UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Applied Mathematics

This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three from each part will be graded.

All problems are worth 10 points and a passing score is 40.

August 13, 2019.

Part A.

A1. Let (u_n) and (v_n) be two orthonormal sequences in a Hilbert space H. Assume that

$$\sum_{n=1}^{\infty} \|u_n - v_n\| < 1.$$

Show that if (u_n) is a total orthonormal sequence, then (v_n) is also a total orthonormal sequence.

Hint: Assume for contradiction that (v_n) is not total.

A2. Let (e_n) be a total orthonormal sequence in a Hilbert space H. Let $(x_n) \subset H$ be a sequence. Prove that the two following statements are equivalent.

(i) $x_n \to x$ **weakly**.

(ii) The sequence (x_n) is bounded and for any $j \geq 1$, $\langle x_n, e_j \rangle \to \langle x, e_j \rangle$ as $n \to \infty$.

Hint: Show that if (ii) holds and $y_N \in \operatorname{span}\{e_1, \dots, e_N\}$ then $\langle x_n, y_N \rangle \to \langle x, y_N \rangle$ as $n \to \infty$.

A3. Consider the nonlinear integral equation

$$u(x) - \lambda \int_a^b w(x,y) \sin(u(y)) dy = \phi(x),$$

where $\lambda \in \mathbb{R}$, ϕ is continuous on the interval $[a,b]$ and $w(x,y)$ is continuous on $[a,b] \times [a,b]$.

Find a constant $C > 0$ such that $|\lambda| < C$ implies that the nonlinear integral equation has a unique solution u.

Hint: You may assume the inequality $|\sin(x) - \sin(y)| \leq |x - y|$, for any $x, y \in \mathbb{R}$.

A4. Consider the left shift operator $L : \ell^2 \to \ell^2$, defined for a sequence $(x_n) \in \ell^2$ by

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

(a) Prove that $\|L\| = 1$.

(b) Explain why if $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, then $\lambda \in \rho(L)$, the resolvent set of L.

(c) Prove that that if $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then $\lambda \in \sigma_p(L)$, the point spectrum of L.

(d) Deduce what is $\sigma(L)$, the spectrum of L.

A5. Let L be a partial differential operator of the form:

$$L = \sum_{j=0}^{m-1} c_j \partial^j,$$

where the c_j are arbitrary constants. Show that $x^m(L\delta) = 0$ as distributions in $\mathscr{D}'(\mathbb{R})$. Here $\delta \in \mathscr{D}'(\mathbb{R})$ is the usual "delta function".

Part B.

B1. If a complex function is differentiable and has constant modulus, show that the function itself is constant.

B2. Let $f(z)$ be an entire function such that there exists a constant M, an $R > 0$ and an integer $n \geq 1$ with $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that $f(z)$ is a polynomial of degree at most n.

B3. (a) Find the Laurent expansion of $f(z) = \dfrac{z}{(z+i)(z-2)}$ for $1 < |z| < 2$.

(b) Classify the singularities of the function $f(z) = \dfrac{z+1}{z\sin(z)}$.

B4. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(x+1)^2 + 4} dx.$$

You must clearly show the contour of integration and treat integration over each contour separately.

B5. Find the leading non constant behavior as $x \to 0^+$ of

$$\int_1^{\infty} \cos(xt) t^{-1} dt.$$

You must express the constant term in terms of known quantities and/or constants.