PRELIMINARY EXAMINATION IN ALGEBRA

January 7, 2010

Instructions: Answer as many questions or parts of questions as you wish. A passing score consists of four complete answers or a reasonable equivalent.

1. Show that for any positive integer n, every element of order 2 in the alternating group A_n is the square of an element of order 4 in the symmetric group S_n.
2. Let G be a finite p-group, with $|G| > p$. Prove that the order of $\mathsf{Aut}(G)$ is divisible by p.
3. Let R be a ring with 1. A left R-module M is called *simple* if $M \neq 0$ and if the only submodules of M are M and 0. Show that every simple module is isomorphic to R/I for some maximal left ideal I and that I is unique if R is commutative.
4. In the category of $\mathbb{Z}$-modules, is the module $\mathbb{Q}/\mathbb{Z}$ (a) projective? (b) injective? (c) flat?

   Justify your answer.
5. Let G be a group of order p^2q, where p and q are distinct primes. Show that G has a normal Sylow subgroup.
6. Let M be a 5 by 5 matrix with real coefficients such that $M^2 = 2M - I$. Show that the subspace of $\mathbb{R}^5$ consisting of vectors fixed by M has dimension at least 3.
7. Let R be a commutative ring with 1. Show that every R-module is free if and only if R is a field.
8. Compute the number of monic irreducible polynomials of degree 3 over the field $\mathbb{Z}_7$.
9. Let F be a field that contains a primitive nth root of unity. Show that if a is an element of F and the field E is obtained from F by adjoining an nth root of a, then E is a Galois extension of F with cyclic Galois group.
10. State and prove Hilbert's basis theorem.