University of Utah, Department of Mathematics
January 2014, Algebra Qualifying Exam

Show all your work, and provide reasonable proofs or justification. You may attempt as many problems as you wish. ***Five*** *correct solutions count as a pass; ten half-correct solutions may not!*

1. Let G be a group of order p^3, where p is a prime. Suppose G is *not* abelian. Show that the center Z of G is isomorphic to $\mathbb{Z}/(p)$ and that $G/Z \cong \mathbb{Z}/(p) \times \mathbb{Z}/(p)$.

2. What is the number of elements of order 11 in a simple group of order 660?

3. Let M be the cokernel of the map $\mathbb{Z}^2 \longrightarrow \mathbb{Z}^3$ given by

$$\begin{pmatrix} 3 & 6 \\ 4 & 10 \\ 10 & 22 \end{pmatrix}.$$

   Write M as a direct sum of cyclic groups.

4. Let M be an $n \times n$ matrix with entries from a field $\mathbb{F}$ such that $M^3 = I$. Is M necessarily diagonalizable if (a) $\mathbb{F} = \mathbb{Q}(\sqrt{3})$, (b) $\mathbb{F} = \mathbb{Q}(i)$, (c) $\mathbb{F} = \mathbb{F}_3$?

5. Determine all 3×3 matrices M with entries from $\mathbb{Q}$ such that $M^8 = I$ and $M^4 \neq I$.

6. Find all positive integers n such that $\cos(2\pi/n)$ is rational.

7. Is the polynomial $x^8 + x + 1$ irreducible in $\mathbb{F}_2[x]$?

8. Which of the following is a principal ideal domain? (a) $\mathbb{Z}[i]$, (b) $\mathbb{Z}[2\sqrt{2}]$, (c) $\mathbb{Q}[3\sqrt{3}]$.

9. Let $\mathbb{F}$ be a field with $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{C}$ such that $[\mathbb{F} : \mathbb{Q}]$ is odd.

   (a) If $\mathbb{F}/\mathbb{Q}$ is Galois, prove that $\mathbb{F}$ is contained in $\mathbb{R}$.

   (b) Find an extension with $[\mathbb{F} : \mathbb{Q}] = 3$ such that $\mathbb{F}$ is not contained in $\mathbb{R}$.

10. Prove that each element of a finite field can be written as a sum of two squares.