University of Utah, Department of Mathematics
January 2015, Algebra Qualifying Exam

Show all your work and provide reasonable justification. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Let σ be a 5-cycle in S_5. How many elements of S_5 commute with σ? How many elements of the subgroup A_5 commute with σ? How many conjugacy classes of 5 cycles are there in A_5?

2. Prove that, up to isomorphism, there are at most four groups of order 30.

3. Let H be a proper subgroup of a **finite** group G. Prove that $\bigcup_{g \in G} gHg^{-1}$ does not equal G.

4. Set $G = \mathrm{GL}_2(\mathbb{C})$. Construct a proper subgroup H of G such that $\bigcup_{g \in G} gHg^{-1}$ equals G.

5. Determine, up to conjugacy, the elements of order 4 in $\mathrm{GL}_3(\mathbb{Q})$.

6. Let M be 3×3 matrix over $\mathbb{C}$ with $\mathrm{trace}(M^k) = 0$ for $k = 1, 2, 3$. Prove that M is nilpotent.

7. Consider the polynomial ring $\mathbb{Q}[x, y]$ where x, y are indeterminates over $\mathbb{Q}$. Determine a finite generating set for the ideal of polynomials $f(x, y)$ with $f(i, i) = 0$.

8. Suppose $K \subset L \subset M$ are fields with $[M : L] = 2 = [L : K]$. Prove that $M = K(\alpha)$, where α is a root of an irreducible polynomial in $K[x]$ of the form $x^4 + bx^2 + c$.

9. Prove that $\mathbb{Q}(\sqrt{5 + \sqrt{5}})$ is Galois over $\mathbb{Q}$, and compute the Galois group.

10. Let K be a field of characteristic $p > 0$, and let t be an indeterminate. Consider the automorphism $\sigma \in \mathrm{Aut}_K K(t)$ with $\sigma(t) = t + 1$. Determine the subfield of $K(t)$ that is fixed by σ.