# University of Utah, Department of Mathematics
# Fall 2012, Algebra Qualifying Exam

*Show all your work and provide reasonable proofs/justification. You may attempt as many problems as you wish. Four correct solutions count as a pass; eight half-correct solutions may not!*

(1) Suppose $G$ is a group acting transitively on a finite set $S$, $|S| \geq 2$. Prove that there exists an element $\sigma \in G$ such that $\sigma(s) \neq s$, for all $s \in S$.

(2) Let $S^1$ be the circle group, i.e., the group of all complex numbers of norm 1 with multiplication. If $A$ is a finite abelian group, define the dual group $\widehat{A}$ to be the multiplicative group of all group homomorphisms $A \to S^1$. Prove that $A \cong \widehat{A}$.

(3) Let $R$ be a commutative ring with 1 and $M_n(R)$ the ring of $n \times n$ matrices with coefficients in $R$. Prove that every ideal of $M_n(R)$ is of the form $M_n(I)$, for some ideal $I$ of $R$.

(4) Let $M$ be a $5 \times 5$ matrix with real entries. Suppose $M$ has finite order and $\det(M - I_5) \neq 0$. Find $\det(M)$.

(5) Let $\phi$ denote the Frobenius map $x \mapsto x^p$ on the finite field $\mathbb{F}_{p^n}$. Determine the Jordan canonical form (over $\overline{\mathbb{F}}_p$) for $\phi$ regarded as an $\mathbb{F}_p$-linear transformation of $\mathbb{F}_{p^n}$.

(6) Determine the splitting field and the Galois group for the polynomial $x^3 - 2$ over $\mathbb{Q}$.

(7) Show that the polynomial $x^4 + 1$ is reducible modulo every prime $p$.

(8) Let $K \subseteq L$ be fields, and let $f(x)$ be an irreducible polynomial in $K[x]$. If there exists $a$ in $L$ with $f(a) = 0 = f(a^2)$, prove that $f(x)$ splits in $L[x]$.

(9) Prove that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.

(10) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ as left $\mathbb{Q}$-modules.