University of Utah, Department of Mathematics
May 2017, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Prove that $a^{561} \equiv a \bmod 561$ for each integer a.

2. Determine the integers n for which there exists a surjective homomorphism of symmetric groups $S_{n+1} \longrightarrow S_n$.

3. Suppose G is a finite group with $\operatorname{Aut} G$ solvable. Prove that G is solvable.

4. Suppose a finite group $G \neq \{e\}$ has c conjugacy classes. Prove that G contains an element other than the identity of order at most c.

5. Let $R = \mathbb{Q}[x]$, and let M be the cokernel of the map

$$R^2 \xrightarrow{\begin{pmatrix} x+1 & 0 \\ 2x & x^2 \\ 1 & 1 \end{pmatrix}} R^3.$$

   Write M as a direct sum of cyclic R-modules.

6. Determine, up to conjugacy, all *real* 3×3 matrices A satisfying $A^8 = I$ and $A^4 \neq I$.

7. Let p be a prime number and n a positive integer. How many elements α are in $\mathbb{F}_{p^n}$ such that $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^6}$?

8. Let k be a field and $k(x)$ the field of rational functions over k, in the indeterminate x. Consider the automorphisms σ and τ of $k(x)/k$ defined by

$$\sigma(x) = \frac{1}{1-x} \quad \text{and} \quad \tau(x) = \frac{1}{x}.$$

   (a) Prove that the subgroup $G = \langle \sigma, \tau \rangle$ of $\operatorname{Aut}(k(x)/k)$ is isomorphic to S_3.

   (b) Prove that the fixed subfield under the action of G equals $k(t)$, where

$$t = \frac{(x^2-x+1)^3}{x^2(x-1)^2}$$

9. Let n a positive integer, and let $\mathbf{x} := x_1, \ldots, x_n$ be indeterminates over $\mathbb{Q}$. For each positive integer i set

$$\alpha_i = x_1^i + \cdots + x_n^i$$

   (a) Clearly $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Q}(s_1, \ldots, s_n)$, the field of symmetric functions in the $\mathbf{x}$. What is the degree of this extension?

   (b) Prove the characteristic polynomial of an $n \times n$ matrix A over $\mathbb{Q}$ is completely determined by the elements $\operatorname{trace}(A), \ldots, \operatorname{trace}(A^n)$.

10. Let $E := \mathbb{Q}(\zeta + \zeta^{-1})$ where $\zeta \in \mathbb{C}$ is a primitive nth root of 1, for some integer $n \geqslant 3$. Determine the Galois group of E over $\mathbb{Q}$.