Unit Circle starting at the point $(0,-1)$ can be modeled by the equations

$$\begin{cases} x = -\sin(\theta) \\ y = -\cos(\theta) \end{cases} \text{ for } 0 \le \theta < 2\pi$$

(We have renamed the parameter as θ to match the context of this problem.) To model this motion on a circle of radius *r*, all we need to do[3] is multiply both *x* and *y* by the factor *r* which yields

$$\begin{cases} x = -r\sin(\theta) \\ y = -r\cos(\theta) \end{cases}$$

We now need to adjust for the fact that the circle isn't stationary with center $(0,0)$, but is rolling along the positive *x*-axis. Since the velocity *v* is constant, we know that at time *t*, the center of the circle has traveled a distance vt down the positive *x*-axis. Furthermore, since the radius of the circle is *r* and the circle isn't moving vertically, we know that the center of the circle is always *r* units above the *x*-axis. Putting these two facts together, we have that at time *t*, the center of the circle is at the point (vt, r).

---

[3] If we replace *x* with $\frac{x}{r}$ and *y* with $\frac{y}{r}$ in the equation for the Unit Circle, $x^2 + y^2 = 1$, we obtain $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$, which reduces to $x^2 + y^2 = r^2$. Note that we are 'stretching' the graph by a factor of *r* in both the *x*- and *y*-directions. Hence, we multiply both the *x*- and *y*-coordinates of points on the graph by *r*.

From **Section 1.3**, we know the angular velocity is $\omega = \dfrac{\theta}{t}$ and the linear velocity is $v = r\omega$. Putting these together, we have $v = \dfrac{r\theta}{t}$, or $vt = r\theta$. Hence, the center of the circle, in terms of the parameter θ, is $(r\theta, r)$.

As a result, we need to modify the equations $x = -r\sin(\theta)$ and $y = -r\cos(\theta)$ by shifting the *x*-coordinates to the right $r\theta$ units (by adding $r\theta$ to the expression for *x*) and the *y*-coordinate up *r* units (by adding *r* to the expression for *y*). We get $x = -r\sin(\theta) + r\theta$ and $y = -r\cos(\theta) + r$, which can be written as

$$\begin{cases} x = r(\theta - \sin(\theta)) \\ y = r(1 - \cos(\theta)) \end{cases}$$

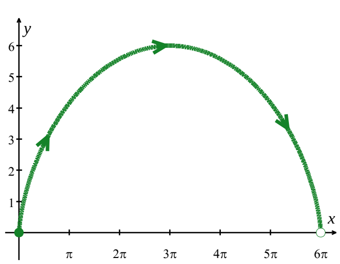
Since the motion starts at $\theta = 0$ and proceeds indefinitely, we set $\theta \geq 0$.

Example 9.5.10. Find the parametric equations of a cycloid which results from a circle of radius 3 rolling down the positive *x*-axis.

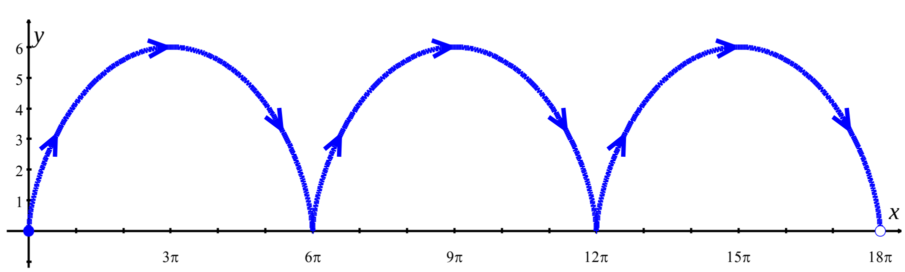
Solution. We completed the major part of our work above. With $r = 3$, we have the equations

$$\begin{cases} x = 3(t - \sin(t)) \\ y = 3(1 - \cos(t)) \end{cases} \text{ for } t \geq 0$$

(Here we have returned to the convention of using *t* as the parameter.) We know that one full revolution of the circle occurs over the interval $0 \leq t < 2\pi$. As *t* ranges between 0 and 2π, we see that *x* ranges between 0 and 6π. The values of *y* range between 0 and 6.



$0 \leq t < 2\pi$



$0 \leq t \leq 6\pi$

Above, the first graph of the cycloid is over the interval $0 \leq t < 2\pi$. For the second graph, we extend *t* to range from 0 to 6π which forces *x* to range from 0 to 18π yielding three arches of the cycloid.

□

9.5 Exercises

In Exercises 1 – 18, eliminate the parameter *t* to rewrite the parametric equation as a Cartesian equation.

1. $\begin{cases} x = 5 - t \\ y = 8 - 2t \end{cases}$

2. $\begin{cases} x = 6 - 3t \\ y = 10 - t \end{cases}$

3. $\begin{cases} x = 2t + 1 \\ y = 3\sqrt{t} \end{cases}$

4. $\begin{cases} x = 3t - 1 \\ y = 2t^2 \end{cases}$

5. $\begin{cases} x = 2e^t \\ y = 1 - 5t \end{cases}$

6. $\begin{cases} x = e^{-2t} \\ y = 2e^{-t} \end{cases}$

7. $\begin{cases} x = 4\log(t) \\ y = 3 + 2t \end{cases}$

8. $\begin{cases} x = \log(2t) \\ y = \sqrt{t-1} \end{cases}$

9. $\begin{cases} x = t^3 - 1 \\ y = 2t \end{cases}$

10. $\begin{cases} x = t - t^4 \\ y = t + 2 \end{cases}$

11. $\begin{cases} x = e^{2t} \\ y = e^{6t} \end{cases}$

12. $\begin{cases} x = t^5 \\ y = t^{10} \end{cases}$

13. $\begin{cases} x = 4\cos(t) \\ y = 4\sin(t) \end{cases}$

14. $\begin{cases} x = 3\sin(t) \\ y = 6\cos(t) \end{cases}$

15. $\begin{cases} x = 2\cos^2(t) \\ y = -\sin(t) \end{cases}$

16. $\begin{cases} x = \cos(t) + 4 \\ y = 2\sin^2(t) \end{cases}$

17. $\begin{cases} x = t - 1 \\ y = t^2 \end{cases}$

18. $\begin{cases} x = -t \\ y = t^3 + 1 \end{cases}$

In Exercises 19 – 22, parameterize (write parametric equations for) each Cartesian equation by setting $x = t$ or by setting $y = t$.

19. $y = 3x^2 + 3$

20. $y = 2\sin(x) + 1$

21. $x = 3\log(y) + y$

22. $x = \sqrt{y} + 2y$

In Exercises 23 – 26, parameterize (write parametric equations for) each Cartesian equation by using $x = a\cos(t)$ and $y = b\sin(t)$. Identify the curve.

23. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

24. $\frac{x^2}{16} + \frac{y^2}{36} = 1$

25. $x^2 + y^2 = 16$

26. $x^2 + y^2 = 10$

In Exercises 27 – 41, find a parametric description for the given oriented curve.

27. the directed line segment from $(3,-5)$ to $(-2,2)$

28. the directed line segment from $(-2,-1)$ to $(3,-4)$

29. the curve $y = 4 - x^2$ from $(-2,0)$ to $(2,0)$

30. the curve $y = 4 - x^2$ from $(2,0)$ to $(-2,0)$

(Shift the parameter so $t = 0$ corresponds to $(2,0)$.)

31. the curve $x = y^2 - 9$ from $(-5,-2)$ to $(0,3)$

32. the curve $x = y^2 - 9$ from $(0,3)$ to $(-5,-2)$

(Shift the parameter so $t = 0$ corresponds to $(0,3)$.)

33. the circle $x^2 + y^2 = 25$, oriented counter-clockwise

34. the circle $(x-1)^2 + y^2 = 4$, oriented counter-clockwise

35. the circle $x^2 + y^2 - 6y = 0$, oriented counter-clockwise

36. the circle $x^2 + y^2 - 6y = 0$, oriented *clockwise*

(Shift the parameter so *t* begins at 0.)

37. the circle $(x-3)^2 + (y+1)^2 = 117$, oriented counter-clockwise

38. the ellipse $(x-1)^2 + 9y^2 = 9$, oriented counter-clockwise

39. the ellipse $9x^2 + 4y^2 + 24y = 0$, oriented counter-clockwise

40. the ellipse $9x^2 + 4y^2 + 24y = 0$, oriented *clockwise*

(Shift the parameter so $t = 0$ correspond to $(0,0)$.)

41. the triangle with vertices $(0,0)$, $(3,0)$, $(0,4)$, oriented counter-clockwise

(Shift the parameter so $t = 0$ corresponds to $(0,0)$.)

42. Use parametric equations and a graphing utility to graph the inverse of $f(x) = x^3 + 3x - 4$.

43. Every polar curve $r = f(\theta)$ can be translated to a system of parametric equations with parameter θ by $\{x = r\cos(\theta) = f(\theta)\cos(\theta),\ y = r\sin(\theta) = f(\theta)\sin(\theta)\}$. Convert $r = 6\cos(2\theta)$ to a system of parametric equations. Check your answer by graphing $r = 6\cos(2\theta)$ by hand using the techniques presented in **Section 8.3** and then graphing the parametric equations you found using a graphing utility.

44. A dart is thrown upward with an initial velocity of 65 feet/second at an angle of elevation of 52°. Consider the position of the dart at any time *t*. Neglect air resistance.

a) Find parametric equations $x(t)$ and $y(t)$ that model the position of the dart. Use g=32 ft./s^2.

b) Find all possible values of *x* that represent the situation.

c) When will the dart hit the ground?

d) Find the maximum height of the dart.

e) At what time will the dart reach maximum height?

45. Carl's friend Jason competes in Highland Games Competitions across the country. In one event, the 'hammer throw', he throws a 56 pound weight for distance. If the weight is released 6 feet above the ground at an angle of 42° with respect to the horizontal, with an initial speed of 33 feet per second, find the parametric equations for the flight of the hammer. (Use g=32 ft./s^2.) When will the hammer hit the ground? How far away will it hit the ground? Check your answer using a graphing utility.

46. Eliminate the parameter in the equations for projectile motion to show that the path of the projectile follows the curve

$$y = -\frac{g\sec^2(\theta)}{2v_0^2}x^2 + \tan(\theta)x + s_0$$

Recall that for a quadratic function $f(x) = ax^2 + bx + c$, the vertex can be determined using the formula $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. Use this formula to show the maximum height of the projectile is

$$y = \frac{v_0^2\sin^2(\theta)}{2g} + s_0 \text{ when } x = \frac{v_0^2\sin(2\theta)}{2g}$$

47. In another event, the 'sheaf toss', Jason throws a 20 pound weight for height. If the weight is released 5 feet above the ground at an angle of 85° with respect to the horizontal and the sheaf reaches a maximum height of 31.5 feet, use your results from **Exercise 48** to determine how fast the sheaf was launched into the air. (Once again, use g=32 ft./s^2.)

In Exercises 48 – 51, we explore the **hyperbolic cosine** function, denoted $\cosh(t)$, and the **hyperbolic sine** function, denoted $\sinh(t)$, defined below:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \text{ and } \sinh(t) = \frac{e^t - e^{-t}}{2}$$

48. Using a graphing utility as needed, verify that the domain of $\cosh(t)$ is $(-\infty, \infty)$ and the range of $\cosh(t)$ is $[1, \infty)$.

49. Using a graphing utility as needed, verify that the domain and range of $\sinh(t)$ are both $(-\infty, \infty)$.

50. Show that $\{x(t)\cosh(t), y(t) = \sinh(t)\}$ parameterize the right half of the 'unit' hyperbola $x^2 - y^2 = 1$. (Hence the use of the adjective 'hyperbolic'.)

51. Four other hyperbolic functions are waiting to be defined: the hyperbolic secant $\operatorname{sech}(t)$ the hyperbolic cosecant $\operatorname{csch}(t)$, the hyperbolic tangent $\tanh(t)$ and the hyperbolic cotangent $\coth(t)$. Define these functions in terms of $\cosh(t)$ and $\sinh(t)$, then convert them to formulas involving e^t and e^{-t}. Consult a suitable reference and spend some time reliving the thrills of trigonometry with these hyperbolic functions.