A TOUR TO STABILITY CONDITIONS ON DERIVED CATEGORIES

AREND BAYER

ABSTRACT. Lecture notes for the Minicourse on derived categories, Utah 2007. Preliminary version with many gaps, omissions, errors. Corrections welcome.

These lecture notes are a brief tour to Bridgeland's space of stability conditions on derived categories, introduced in [Bri02]. A more complete version will be made available on the website of the Minicourse and/or my homepage.[1]

1. STABLE VECTOR BUNDLES AND COHERENT SHEAVES

Stability in algebraic geometry is a very classical concept, in the two different (but closely related) contexts of geometric invariant theory, and stability of vector bundles and coherent sheaves. We will say nothing about the former, and take a very fast tour through the latter.

Let X be a smooth, projective curve over $\mathbb{C}$ (a Riemann surface). If E is a vector bundle, $d(E)$ its degree and $r(E)$ its rank, we call

$$\mu(E) = \frac{d(E)}{r(E)}$$

its slope.

The following lemma is extremely crucial:

1.1. **Lemma.** *Let $0 \to A \to E \to B \to 0$ be a short exact sequence of vector bundles. Then*

$$\begin{aligned} \mu(A) < \mu(E) &\quad \leftrightarrow \quad \mu(E) < \mu(B) \\ \mu(A) > \mu(E) &\quad \leftrightarrow \quad \mu(E) > \mu(B) \end{aligned}$$

This follows by simple algebra from $r(E) = r(A) + r(B)$ and $d(E) = d(A) + d(B)$, but even more convincingly from the picture in figure 1, where we set $Z(X) = r(X) + id(X)$ for $X = A, E, B$.

1.2. **Remark.** What we used in the above "proof by picture" are just two properties of the function Z:

(1) Z is additive on short exact sequences; in other words, Z is actually a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$.
(2) The image of Z is contained in some half-plane in $\mathbb{C}$, so that we can meaningfully compare the slopes of objects.

[1] http://math.utah.edu/∼bayer/

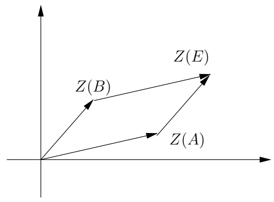


Figure 1: See-saw property

1.3. **Definition.** *A vector bundle is (semi-)stable if for all subbundles $A \hookrightarrow E$ we have $\mu(A) < \mu(E)$.*

Equivalently (by the see-saw property), we could ask that for all quotients $E \twoheadrightarrow B$ we have $\mu(E) > \mu(B)$.

1.4. **Examples.**

(1) Any line bundle is stable.
(2) An extension $0 \to \mathcal{O}_X \to E \to L_1 \to 0$ between the structure sheaf and a line bundle L_1 of degree one is stable if and only if the extension does not split. (Exercise!)

1.5. **Lemma.** *If E, E' are semistable and $\mu(E) > \mu(E')$, then $\mathrm{Hom}(E, E') = 0$.*

Proof. Factor any non-zero map via its image in E', and use the definition and see-saw property. □

Most interest in stable vector bundles is due to the fact that stability allows a meaningful study of moduli of vector bundles (in particular, the moduli space of *stable* vector bundles of fixed Chern class is bounded, which is obviously not true for the moduli of arbitrary vector bundles). However, for our purposes the existence of Harder-Narasimhan filtration is the most interesting aspect:

1.6. **Theorem.** *For any vector bundle E there is a unique increasing filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

such that the filtration quotients E_i/E_{i-1} are semistable of slope μ_i, with $\mu_1 > \mu_2 > \cdots > \mu_n$.

We will only sketch a small part of the proof here: Consider the short exact sequence $0 \to E_{n-1} \to E_n \to E_n/E_{n-1} \to 0$. If the Harder-Narasimhan filtration exists, it follows easily from the definitions and lemma 1.5 that E_n/E_{n-1} has the following property:

1.7. **Definition.** *A maximal destabilizing quotient (mdq) is a quotient $E \twoheadrightarrow B$ such that for every other quotient $E \twoheadrightarrow B'$, we have $\mu(B') \geq \mu(B)$, and equality only if the map factors via $E \twoheadrightarrow B \twoheadrightarrow B'$.*

The mdq (if it exists) is obviously semistable and unique. If we can show it always exists, this would proof the theorem:

If $E = B$, then E is semistable and we are done, otherwise let $E' \hookrightarrow E \twoheadrightarrow B$ be the kernel. Since $\mu(E) > \mu(B)$, we have (see-saw!) $\mu(E') > \mu(E)$, hence the rank of E' is strictly smaller than the rank of E, and by induction (*) we can assume the existence of a HN-filtration for E'.

The HN-filtration of E' then extends to a HN filtration of E.

The complete proof (see e.g. [Bri02, section 2]) works in any category and for any slope function ϕ with the see-saw property and the following two properties:

(1) There is no infinite chain of subobjects

$$\ldots \hookrightarrow E_3 \hookrightarrow E_2 \hookrightarrow E_1 \hookrightarrow E_0$$

with $\phi(E_{i+1}) > \phi(E_i)$ for all i.

(2) There is no infinite chain of quotients

$$E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots$$

with $\phi(E_i) > \phi(E_{i+1})$ for all i.

In particular, the proof always works in a category of finite length.[2]

2. T-STRUCTURES

2.1. **Digression: Octahedral axioms.** There seem to exist two myths about the octahedral axiom in a triangulated category $\mathcal{D}$:

(1) It is not important.
(2) Since it is difficult to draw, it must be scary and difficult to understand and apply.

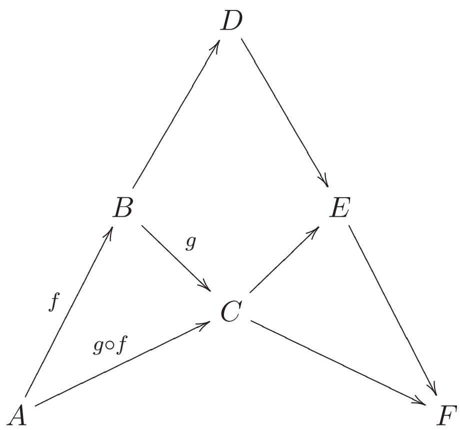
While the first one may be true to some extent, it definitely ceases to be true when dealing with t-structures; however, fortunately the second myth is definitely wrong.

The octahedral axiom answers a simple question: Given a composition

$$A \to^f B \to^g C,$$

is there any way to relate the three cones $\mathrm{cone}(f)$, $\mathrm{cone}(g)$ and $\mathrm{cone}(g \circ f)$? Phrased this way, the axiom is easy to guess: they form an exact triangle. More precisely, there is the following commutative diagram, where all the (almost) straight lines are part of an exact triangle:

[2] An abelian category has finite length if even arbitrary infinite chains of subobjects and quotients do not exist.



In the special case where $\mathcal{D} = D^b(\mathcal{A})$ is the derived category of an abelian category $\mathcal{A}$, and A, B, C are objects concentrated in degree zero (identified with objects in $\mathcal{A}$, and f, g are inclusions, then the octahedral axiom specializes to a very familiar statement:

$$(C/A)/(B/A) = C/B$$

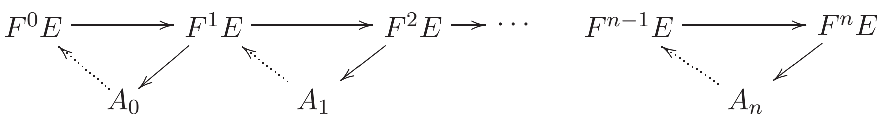
More generally, in the same spirit where exact triangles are the replacement of exact sequences, the use of the octahedral axiom replaces all abelian category proofs using diagram chasing, referring to the snake lemmma, lemma of 9 etc.

2.2. **Exercise.** *Translate the lemma of 9 to a triangulated category, and prove it!*

2.3. **Definition of a t-structure.** The notion of a t-structure can be motivated by the following question: Assuming we have an equivalence of derived categories $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$, can we understand the image of $\mathcal{A} = \mathcal{A}[0]$ in $D^b(\mathcal{B})$? For interesting examples (almost all Fourier-Mukai transforms, etc.) $\mathcal{A}$ does not get mapped to $\mathcal{B}$, so we would like to understand what structure the image of $\mathcal{A}$ in $D^b(\mathcal{B})$ satisfies.

2.4. **Definition.** *The heart of a bounded t-structure in a triangulated category $\mathcal{D}$ is a full additive subcategory such that*

(1) *For $k_1 > k_2$, we have $\mathrm{Hom}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$.*
(2) *For every object E in $\mathcal{D}$ there are integers $k_1 > k_2 > \cdots > k_n$ and a sequence of exact triangles*



with $A_i \in \mathcal{A}[k_i]$.

The concept of t-structures was introduced in [BBD82], which is required reading for anyone interested in details about t-structures.

2.5. **Remarks.**

(1) A bounded t-structure is uniquely determined by its heart, which allows us to omit the definition bounded t-structure in this note.
(2) $\mathcal{A}[0] \subset D^b(\mathcal{A})$ is the heart of a t-structure. Property (1) says that there are no Ext-groups in negative degree,[3] and property (2) is the filtration of a complex by its cohomology objects, induced by successive application of the truncation functor $\tau_{\geq n}$.
(3) The core $\mathcal{A}$ is automatically abelian: A morphism $A \to B$ between two objects in $\mathcal{A}$ is defined to be an inclusion if its cone is also in $\mathcal{A}$, and it is defined to be a surjection if the cone is in $\mathcal{A}[1]$.

2.6. **Exercise.** *Use the filtration with respect to the standard t-structure to show that for a smooth projective curve X, every object in $D^b(X)$ is the direct sum of its cohomology sheaves.*

The simplest examples of non-trivial t-structures are given by *tilting at a torsion pair*.

2.7. **Definition.** *A torsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories with*

(1) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(2) For all $E \in \mathcal{A}$ there exists a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}, F \in FF$.

Property (1) implies that that the filtration in (2) is automatically unique and factorial.

2.8. **Examples.** The canonical example of a torsion pair is $\mathcal{A} = \mathrm{Coh}\, X$, where we define $\mathcal{T}$ to be the torsion sheaves and $\mathcal{F}$ the torsion-free sheaves.

For a more interesting example, let $\mathcal{A} = \mathrm{Coh}\, X$ be the category of coherent sheaves on a smooth projective curve X, and $\mu \in \mathbb{R}$ a real number. Let $\mathcal{A}_{\geq \mu}$ be the subcategory generated by torsion sheaves and vector bundles all of whose HN-filtration quotients have slope $\geq \mu$, and $\mathcal{A}_{<\mu}$ the category of vector bundles all of whose filtration quotients have slope $< \mu$. Then $(\mathcal{A}_{\geq\mu}, \mathcal{A}_{<\mu})$ is a torsion pair: property (1) follows from lemma 1.5, and (2) is obtained by collapsing the HN-filtration into two parts: we let $T = E_i$ for i maximal such that $\mu_i \geq \mu$.

2.9. **Proposition.** *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following defines the heart of a bounded t-structure:*

$$\mathcal{A}^\sharp := \left\{ E \in D^b(\mathcal{A}) \,\middle|\, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \quad \textit{for } i \neq 0, -1 \right\}$$

[3]Note that it is *not* enough to obvserve that a morphism $A[0] \to B[-n]$ for some objects $A, B \in \mathcal{A}$ would induce the zero-morphism on cohomology. Any non-trivial element in $\mathrm{Ext}^n(A, B)$ for $n > 0$ yields a non-zero morphism $A \to B[n]$ in $D^b(\mathcal{A})$ that induces the zero morphism on cohomology.

Objects in $\mathcal{A}$ can be interpreted as a pair (T, F), $T \in \mathcal{T}, F \in \mathcal{F}$ and an element in $\mathrm{Ext}^1(T, F)$. Objects in $\mathcal{A}^\sharp$ are instead a pair (F, T) and an element in $\mathrm{Ext}^2(F, T)$.

2.10. **Exercise.** *Let $X = \mathbb{P}^1, \mathcal{A} = \mathrm{Coh}\, X$, and let $\mathcal{A}^\sharp$ be the tilted heart for the torsion pair $(\mathcal{A}_{\geq 0}, \mathcal{A}_{<0})$. Let Q be the Kronecker quiver (the directed quiver with tow vertices and two arrows), and let $\Phi_T \colon D^b(\mathbb{P}^1) \to \mathbb{D}^b(\mathrm{rep}_{\mathbb{C}}(Q))$ be the equivalence induced by the tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$. Show that $\mathcal{A}^\sharp$ is the inverse image of the heart of the standard t-structure.*

2.11. **Exercise.** *Consider an elliptic curve E, and its auto-equivalence $\Phi \colon D^b(E) \to D^b(E)$ given by the Fourier-Mukai transform of the Poincaré line bundle. Determine the image $\Phi(\mathrm{Coh}\, E)$ of the heart of the standard t-structure.*

3. Stability conditions on a triangulated category

3.1. **Definition.** *A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying*

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$
(2) For all $\phi_1 > \phi_2$ we have $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$.
(3) For each $0 \neq E \in \mathcal{D}$ there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers and a sequence of exact triangles

(1)
$$\begin{array}{ccccccccc} F^0E & \longrightarrow & F^1E & \longrightarrow & F^2E & \longrightarrow \cdots & F^{n-1}E & \longrightarrow & F^nE \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & & & \nwarrow \swarrow & \\ & A_0 & & A_1 & & & & A_n & \end{array}$$

with $A_i \in \mathcal{P}(\phi_i)$ (which we call the Harder-Narasimhan filtration of E).

3.2. **Remarks.**

(1) We call the objects in $\mathcal{P}(\phi)$ the semistable objects of phase ϕ.
(2) Given the slicing $\mathcal{P}$, the sequence of ϕ_i and the Harder-Narasimhan filtration are automatically unique. We set $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n$ (where we sometimes omit the subscript $\mathcal{P}$).
(3) If $\phi^-(A) > \phi^+(B)$, the $\mathrm{Hom}(A, B) = 0$.
(4) If $\mathcal{P}(\phi) \neq 0$ only for $\phi \in \mathbb{Z}$, then the slicing is equivalent to the datum of a bounded t-structure, with heart $\mathcal{A} = \mathcal{P}(0)$.
(5) More generally, given a slicing $\mathcal{P}$, let $\mathcal{A} = \mathcal{P}((0, 1])$ be the full extension-closed[4] subcategory generated by all $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$; equivalently, $\mathcal{A}$ is the subcategory of objects E with $\phi^+_{\mathcal{P}}(E) \leq 1$ and $\phi^-_{\mathcal{P}}(E) > 0$. Then $\mathcal{A}$ is the heart of a bounded t-structure. In other words, a slicing is a refinement of a bounded t-structure.

3.3. **Exercise.** *Prove the claim in (5).*

[4] An extension of two objects A, B in a triangulated category is any object E that fits into an exact triangle $A \to E \to B \to^{[1]}$.

While this gives a notion of semistable objects and successfully generalizes Harder-Narasimhan filtrations, it is rather unsatisfying that we have to specify the semistable objects explicitly (instead of defining them implicitly by a slope function as in the case of vector bundles). The remedy for this lies in the following (somewhat surprising) definition:

3.4. **Definition.** *A stability condition on a triangulated category $\mathcal{D}$ is a pair $(Z, \mathcal{P})$ where $Z\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism (called the stability function or central charge) and $\mathcal{P}$ is a slicing, so that for every $0 \neq E \in \mathcal{P}(\phi)$ we have $Z(E) = m(E) \cdot e^{i\pi\phi}$ for some $m(E) \in \mathbb{R}_{>0}$.*

3.5. **Lemma.** *To give a stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded t-structure and a group homomorphism $Z_{\mathcal{A}}\colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z_{\mathcal{A}}([A]) \in \mathbb{H} = \{z \in \mathbb{C}^* \mid 0 < \arg(z) \leq 1\}$ for all objects $A \in \mathcal{A}$, and such that $Z_{\mathcal{A}}$ "has the Harder-Narasimhan property".*

If $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, then $K(\mathcal{D}) = K(\mathcal{A})$ (even though $\mathcal{D}$ might not be equivalent to $D^b(\mathcal{A})$), so it is clear how to go from Z to $Z_{\mathcal{A}}$ and vice versa. Given the stability condition, we set $\mathcal{A} = \mathcal{P}((0, 1])$ as before; by definition of a stability condition, any $\mathcal{P}$-semistable object is sent to $\mathbb{H}$ by Z; since any object in $\mathcal{A}$ is an extension of semistable ones, this is true for all objects in $\mathcal{A}$. Then one can show that the Z-semistable objects in $\mathcal{A}$ are exactly the semistable objects with respect to $\mathcal{P}$.

Conversely, given $\mathcal{A}$ and $Z_{\mathcal{A}}$, we can define $\mathcal{P}(\phi)$ for $0 < \phi \leq 1$ to be the subcategory of $Z_{\mathcal{A}}$-semistable objects in $\mathcal{A}$ of phase ϕ.

3.6. **Example.** If X is a smooth projective curve and $\mathcal{D} = \mathcal{D}^b(X)$, let $\mathcal{A} = \operatorname{Coh} X$ be the heart of the standard t-structure, and $Z(E) = -\deg(E) + i \cdot \operatorname{rk}(E)$. Then Z is a stability function with the Harder-Narasimhan property, and thus induces a stability condition on $\mathbb{D}^b(X)$.

The same construction does not work for higher-dimensional varieties.

3.7. **Example.** Let Q be a quiver with relations R, such that its path algebra is finite-dimensional. Let $\mathcal{A} = \operatorname{Mod}(Q, R)$ be its category of representations. Then $K(\mathcal{A}) = K(D^b(\mathcal{A})) = \bigoplus_{Q_0} \mathbb{Z}$, so a stability function for $\mathcal{A}$ is just given by a complex number $z_q \in \mathbb{H}$ for every vertex $q \in Q_0$ of the quiver. (Given an object in $\mathcal{A}$, its class in the K-group is a *non-negative* linear combination of the classes of the one-dimensional simple representations associated to the vertices, so its image under Z will also lie in the upper half plane.) Since $\mathcal{A}$ has finite length, Z automatically has the Harder-Narasimhan property.

3.8. **Remark.** The condition that Z sends objects of $\mathcal{A}$ to the upper half plane is highly non-trivial. Already for a projective surface S, there is no central charge $K(D^b(S)) \to \mathbb{C}$ that would map objects in $\operatorname{Coh} S$ to the upper half plane.

4. SPACE OF STABILITY CONDITIONS

Given a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ and an object $E \in \mathcal{D}$ with semistable Harder-Narasimhan filtration quotients A_i we define its *mass* with respect to σ to be $m_\sigma(E) = \sum_i |Z(A_i)|$.

We can define a generalized metric on the set of stability conditions on E:

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma^2}(E) - \phi^-_{\sigma^1}(E)|, |\phi^+_{\sigma^2}(E) - \phi^+_{\sigma^1}(E)|, |\log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)}| \right\} \in [0, +\infty]$$

From now on, we assume for simplicity either that

(1) $K(\mathcal{D})$ is finite-dimensional, or
(2) assume that the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$[5] is finite-dimensional, and restrict our attention to stability conditions for which the central charge $Z \colon K(\mathcal{D})$ factors via $\mathcal{N}(\mathcal{D})$.

So in either case, Z is just a linear map from a finite-dimensional vector space to $\mathbb{C}$.

For technical reason, we need to exclude some degenerate stability conditions:

4.1. **Definition.** *A stability conditions $\sigma = (Z, \mathcal{P})$ is called locally finite if there exists $\epsilon > 0$ such that $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is a category of finite length for all $\phi \in \mathbb{R}$.*

Let $\mathrm{Stab}(\mathcal{D})$ be the space of locally finite stability conditions on $\mathcal{D}$ with the topology generated by the generalized metric d above. Set $K = K(\mathcal{D})$ of $K = \mathcal{N}(\mathcal{D})$ accordingly.

4.2. **Theorem.** *The space $\mathrm{Stab}(\mathcal{D})$ of (numerical) stability conditions is a smooth finite-dimensional manifold such that the map*

$$\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}) \to K^\vee, \quad \sigma = (Z, \mathcal{P}) \to Z$$

is a local chart at every point of $\mathrm{Stab}(\mathcal{D})$.[6]

In other words, we can deform a stability condition $(Z, \mathcal{P})$ (uniquely) by deforming Z.

Given $(Z, \mathcal{P})$ and Z' "nearby" Z, we will explain how to determine $\mathcal{P}'$ "nearby" $\mathcal{P}$ such that $(Z', \mathcal{P}')$ is again a stability condition. Given $\phi \in \mathbb{R}$, consider the category $\mathcal{A}_\epsilon = \mathcal{P}((\phi - \epsilon, \phi + \epsilon))$. The central charge Z maps objects in $\mathcal{A}_\epsilon$ to the small sector $\mathbb{R}_{>0} \cdot e^{i\pi \cdot (\phi - \epsilon, \phi + \epsilon)}$. We assume Z' is nearby Z, hence Z' maps $\mathcal{A}_\epsilon$ to some slightly bigger sector in $\mathbb{C}$; it will still be small enough that we can define the phases $\phi_{Z'}(A)$ of objects $A \in \mathcal{A}_\epsilon$ with respect to Z'.

However, since $\mathcal{A}_\epsilon$ is usually not abelian, we need to be a little more careful to define stability with respect to Z': we say $i \colon A \to E \in \mathcal{A}_\epsilon$ is a strict inclusion if

[5] The numerical Grothendieck group is the quotient of $K(\mathcal{D})$ by the null-space of the bilinear form $\chi(E, F) = \chi(\mathbf{R}\mathrm{Hom}(E, F))$.

[6] The precise statement is that there is a subspace U_Σ of $K^\vee$, only depending on the connected component Σ of $\mathrm{Stab}(\mathcal{D})$, such that $\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}) \to U_\Sigma$ is a local homeomorphism. However, when the image of the integral lattice in K under Z is a discrete subset of $\mathbb{C}$ for any Z in the connected component, then $U_\Sigma = K^\vee$.

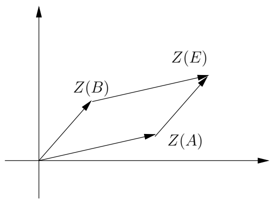
the cone is also in $\mathcal{A}_\epsilon$. We say E is Z'-stable if there is no strict inclusion $A \to E$ with $\phi_{Z'}(A) > \phi_{Z'}(E)$.

Then we define $\mathcal{P}'(\phi')$ (for $\phi' \approx \phi$) to be the subcategory of Z'-semistable objects of phase ϕ' with respect to Z'.

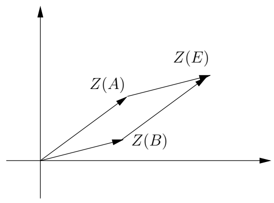
4.3. **Example.** Assume that in $\mathcal{A}_\epsilon$ there is an exact triangle $A \to E \to B$, such that A, B have no strict subobjects in $\mathcal{A}_\epsilon$, and A is the only strict subobject of E. In particular, A, B are stable for $\sigma = (Z, \mathcal{P})$, and will also be stable for $\sigma' = (Z', \mathcal{P}')$.

(1) If the parallelogram $0, Z([A]), Z(E), Z(B)$ has positive orientation, then E is stable.
(2) If the parallelogram has negative orientation, then E is unstable, and $0 \to A \to E$ is the HN filtration of E.

So if the orientation of the parallelogram changes between Z and Z', then E changes from being stable to unstable, or vice versa.



(a) E is stable



(b) E is unstable

Figure 2: A simple wall-crossing

4.4. **Example:** $D^b(\mathbb{P}^1)$**.** Consider the heart $\mathcal{A}^\sharp$ obtained by tilting at the torsion pair $(\mathcal{A}_{\geq 0}, \mathcal{A}_{<0})$. It is generated by $\mathcal{O}(0)$ and $\mathcal{O}(-1)[1]$ and extensions. The short exact sequences $\mathcal{O}(k) \to \mathcal{O}(k+1) \to \mathcal{O}_x$ for $k \in \mathbb{Z}$ give, after appropriate rotation, extensions in $\mathcal{A}^\sharp$ showing that $\mathcal{O}_x$ for $x \in \mathbb{P}^1$ (and thus all torsion sheaves), $\mathcal{O}(n)$ for $n \geq 0$ and $\mathcal{O}(n)[1]$ for $n < 0$ are all objects in $\mathcal{A}^\sharp$. All other objects in $\mathcal{A}^\sharp$ are decomposable. By example 3.7 and exercise 2.10, any choice of $(z_0, z_{-1}) \in \mathbb{H}^2$ gives a stability condition with $Z(\mathcal{O}) = z_0$ and $Z(\mathcal{O}(-1)[1]) = z_{-1}$.

(1) If $\arg(z_0) < \arg(z_{-1})$, then all the (shifts of) line bundles listed above are stable. Up to reparametrization of $\mathbb{C}$ by an element in $\mathrm{GL}_2(\mathbb{R})$ (and accordingly adjusting the phases of stable objects), this stability condition is equivalent to the standard one given in example 3.6. See fig. 3, which shows the images of the stable objects, with arrows denoting inclusions.
(2) If $\arg(z_0) > \arg(z_{-1})$, then $\mathcal{O}, \mathcal{O}(-1)[1]$ are the only stable objects in $\mathcal{A}^\sharp$.

This describes a chamber of the space of stability conditions, and the natural question is what happens when we deform Z so that one of z_0, z_{-1} leaves the upper half-plane.

If we are in case (2) and, say, z_{-1} passes the positive real line, then the stable objects don't change; however, the new heart $\mathcal{A}' = \mathcal{P}((0, 1])$ is generated by the

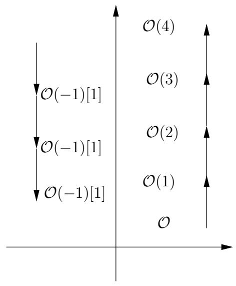


Figure 3: Stable objects in case (1)

stable objects $\mathcal{O}$ and $\mathcal{O}(-1)[2]$. There are no extensions or morphisms between these two objects, so $\mathcal{A}'$ is isomorphic to the category of pairs of vector spaces (representations of the algebra $\mathbb{C} \oplus \mathbb{C}$). This is the easiest example of a heart $\mathcal{A}'$ of a bounded t-structure whose bounded derived category $D^b(\mathcal{A}')$ is not isomorphic to the original derived category.

The most interesting case is when z_{-1} lies on the negative real line, and z_0 passes the positive real axis. Also let us assume that $\Re(z_{-1}) > -\Re(z_0)$. We have to consider the category $\mathcal{A}_\epsilon = \mathcal{P}((-\epsilon, \epsilon))$, where ϵ is small but big compared to the phase of z_0. The stable objects in this interval are

(1) The skyscraper sheaves $\mathcal{O}_x$,
(2) all $\mathcal{O}(k)$ such that $k\Re(z_0) + (k-1)\Re(z_{-1}) > 0$, i.e. all $k \geq k_0 := \lceil \frac{\Re z_{-1}}{\Re(z_0 + z_{-1})} \rceil$, and
(3) all $\mathcal{O}(k)[1]$ for $k < k_0$.

When z_{-1} passes through the real axis, then the strict inclusions $\mathcal{O}(k_0) \hookrightarrow \mathcal{O}(k_0 + 1) \hookrightarrow \dots$ will destabilize all but $\mathcal{O}(k_0)$ (see also fig. 4); similarly the strict inclusions $\mathcal{O}_x \hookrightarrow \mathcal{O}(k)[1] \hookrightarrow \mathcal{O}(k+1)[1]$ for $k \leq k_0 - 2$ will destabilize all but $\mathcal{O}(k_0 - 1)[1]$; finally $\mathcal{O}(k_0) \hookrightarrow \mathcal{O}_x \twoheadrightarrow \mathcal{O}(k_0 - 1)[1]$ will destabilize the skyscraper sheaves.

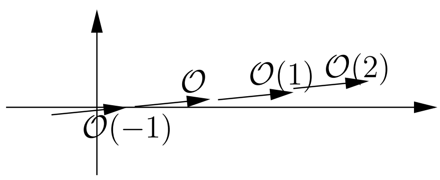


Figure 4: Inclusions in $\mathcal{P}((-\epsilon, \epsilon))$.

Hence the only "surviving" stable objects are $\mathcal{O}(k_0), \mathcal{O}(k_0 - 1)[1]$, and we get a stability condition of case (2) for the heart generated by these two objects.

For a much more detailed study of the space of stability conditions on $D^b(\mathbb{P}^1)$, see [Oka06].

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E-mail address: `bayer@math.utah.edu`

AREND BAYER, UNIVERSITY OF UTAH