University of Utah, Department of Mathematics
Fall 2010, Algebra Qualifying Exam

Show all your work, and provide reasonable proofs/justification. You may attempt as many problems as you wish. Four correct solutions count as a pass; eight half-correct solutions may not!

1. Determine the number of 5-Sylow subgroups of $\mathrm{SL}_2(\mathbb{F}_5)$.

2. Let G be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices of the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$. Is G solvable?

3. Consider the automorphisms σ, τ of $\mathbb{Q}(x)$ with $\sigma\colon x \longmapsto 1/x$ and $\tau\colon x \longmapsto 1-x$. What is the order of the group generated by these two elements? Determine the group.

4. Set $R = \mathbb{Q}[x]$, and consider the submodule M of R^2 generated by the elements $(1-2x,\ -x^2)$ and $(1-x,\ x-x^2)$. Express R^2/M as a direct sum of cyclic modules.

5. Recall that a Hermitian matrix is a complex matrix which equals its conjugate transpose. Determine the conjugacy classes of 5×5 Hermitian matrices A satisfying $A^5 + 2A^3 + 3A = 6I$.

6. Determine the number of conjugacy classes of 4×4 complex matrices satisfying $A^3 - 2A^2 + A = 0$.

7. Let α be the positive real root of $x^6 - 7$. What is the number of elements of the ring $\mathbb{Z}[\alpha]/(\alpha^2)$? Is every ideal in this ring principal?

8. Find the degree of the splitting field of $x^6 - 3$ over (i) $\mathbb{Q}(\sqrt{-3})$ and (ii) $\mathbb{F}_5$.

9. Prove that $x^4 + 1$ is reducible over any field of positive characteristic.

10. For p a prime, determine the Galois group of $x^p - 2$ over $\mathbb{Q}$. What is its order? Is it abelian?