University of Utah, Department of Mathematics
Fall 2011, Algebra Qualifying Exam

Show all your work, and provide reasonable proofs or justification. You may attempt as many problems as you wish. Four correct solutions count as a pass; eight half-correct solutions may not!

1. Determine, up to isomorphism, the groups of order 154.

2. Let P be a p-Sylow subgroup of a group G, and N a normal subgroup of G. Prove that $P \cap N$ is a p-Sylow subgroup of N.

3. Determine the number of 2×2 nilpotent matrices over a finite field $\mathbb{F}_q$.

4. Determine, up to conjugacy, all 4×4 matrices M over $\mathbb{Q}$ with $M^5 = -M^3$ and $M^3 \neq 0$.

5. Let M be an $n \times n$ matrix over $\mathbb{Q}$ in which each entry is 1. What is the Jordan form of M?

6. Let $f(x)$ be a monic polynomial with integer coefficients, such that $f(\alpha) = 0 = f(2\alpha)$ for some complex number α. Prove that $f(0) \neq 1$.

7. Let R be a commutative ring with 1.

   (a) If a maximal ideal $\mathfrak{m}$ of R is principal, prove that there is no ideal I with $\mathfrak{m}^2 \subsetneq I \subsetneq \mathfrak{m}$.

   (b) Give an example where $\mathfrak{m}$ is maximal, but there exists an ideal I with $\mathfrak{m}^2 \subsetneq I \subsetneq \mathfrak{m}$.

8. Let α be a complex root of $x^6 + 3$. Set $K = \mathbb{Q}(\alpha)$.

   (a) Prove that K contains a primitive 6-th root of unity.

   (b) Compute the Galois group of K over $\mathbb{Q}$.

9. Let ζ be a primitive 16-th root of unity over a field K. Determine $[K(\zeta) : K]$ when K is:

   (a) $\mathbb{F}_7$ (b) $\mathbb{F}_9$ (c) $\mathbb{F}_{17}$

10. Explain (preferably in a sentence!) why $\mathbb{F}_3[x]/(x^2 - 2)$ and $\mathbb{F}_3[x]/(x^2 - 2x - 1)$ are isomorphic. Then construct an explicit isomorphism.