PhD Preliminary Qualifying Examination
Applied Mathematics
August 21, 2006

Instructions: This exam is in two parts. Do three problems from each of the two parts. Indicate clearly which questions you wish to have graded, as no more than three questions from each part will be graded.

Part A.

A.1 (a) State the Riesz Representation Theorem (a proof is not required).

(b) Use the Riesz Representation theorem to prove that a bounded linear operator on a Hilbert Space has an adjoint. Prove that the adjoint is a bounded linear operator on the Hilbert space.

(c) Find the adjoint for the operator

$$(Ku)(y) = \int_0^1 \min(x, y)u(x)dx.$$

A.2 (a) What is the weak formulation for the boundary value problem $u'' = f(x)$ subject to boundary conditions $u(0) = u(1) = 0$?

(b) Let $\phi(x) = N_2(x + 1)$ where $N_2(x)$ is the piecewise linear function

$$N_2(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Divide the unit interval $[0, 1]$ interval N uniform subintervals with endpoints at $x_k = \frac{k}{N}$, $k = 0, \ldots, N$. Use $\phi(x)$ to define a finite element basis for continuous piecewise linear functions on the unit interval which interpolates function values at x_k. Suppose $u_k = u(x_k)$ is known. Express $u(x)$ in terms of $\phi(x)$.

(c) Use the Galerkin method to find equations for $u_k = u(x_k)$ that approximately solve the boundary value problem in part (a), using the piecewise linear finite element basis.

A.3 (a) Find all conditions on α, β, γ, and $f(x)$ for which solutions of

$$u'' + \alpha^2 u = f(x), \qquad u(0) = \beta, \qquad \alpha u'(0) - u'(1) = \gamma$$

exist.

(b) Find an integral representation for the solution in the case that $\alpha = 0$.

A.4 The n data points $(x_i, y_i), i = 1, 2, \ldots, n$ are believed to lie on an exponential curve $y_i = A\exp(\lambda x_i)$. Formulate a *linear* regression problem to estimate λ.

A.5 The Legendre polynomials are given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$, and are eigenfunctions of the differential operator $LU = -\left((1 - x^2)u'\right)'$ on the interval $-1 < x < 1$.

(a) Find the eigenvalue λ_n corresponding to the eigenfunction $P_n(x)$.

(b) In what sense, if any, are the functions $\{P_n(x)\}$ complete?

(c) Prove or disprove that L is a positive definite operator.

Part B.

1. Evaluate the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

Explain your reasoning.

2. Solve the Laplace equation $\Delta\phi = 0$ subject to the boundary conditions:

$$\begin{aligned} \phi &= a \quad \text{on} \quad \{z : |z| = 2, z \neq 2\} \\ \phi &= b \quad \text{on} \quad \{z : |z-1| = 1, z \neq 2\} \end{aligned}$$

(a and b are real constants; $z = x + iy$).

3. Formulate and prove the Fundamental theorem of Algebra.

4. Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha dx}{(x-1)^2 + 1}$$

(where α is a constant, such that the integral converges: $|\alpha| < 1$).

5. Find the two-term asymptotic expansion of the integral

$$I(s) = \int_0^3 \frac{e^{s(-t^2+4t-3)}}{5+t} dt, \qquad s \text{ is real, and } s \to +\infty.$$

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

August, 2005

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to be graded.

Part A.

A.1 Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{r\times m}$, and $b \in \mathbb{R}^n$ be given. Assume $R(B^*) = N(A)$. For fixed $\mu \geq 0$, consider the problem

$$\min_{x\in\mathbb{R}^m} \|Ax - b\|_2^2 + \mu\|Bx\|_2^2.$$

(a) Find the necessary conditions (analogous to the normal equations for the standard least-squares problem) and prove that the necessary conditions have a solution when $\mu = 0$.

(b) Prove that the necessary conditions have a unique solution if $\mu > 0$.

A.2 Consider the distribution g defined by $\langle g, \phi\rangle = \int_{-\infty}^{\infty} e^{-|x|}\phi(x)\,dx$, for all test functions ϕ.

(a) Calculate the second derivative g'' explicitly. Prove g'' is a distribution of order zero.

(b) Find a differential equation for which g is a Green's function.

A.3 Let $K : H \to H$ be a compact linear operator on the infinite-dimensional Hilbert space H.

(a) Prove that K^{-1}, if it exists, is not bounded.

(b) Given $T > 0$, let $H = L^2(0, T)$. For $u \in H$, define

$$(Ku)(t) = \int_0^T e^{-st}u(s)\,ds, \quad 0 \leq t \leq T.$$

Prove that $K : H \to H$ is a compact operator. What are the consequences of this fact for the utility of the Laplace transform as a numerical method?

A.4 Consider the fourth-order differential equation

$$\begin{aligned} u^{(4)} &= f, \qquad \text{on } (0,1), \\ u(0) &= u(1), \quad u'(0) = u'(1), \\ u''(1) &= -2, \quad u^{(3)}(1) = 1. \end{aligned}$$

(a) Find the least restrictive constraint(s) on $f \in L^2(0,1)$ under which a solution to the problem exists.

(b) With fixed f chosen such that a solution exists, is the solution unique? If the solution is unique, prove it. Otherwise provide a counterexample.

A.5 Prove that the nonlinear integral equation

$$u(x) - \int_0^1 \frac{u^2(y)}{4 + x^2 - y}\, dy = \frac{x^3}{3}, \qquad 0 \le x \le 1,$$

has exactly one continuous solution u satisfying $\max_{0 \le x \le 1} |u(x)| \le 1$. Describe, but do not carry out, an iterative procedure for obtaining an approximation to this solution.

Part B.

B.1 Use contour integration to show that

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2}dx = \frac{\pi\alpha}{\sin \pi\alpha}$$

where $0 < \alpha < 1$.

B.2 Consider the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}, \quad 0 \le x \le l$$

with the following boundary conditions

$$\phi(x,0) = 0, \quad \frac{\partial \phi}{\partial t}(x,0) = 0$$

$$\phi(0,t) = 0, \quad \phi(l,t) = f(t)$$

(a) Show that the Laplace transform of the solution $\widehat{\Phi}(x,s)$ is given by

$$\widehat{\Phi}(x,s) = \frac{\widehat{F}(s)\sinh sx}{\sinh sl}$$

where $\widehat{F}(s)$ is the Laplace transform of $f(t)$.

(b) Suppose that $\phi_0(x,t)$ is the solution when $f(t) = 1$. Show that the solution for general f is given by

$$\phi(x,t) = \int_0^t \frac{\partial \phi_0}{\partial t'}(x,t')f(t-t')dt'$$

B.3 Discuss the flow pattern associated with the complex potential

$$\Omega(z) = Q_0 z + \frac{\bar{Q}_0 a^2}{z} + \frac{i\gamma}{2\pi}\log z$$

with $a > 0$, γ real and $Q_0 = U_0 + iV_0$. Sketch the flow when $\gamma = 0$. Given that the complex force on a cylindrical obstacle with boundary C is

$$\bar{F} = \frac{1}{2}i\rho \int_C \left(\frac{d\Omega}{dz}\right)^2 dz,$$

where ρ is the density of fluid, determine the lift when $\gamma \neq 0$.

B.4 (a) Introducing the discrete Fourier transform $U(k,t) = \sum_{n=-\infty}^{\infty} e^{ikn} u_n(t)$, $0 \le k < 2\pi$, show that the general solution to the discretized heat equation

$$\frac{du_n}{dt} = \frac{1}{h^2}\left[u_{n+1} - 2u_n + u_{n-1}\right], \quad -\infty < n < \infty$$

can be written in the form

$$u_n(t) = \int_0^{2\pi} U(k,0) e^{-\epsilon(k)t} e^{-ikn} \frac{dk}{2\pi}$$

and calculate the function $\epsilon(k)$.

(b) Using the fact that $U(k,t)$ and $u_n(t)$ form a transform pair for fixed t, prove the identity $\sum_{n=-\infty}^{\infty} e^{i(k-k')n} = 2\pi\delta(k-k')$. Hence, find $u_n(t)$ given the initial data $u_n(0) = \sin(2\pi n/p)$ for some fixed integer p.

B.5 Suppose that $\psi_+(z)$ is analytic on the upper half plane and $\psi_-(z)$ is analytic on the lower half plane such that on the real axis

$$\psi_+(x) - \psi_-(x) = f(x)$$

Assuming that the integrable function $f(x)$ can be analytically extended in a neighborhood of the real axis, show that a solution to this problem is given by the Cauchy integral

$$\Psi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z} d\xi$$

with $\psi_\pm(x) = \lim_{\epsilon\to 0^+} \Psi(x \pm i\epsilon)$. Hence, establish that

$$\psi_+(x) + \psi_-(x) = \frac{1}{\pi i}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi$$

where PV denotes the principal value.

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

August 16th, 2004

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to be graded.

Part A.

1. The dilation equation is

$$\phi(x) = \sum_k c_k \phi(2x - k)$$

and the associated wavelet is

$$W(x) = \sum_k (-1)^k c_{1-k} \phi(2x - k)$$

(a) Show that the function

$$N_2(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

satisfies the dilation equation and determine the values of c_k? Construct the associated wavelet.

(b) Prove that if the functions $\{\phi(x - k)\}_k$ form an orthonormal set then the functions $\{2^{j/2} W(2^j x - k)\}_k$ form an orthonormal set for each fixed integer j.

2. Consider a linear chain of $2N$ atoms with the same spacing a and force constant β but of two different masses m, M $M > m$, placed alternately. Label the light atoms by even integers $2n, n = 0, ..., N - 1$ and the heavy atoms by odd integers $2n - 1, n = 1, ..., N$. Denoting their displacements from equilibrium by the variables U_{2n} and V_{2n-1}, Newton's law of motion gives

$$m\ddot{U}_{2n} = \beta \left[V_{2n-1} + V_{2n+1} - 2U_{2n}\right]$$
$$M\ddot{V}_{2n-1} = \beta \left[U_{2n} + U_{2n-2} - 2V_{2n-1}\right]$$

Determine the modes of vibration of the system and sketch the resulting pair of dispersion curves. [Assume periodic boundary conditions $U_0 = U_{2N}$ and $V_1 = V_{2N+1}$].

3. Find the eigenvalues and eigenfunctions of the integral operator

$$(Ku)(x) = \int_0^1 k(x,y)u(y)dy$$

where

$$k(x,y) = \min\{x,y\}$$

Solve the corresponding integral equation

$$u(x) - \lambda \int_0^1 k(x,y)u(y)dy = f(x)$$

using the eigenfunctions of K assuming that $\lambda \neq 1/a_n^2$ for any integer n, where $a_n = (2n+1)\pi/2$ (i.e. determine the resolvent operator). Write down the Fredholm alternative condition for f when $\lambda = 1/a_n^2$.

4. (a) Prove that the eigenvalues of a self-adjoint linear operator acting in a Hilbert space H are real and that the eigenfunctions of distinct eigenvalues are orthogonal.

   (b) Prove that if L is a bounded linear operator in H then the adjoint L^* is also a bounded linear operator.

5. Construct the Green's function for the equation

$$u''(x) + \alpha^2 u(x) = f(x), \; u(0) = u(1), \; u'(0) = u'(1)$$

and express the solution in terms of the Green's function. For what values of α does the Green's function fail to exist?

Part B.

1. Show that

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x} dx = \frac{1}{2} \sec\left(\frac{a}{2}\right), \quad |a| < \pi$$

Use a rectangular contour with corners at $\pm R$ and $\pm R + i$.

2. Obtain the inverse Laplace transform of the function

$$\widehat{F}(s) = s^{-a}, \quad 0 < a < 1$$

by evaluating the associated Bromwich contour integral with a branch cut along the negative real axis. Express the answer in terms of the gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$

3. Consider the flow around a thin airfoil of length $2a$ that is inclined at an angle α to the x–axis. Assume that there is uniform flow at ∞ with velocity $U_x = U, U_y = 0$.

(a) Show that the conformal map

$$z = \frac{ae^{-i\alpha}}{2}\left(\omega + \frac{1}{\omega}\right)$$

maps the unit circle in the ω–plane to the airfoil in the z–plane.

(b) Consider the complex potential

$$\Gamma(\omega) = \frac{aU}{2}\left(e^{-i\alpha}\omega + \frac{e^{i\alpha}}{\omega}\right) - \frac{i\gamma}{2\pi}\log\omega$$

Calculate the corresponding potential $\Omega(z) = \Gamma(\omega(z))$ where $\omega(z)$ is the inverse of the conformal map in part (a). What does the term in γ represent?

(c) Calculate $\Omega'(z)$ and show that it satisfies the correct asymptotic flow. Determine what happens at the end points of the airfoil, and use this to calculate γ.

4. Find the solution of Laplace's equation on the circular domain $0 \leq r \leq R$, $0 \leq \theta < 2\pi$, with Dirichlet boundary data $u(R, \theta) = f(\theta)$.

5. Liouville's theorem states that a function analytic and bounded in the complex plane is necessarily a constant. Prove this theorem. Using it prove the Fundamental theorem of algebra: every polynomial $P(z)$ which is not a constant has at least one complex root.

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

January 2004

Instructions: Answer three questions from part A and three questions From part B. Indicate clearly which questions you wish to be graded.

Part A.

A.1 (a) Prove that $x\delta(x) = 0$ in the sense of distribution.

(b) Formulate the differential equation $xu' - u = 0$ as a problem in the sense of distribution and find its general solution, also in the sense of distribution.

A.2 The n data points $(x_i, y_i), i = 1, 2, \ldots, n$ are believed to lie on an exponential curve $y_i = A\exp(\lambda x_i)$. Estimate λ.

A.3 The Chebyshev functions are given by $T_n(x) = \cos(n\cos^{-1} x)$, and are eigenfunctions of the differential operator $LU = -\sqrt{1-x^2}\left(\sqrt{1-x^2}u'\right)'$ on the interval $-1 < x < 1$.

(a) Find the eigenvalue λ_n corresponding to the eigenfunction $T_n(x)$.

(b) In what sense, if any, are the functions $\{T_n(x)\}$ complete?

(c) Prove or disprove that L is a positive definite operator.

A.4 The "best" least squares solution of the boundary value problem $u'' + u = 0$, $u(0) = \alpha$, $u(\pi) = 0$ is the unique exact solution of what problem?

A.5 Prove that if L is a bounded linear operator in a Hilbert Space H, then the adjoint operator L^* exists.

Part B.

B.1 (a) Using Jordan's lemma show that

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx = \frac{\pi}{2}e^{-a}$$

(b) Show that

$$\int_0^{2\pi} \frac{d\theta}{A + B\sin\theta} = \frac{2\pi}{\sqrt{A^2 - B^2}}$$

where $A^2 > B^2, A > 0$.

B.2 Use contour integration to show that

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2} = \frac{\pi\alpha}{\sin \pi\alpha}$$

where $|\text{Re}\ \alpha| < 1$.

B.3 Discuss the flow pattern around a circular obstacle associated with the complex potential

$$\Omega(z) = u_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi}\log z$$

Show that $r = a$ is a streamline. Determine the asymptotic velocity when $z \to \infty$. Calculate the stagnation points. Sketch the flow when $\gamma = 0$. Use the Blasius formula

$$F_x - iF_y = \frac{1}{2}i\rho\int_C \left(\frac{d\Omega}{dz}\right)^2 dz$$

to determine the lift on the obstacle when $\gamma \neq 0$.

B.4 Use Fourier transforms to show that the solution of the diffusion equation

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}, \qquad -\infty < x < \infty$$

with initial data $\phi(x,0) = h(x)$ takes the form

$$\phi(x,t) = \int_{-\infty}^{\infty} g(x - x', t)h(x')dx'$$

where

$$g(x,t) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}}$$

Determine $\lim_{t\to 0} g(x,t)$.

B.5 Consider the bilinear transformation

$$\omega = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

(a) Show that this transformation maps circles to circles.

(b) Show that the bilinear transformations form a group

(c) In the case $(a, b, c, d) = (1, -i, 1, i)$ show that the upper-half plane is mapped on to the unit disc.

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

August, 2003

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

A.1 Suppose that A is an $n \times n$ matrix with n distinct eigenvalues.

(a) How are the eigenvalues of A and A^* related?

(b) Show that the eigenvectors of A and A^* form a biorthogonal set, that is, if $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are the eigenvectors of A and A^*, appropriately ordered, then $\langle \phi_i, \psi_j \rangle = 0$ if $i \neq j$.

(c) Establish that $\langle \phi_i, \psi_i \rangle \neq 0$.

A.2 (a) Determine the conditions on $f(x)$, α and β for which there solutions of

$$\frac{d^2u}{dx^2} = f(x), \qquad u(0) = \alpha, \qquad u'(1) - u(1) = \beta.$$

(b) Find the "best" solution of

$$\frac{d^2u}{dx^2} = 1, \qquad u(0) = 1, \qquad u'(1) - u(1) = 1.$$

A.3 Consider the non-local differential operator

$$Lu = Du'' - u(g(x))$$

with boundary conditions $u(0) = u(1) = 0$ and where $g(x)$ is a monotone increasing function with $g(0) = 0$, $g(1) = 1$, and $g(x) < x$ for $0 < x < 1$.

(a) Find the adjoint of the operator L.

(b) Prove that the operator L is invertible.

(c) Devise an iteration scheme that could be used to solve $Lu = f(x)$. Under what conditions on D is this iteration scheme a contraction mapping?

A.4 Find the weak formulation for the boundary value problem

$$u'' - \delta(x)u = 1, \qquad u(-1) = u(1) = 0, \tag{0.1}$$

and then solve the problem. Verify that your solution is a weak solution. Why is it not a strong solution?

A.5 Use piecewise linear finite element basis functions with uniform grid size h and the Galerkin method to find an approximate representation for the differential equation $a\frac{d^2u}{dx^2} + b\frac{du}{dx} + u(x) = f(x)$, with $u(0) = 0$, $u(1) = 0$, on the interval $[0, 1]$. (It is only necessary to find the approximating equations, not to solve them.)

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

August, 2003

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

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(a) How are the eigenvalues of A and A^* related?

(b) Show that the eigenvectors of A and A^* form a biorthogonal set, that is, if $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are the eigenvectors of A and A^*, appropriately ordered, then $\langle \phi_i, \psi_j \rangle = 0$ if $i \neq j$.

(c) Establish that $\langle \phi_i, \psi_i \rangle \neq 0$.

A.2 (a) Determine the conditions on $f(x)$, α and β for which there solutions of

$$\frac{d^2u}{dx^2} = f(x), \qquad u(0) = \alpha, \qquad u'(1) - u(1) = \beta.$$

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$$Lu = Du'' - u(g(x))$$

with boundary conditions $u(0) = u(1) = 0$ and where $g(x)$ is a monotone increasing function with $g(0) = 0$, $g(1) = 1$, and $g(x) < x$ for $0 < x < 1$.

(a) Find the adjoint of the operator L.

(b) Prove that the operator L is invertible.

(c) Devise an iteration scheme that could be used to solve $Lu = f(x)$. Under what conditions on D is this iteration scheme a contraction mapping?

Part B.

B.1 (a) Evaluate the following real integrals by contour integration:

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$$

(b) Let

$$e^{t/2(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Show from the definition of Laurent series and properties of complex integration that the Bessel function

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - t\sin\theta)d\theta$$

B.2 Consider the contour integral

$$J = \int_C \frac{(z^2-1)^{1/2}}{1+z^2} dz$$

with the contour C shown in Fig. 1. By calculating the various contributions to J and using the residue theorem, show that

$$I := \int_{-1}^{1} \frac{(1-x^2)^{1/2}}{1+x^2} dx = \pi(\sqrt{2}-1)$$

The branch of the multivalued function $(z^2-1)^{1/2}$ should be fixed by choosing polar ccordinates $z = 1+\rho_1 e^{i\phi_i}$ and $z = -1+\rho_2 e^{i\phi_2}$ with $0 \le \phi_1, \phi_2 \le 2\pi$.

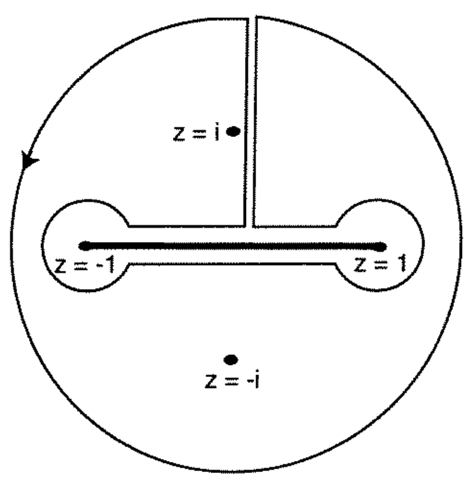

Figure 1: Contour C of question B.2

B.3 Discuss the flow pattern around a circular obstacle associated with the complex potential

$$\Omega(z) = u_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi}\log z$$

Show that $r = a$ is a streamline. Determine the asymptotic velocity when $z \to \infty$. Calculate the stagnation points. Sketch the flow when $\gamma = 0$. Use the Blasius formula

$$F_x - iF_y = \frac{1}{2}i\rho\int_C \left(\frac{d\Omega}{dz}\right)^2 dz$$

to determine the lift on the obstacle when $\gamma \neq 0$.

B.4 Consider the wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}, \quad 0 \leq x \leq l$$

with the following boundary conditions

$$\phi(x,0) = 0, \quad \frac{\partial\phi}{\partial t}(x,0) = 0$$

$$\phi(0,t) = 0, \quad \phi(l,t) = 1$$

(a) Show that the Laplace transform of the solution $\widehat{\Phi}(x,s)$ is given by

$$\widehat{\Phi}(x,s) = \frac{\sinh sx}{s\sinh sl}$$

(b) Obtain the solution $\phi(x,t)$ by inverting the Laplace transform using a Bromich integral and show that it can be expressed as the infinite series

$$\phi(x,t) = \frac{x}{l} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi t}{l}\right)$$

B.5 Consider the bilinear transformation

$$\omega = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

(a) Show that this transformation maps circles to circles.

(b) Show that the bilinear transformations form a group

(c) In the case $(a,b,c,d) = (1,-i,1,i)$ show that the upper–half plane is mapped on to the unit disc.

Part B.

B.1 (a) Evaluate the following real integrals by contour integration:

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$$

(b) Let

$$e^{t/2(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Show from the definition of Laurent series and properties of complex integration that the Bessel function

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - t\sin\theta)d\theta$$

B.2 Consider the contour integral

$$J = \int_C \frac{(z^2-1)^{1/2}}{1+z^2}dz$$

with the contour C shown in Fig. 1. By calculating the various contributions to J and using the residue theorem, show that

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The branch of the multivalued function $(z^2-1)^{1/2}$ should be fixed by choosing polar ccordinates $z = 1+\rho_1 e^{i\phi_i}$ and $z = -1+\rho_2 e^{i\phi_2}$ with $0 \le \phi_1, \phi_2 \le 2\pi$.

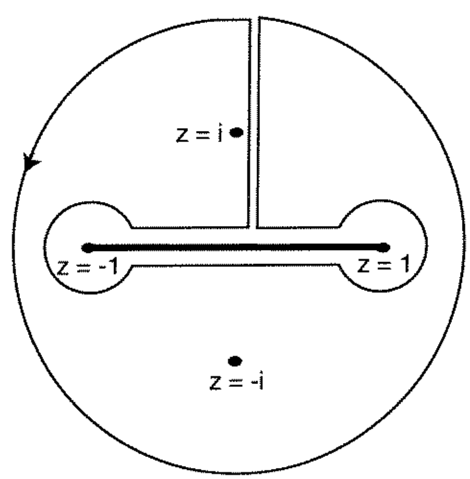

Figure 1: Contour C of question B.2

PhD Preliminary Qualifying Examination: Applied Mathematics (6710/20)

August 12th, 2002

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to be graded.

Part A.

1. Consider the error function

$$E(\mathbf{w}) = \sum_{p=1}^{P} (y_p - \mathbf{w}.\mathbf{x}_p)^2$$

where $\mathbf{w}$ and $\mathbf{x}_p$ are N-component vectors, $N \geq P$, and $y_p, \mathbf{x}_p$ are prescribed.

(a) Show that if $\mathbf{w}_0$ is a minimizer of E then

$$[\mathbf{X}^T\mathbf{X}]\mathbf{w}_0 = \mathbf{X}^T\mathbf{y}$$

where $X_{pj} = [\mathbf{x}_p]_j$, $j = 1, \ldots N$, $\mathbf{X}^T$ is the transpose of the matrix $\mathbf{X}$, and $\mathbf{y} = (y_1, \ldots y_P)^T$.

(b) Suppose that the weight vector $\mathbf{w}$ is updated at each time step m using the gradient decent algorithm

$$w_j(m+1) = w_j(m) - \eta \frac{\partial E(\mathbf{w}(m))}{\partial w_j(m)}$$

Show that $\mathbf{w}(m) \to \mathbf{w}_0$ as $m \to \infty$ provided that the rate parameter η satisfies

$$|1 - \eta\lambda_j| < 1, \quad j = 1, \ldots, N$$

where λ_j are the eigenvalues of the symmetric $N \times N$ matrix $\mathbf{X}^T\mathbf{X}$. [Hint: Explicitly calculate $\partial E/\partial w_j$, use part (a) to rewrite the difference equation in terms of $\Delta\mathbf{w}(m) = \mathbf{w}(m) - \mathbf{w}_0$, and then diagonalize. Assume that $\mathbf{X}^T\mathbf{X}$ is invertible so that $\mathbf{w}_0$ is unique].

2. Consider a linear chain of $2N$ atoms with the same spacing a and force constant β but of two different masses m, M $M > m$, placed alternately. Label the light atoms by even integers $2n, n = 0, ..., N-1$ and the heavy atoms by odd integers $2n-1, n = 1, ..., N$. Denoting their displacements from equilibrium by the variables U_{2n} and V_{2n-1}, Newton's law of motion gives

$$m\ddot{U}_{2n} = \beta\left[V_{2n-1} + V_{2n+1} - 2U_{2n}\right]$$
$$M\ddot{V}_{2n-1} = \beta\left[U_{2n} + U_{2n-2} - 2V_{2n-1}\right]$$

Determine the modes of vibration of the system and sketch the resulting pair of dispersion curves. [Assume periodic boundary conditions $U_0 = U_{2N}$ and $V_1 = V_{2N+1}$].

3. Find the eigenvalues and eigenfunctions of the integral operator

$$(Ku)(x) = \int_0^1 k(x,y)u(y)dy$$

where

$$k(x,y) = \begin{cases} x(1-y) & 0 \le x < y \le 1 \\ y(1-x) & 0 \le y < x \le 1 \end{cases}$$

Solve the corresponding integral equation

$$u(x) - \lambda \int_0^1 k(x,y)u(y)dy = f(x)$$

using the eigenfunctions of K assuming that $\lambda \neq n^2\pi^2$ for any integer n (i.e. determine the resolvent operator). Write down the Fredholm alternative condition for f when $\lambda = m^2\pi^2$ for some integer m.

4. Suppose that the real scaling function ϕ satisfies the dilation equation

$$\phi(x) = \sum_k c_k \phi(2x - k)$$

Also assume that $\{\phi(x-k), k \in \mathbf{Z}\}$ forms an orthonormal set with respect to the inner product $\langle u, v\rangle = \int_{-\infty}^{\infty} u(x)v(x)dx$. Define the corresponding wavelet function according to

$$\psi(x) = \sum_k (-1)^k c_{1-k}\phi(2x - k)$$

(a) Prove that

$$\sum_{k \in \mathbf{Z}} c_k c_{k-2p} = 2\delta_{0,p}$$

(b) Show that $\{2^{j/2}\psi(2^j x - k), k \in \mathbf{Z}\}$ forms an orthonormal set.

(c) Briefly explain signal decomposition and reconstruction within the framework of multiresolution analysis.

5. Consider the differential operator $Lu = u'' + k^2 u$, $k \neq 0$, subject to homogeneous boundary conditions $u(0) = u(1) = 0$. Show by direct calculation that the associated Green's

function is

$$g(x,y) = \begin{cases} \dfrac{\sin kx \sin k(y-1)}{k \sin k} & 0 \le x < y \le 1 \\ \\ \dfrac{\sin ky \sin k(x-1)}{k \sin k} & 0 \le y < x \le 1 \end{cases}$$

provided $\sin k \neq 0$. Using the fact that L is self-adjoint, show that the solution to the equation

$$u''(x) + k^2 u(x) = f(x)$$

with *inhomogeneous* boundary data $u(0) = \alpha,\ u(1) = \beta$ is

$$u(x) = \int_0^1 g(x,y) f(y) dy - \frac{\alpha}{\sin k} \sin k(x-1) + \frac{\beta}{\sin k} \sin kx$$

provided $\sin k \neq 0$. What happens when $k = m\pi$?

Part B.

1. (a) Using Jordan's lemma show that

$$\int_0^\infty \frac{\cos ax}{1+x^2} dx = \frac{\pi}{2} e^{-a}$$

(b) Show that

$$\int_0^{2\pi} \frac{d\theta}{A + B\sin\theta} = \frac{2\pi}{\sqrt{A^2 - B^2}}$$

where $A^2 > B^2, A > 0$.

2. Use contour integration to show that

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2} = \frac{\pi\alpha}{\sin \pi\alpha}$$

where $|\text{Re } \alpha| < 1$.

3. Discuss the flow pattern around a circular obstacle associated with the complex potential

$$\Omega(z) = u_0 \left(z + \frac{a^2}{z} \right) + \frac{i\gamma}{2\pi} \log z$$

Show that $r = a$ is a streamline. Determine the asymptotic velocity when $z \to \infty$. Calculate the stagnation points. Sketch the flow when $\gamma = 0$. Use the Blasius formula

$$F_x - iF_y = \frac{1}{2} i\rho \int_C \left(\frac{d\Omega}{dz} \right)^2 dz$$

to determine the lift on the obstacle when $\gamma \neq 0$.

4. Use Fourier transforms to show that the solution of the diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}, \qquad -\infty < x < \infty$$

with initial data $\phi(x, 0) = h(x)$ takes the form

$$\phi(x, t) = \int_{-\infty}^{\infty} g(x - x', t) h(x') dx'$$

where

$$g(x, t) = \frac{\mathrm{e}^{-x^2/4Dt}}{2\sqrt{\pi D t}}$$

Determine $\lim_{t \to 0} g(x, t)$.

5. Show that the conformal transformation

$$w = \frac{z - a}{z + a}, \qquad a = \sqrt{c^2 - \rho^2}$$

where $0 < \rho < c$, maps the domain bounded by the circle $|z - c| = \rho$ and the imaginary axis onto an annular region centered about the origin with outer radius $|\omega| = 1$ and inner radius $|\omega| = \delta$. Find the radius δ.

PhD Preliminary Qualifying Examination
Applied Mathematics
August 13, 2001

Instructions: This exam is in two parts. Do three problems from each of the two parts. Indicate clearly which questions you wish to have graded, as no more than three questions from each part will be graded.

Part A.

1. (a) Describe the Gram-Schmidt orthogonalization procedure for a set of n-vectors $\{u_1, u_2, \ldots, u_k\}$.

   (b) Suppose $k < n$ and that the vectors $\{u_1, u_2, \ldots, u_k\}$ are linearly independent. Show that the Gram-Schmidt procedure is equivalent to factoring the matrix U with column vectors $\{u_1, u_2, \ldots, u_k\}$ as $U = QR$ where R is triangular. What is the structure of Q?

   (c) Use this factorization of U to find the least-squares solution of $Ux = b$.

2. The dilation equation is

$$\phi(x) = \sum_k c_k \phi(2x - k).$$

   Show that the function

$$N_2(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   satisfies the dilation equation. What are the values of c_k? Are the wavelets formed from $N_2(x)$ an orthogonal set? Why or why not?

3. (a) State the Riesz Representation Theorem.

   (b) Use the Riesz Representation theorem to prove that a bounded linear operator on a Hilbert Space has an adjoint. Prove that the adjoint is a bounded linear operator on the Hilbert space.

   (c) Find the adjoint for the operator

$$(Ku)(y) = \int_0^1 \min(x, y) u(x) dx.$$

4. (a) What is the weak formulation for the boundary value problem $u'' = f(x)$ subject to boundary conditions $u(0) = u(1) = 0$?

(b) Let $\phi(x) = N_2(x+1)$ where $N_2(x)$ is the piecewise linear function given in Problem A.2. Divide the unit interval $[0,1]$ interval N uniform subintervals with endpoints at $x_k = \frac{k}{N}$, $k = 0, \ldots, N$. Use $\phi(x)$ to define a finite element basis for continuous piecewise linear functions on the unit interval which interpolates function values at x_k. Suppose $u_k = u(x_k)$ is known. Express $u(x)$ in terms of $\phi(x)$.

(c) Use the Galerkin method to find equations for $u_k = u(x_k)$ that approximately solve the boundary value problem in part (a), using the piecewise linear finite element basis.

5. (a) Find all conditions on α, β, γ, and $f(x)$ for which solutions of

$$u'' + \alpha^2 u = f(x), \qquad u(0) = \beta, \qquad \alpha u'(0) - u'(1) = \gamma$$

exist.

(b) Find an integral representation for the solution in the case that $\alpha = 0$.

Part B.

1. Evaluate $\int_0^\infty \frac{\sqrt{x}}{x^2+1} dx$.

2. Remark: For this problem you may use without proof that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu x} d\mu = \delta(x)$.

   (a) Suppose $f(x)$ has the Fourier transform $F(\mu)$. Evaluate

   $$\int_{-\infty}^{\infty} f(x)\overline{f(x-k)}dx$$

   in terms of $F(\mu)$.

   (b) Use this identity to prove that the set $\{\text{sinc}(t-k)\}_{k=-\infty}^{\infty}$ forms an orthonormal set.

3. (a) Derive the "diffusion" or "heat" equation in several spatial dimensions.

   (b) In the derivation of this equation, it is routinely assumed that $\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega \frac{\partial u}{\partial t} dx$. Under what conditions is this interchange of order valid? Prove your result. Hint: Examine the function $w(t) = \int_0^t \left(\int_\Omega \frac{\partial u(x,t)}{\partial t} dx\right) dt$.

4. Find an integral representation for the solution of the telegrapher's equation

   $$u_{tt} + au_t + bu = u_{xx}$$

   with $u(0,t) = f(t)$ and $u_x(0,t) = g(t)$ specified for all time.

5. (a) Find the leading order behavior of

   $$G(x) = \int_0^\infty \cos\left(x\left(\frac{t^3}{3} - t\right)\right) dt$$

   for large x.

   (b) Outline the method to find more terms in the asymptotic expansion of $G(x)$.

   Useful information: $\int_{-\infty}^{\infty} \cos(ax^2)dx = \int_{-\infty}^{\infty} \sin(ax^2)dx = \sqrt{\frac{\pi}{2a}}$.

Applied Math Qualifying Exam
August 15, 2000

For each of the following two sections, do three out of five problems.

Section A

A.1 (a) State the Singular Value Decomposition Theorem.

(b) For any $n \times m$ real matrices A and B, show the $Tr(AB^*) = Tr(BA^*)$.

(c) Show that $||A|| = \sqrt{Tr(AA^*)}$ is a norm.

(d) Suppose A and B are $n \times m$ real matrices. Find the orthogonal matrix Q that minimizes $||QA - B||$ using the norm of part c.

A.2 Suppose the function $\phi(x)$ is Lipschitz continuous on the unit interval $0 < x < 2\pi$ and that $\lim_{x\to 0}\phi(x) = \phi(0^+)$ exists. Suppose $\int_0^\infty \frac{\sin x}{x}dx = \alpha$ is known. Evaluate $\lim_{k\to\infty}\int_0^{2\pi}\phi(x)\frac{\sin kx}{x}dx$ (with k restricted to be integer-valued).

A.3 (a) Show that the functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ form an orthogonal set in $L^2[0, 2\pi]$.

(b) Suppose $f(x) \in L^2[0, 2\pi]$. Let $\alpha_n = \int_0^{2\pi} f(x)e^{inx}dx$. Prove that $\{\alpha_n\}_{n=-\infty}^{\infty}$ is convergent in l^2 and also that $\sum_{n=-\infty}^{\infty}\alpha_n e^{inx}$ is convergent in $L^2[0, 2\pi]$. What does $\sum_{n=-\infty}^{\infty}\alpha_n e^{inx}$ converge to?

(c) Suppose $f(x)$ and $g(x)$ are 2π periodic functions and square integrable on the interval $0 < x < 2\pi$. State and prove the Convolution Theorem for these functions, that is, find the Fourier coefficients of the function $h(x) = \int_0^{2\pi} f(x-t)g(t)dt$ in terms of the Fourier coefficients of $f(x)$ and $g(x)$.

A.4 Under what conditions does a solution of the boundary value problem $u'' + u = f(x)$ with $u(0) = \alpha, u'(\frac{\pi}{2}) = \beta$ exist?

A.5 Find the representation of the delta function and the appropriate transform pair for the differential operator $Lu = -u''$ on the interval $0 < x < \pi$ subject to boundary conditions $u'(0) = 0, u(\pi) = 0$.

Section B

B.1 Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.

B.2 (a) Find the Fourier transform of $\frac{\sin x}{x}$.

(b) Suppose that the Fourier transform of the function $f(t)$ is 0 for frequencies whose absolute value is greater than π/h . Find the representation of $f(t)$ in terms of sinc functions.

B.3 Approximate the integral

$$I(k) = \int_C \frac{e^{k(z-2)^2}}{z-1} dz \qquad (1)$$

for large k, where the contour C is the vertical imaginery axis from $z = -i\infty$ to $z = i\infty$.

B.4 Solve the discretized heat equation $u_{n_t} = \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1})$ with $-\infty < n < \infty$ and initial data $u_n(t=0) = \sin \frac{2\pi n}{k}$ where k is a fixed integer.

B.5 Suppose Ω is some closed, bounded domain in two spatial dimensions.

(a) Under what conditions does the equation $\nabla^2 \phi = 0$ in Ω, $\mathbf{n} \cdot \nabla \phi = f$ on $\partial\Omega$, have a solution? Is it unique?

(b) Find an integral equation for ϕ on the boundary $\partial\Omega$. Be as explicit as possible.

(c) Under what conditions does this integral equation have a solution?

August 17, 1999

Department of Mathematics
University of Utah

Written Qualifying Examination in

Applied Mathematics

Instructions: The examination has two parts consisting of six problems each. You need to solve three problems from Part A and three problems from Part B. If you work on more than six problems, then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the qualifying examination your overall score must be at least 60%

Applied Math Qualifying Exam Part A

(Solve three of the six problems for full credit on Part A.)

Problem A1.

Construct the Green function for the equation

$$-u'' + u = f(x) \ \ (-\infty \leq x \leq +\infty), \ \ u(x) \to 0 \text{ as } x \to \pm\infty.$$

Problem A2.

Solve the differential equation in the sense of distributions

$$xu' = \delta(x^4 - 1), \quad x \in [-2, 2], \quad u(-2) = u(2) = 0.$$

Discuss why *two* boundary conditions are provided for the *first* order differential equation.

Problem A3.

Suppose that T is a transformation of a complete metric space X, and T^n (for a certain positive integer n) is a contraction with fixed point a. Show that

(1) a is a fixed point of T;

(2) T has only one fixed point;

(3) the sequence of successive iterations $u_0, \ u_1 = Tu_0, \ u_2 = Tu_1, \ \ldots$ tends to a for any initial element $u_0 \in X$.

Problem A4.

Let C be a closed convex set in a Hilbert space H, and let u be an arbitrary vector in H. Show that there exists one and only one element in C which is closest to u.

Problem A5.

Prove the following statement. In an arbitrary real Hilbert space H an orthonormal system $(\phi_1, \phi_2, \ldots, \phi_n, \ldots)$ is complete if and only if the equality

$$||f||^2 = \sum_{n=1}^{\infty} < f, \phi_n >^2$$

holds for any $f \in H$.

Problem A6.

For a function $f(x)$ "sufficiently smooth" and "sufficiently fast decaying" as $x \to \pm\infty$, the Fourier transform of f is defined as $g(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$. Suppose that the function $f(x)$ is normalized such that its L_2-norm is 1. Let

$$E_x = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx, \quad E_k = \int_{-\infty}^{\infty} k^2 |g(k)|^2 dk$$

Prove and interpret the inequality

$$E_x\, E_k \geq \frac{1}{4}.$$

(Hint: Apply Schwartz's inequality to $xf(x)f'(x)$; use Plancherel's equality.)

Applied Math Qualifying Exam Part B

(Solve three of the six problems for full credit on Part B.)

Problem B1.

Evaluate the integral

$$I = \int_0^\infty \frac{\sin x}{x} dx.$$

Problem B2.

Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha}{x^2+1} dx$$

where α is a parameter, $-1 < \alpha < 1$.

Problem B3.

Find a conformal transformation of the domain D between the unit circle $\Gamma_1 = \{z = x + iy : |z| = 1\}$ and the vertical line $\Gamma_2 = \{z = x + iy : \ x = 5/3\}$ onto a domain between two concentric circles.

Problem B4.

Liouville's theorem states that a function analytic and bounded in the complex plane is necessarily a constant.

Prove this theorem. Using it prove the Fundamental theorem of algebra: Every polynomial $P(z)$ which is not a constant has at least one complex root.

Problem B5.

Find the bounded solution $u(x, y)$ of the Laplace equation in the sector

$$D = \{(r, \theta) : 0 < \theta < \frac{3\pi}{4}\} \quad (x + iy = z = re^{i\theta})$$

satisfying the boundary conditions

$$u = 0 \text{ when } \theta = 0, \quad u = 5 \text{ when } \theta = \frac{3\pi}{4}.$$

Problem B6.

Find the leading behavior of the integral

$$I(x) = \int_0^{10} \frac{e^{-xt}}{1 + t\log t} dt \qquad \text{as} \quad x \to +\infty.$$

Explain your rationale.

Department of Mathematics
University of Utah

Written Qualifying Examination in

Applied Mathematics

Instructions: The examination has three parts consisting of four problems each. You are to solve two problems from Part A, two problems from Part B, two problems from Part C. If you work on more than six problems, then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the qualifying examination your overall score must be at least 60%

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Applied Math Qualifying Exam Part A

Solve two of the four problems for full credit on Part A.

Problem A.1.

Prove the following statement. In an arbitrary real Hilbert space H an orthonormal system $(\phi_1, \phi_2, \ldots, \phi_n, \ldots)$ is complete if and only if the equality

$$||f||^2 = \sum_{n=1}^{\infty} < f, \phi_n >^2$$

holds for any $f \in H$.

Problem A.2.

Consider the integral equation

$$u(x) = f(x) + \frac{1}{2}\int_0^1 xtu(t)dt, \quad \text{for } 0 \le x \le 1$$

where $f(x)$ is a prescribed function.

a). Show that the equation has at most one solution.

b). Derive a formula for the solution.

c). Calculate the solution for the special case $f(x) = 5x/6$.

Problem A.3.

Construct the Green function for the equation

$$u''(x) = f(x) \text{ on the interval } 0 \le x \le 1$$

if u satisfies the auxiliary conditions

$$\int_0^1 xu(x)dx = 0 \quad \text{and} \quad u(0) + u'(1) = 0.$$

Problem A.4.

Solve the differential equation in the sense of distributions

$$xu' = \delta(4x^2 - 1), \quad x \in [-1, 1], \quad u(-1) = 1, \quad u(1) = 0.$$

Discuss why *two* boundary conditions are provided for the *first* order differential equation.

Applied Math Qualifying Exam Part B

Solve two of the four problems for full credit on Part B.

Problem B.1.

Using complex integration, evaluate the integral

$$I = \int_0^{2\pi} \frac{1}{1 + \frac{1}{3}\cos\phi} d\phi.$$

Problem B.2.

Evaluate by contour integration

$$I = \int_0^{\infty} \frac{x \sin 3x}{4 + x^2} dx.$$

Problem B.3.

Using conformal mapping find the electrostatic potential $\phi(x, y)$ in the region between the circumferences

$$\gamma = \{z : \ |z| = 1\} \quad \text{and } \Gamma = \{z : \ |z - 1| = 2.5\}$$

satisfying the boundary conditions

$$\phi \equiv 0 \text{ on } \gamma, \quad \phi \equiv 1 \text{ on } \Gamma.$$

Problem B.4.

Liouville's theorem states that a function analytic and bounded in the complex plane is necessarily a constant.
Prove this theorem. Using it prove the Fundamental theorem of algebra: every polynomial $P(z)$ which is not a constant has at least one complex root.

Applied Math Qualifying Exam Part C

Solve two of the four problems for full credit on Part C.

Problem C.1.

The so called "Brachistochrone problem" admits such formulation. "A particle is sliding along a trough (in a vertical plane from the point $(0,0)$ to some given point (x_1, y_1) $(x_1 > 0,\ y_1 < 0)$ under the effect of gravity without friction (having zero initial velocity). Find the form $y = Y(x)$ of the trough which minimizes the time of descent of this particle."

a). Find the functional that ought to be minimized.

b). Using this functional, write the boundary value problem that ought to be solved for finding the solution of the "Brachistochrone problem". You do not need to solve this boundary value problem.

Problem C.2.

The z-transform is defined on sequences in l^2 by the following formulas

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^n, \qquad u_n = \frac{1}{2\pi i}\int_C U(z) z^{-n-1} dz$$

where C is a closed contour enclosing the origin, and integration is counterclockwise.

If z is restricted to lie on the unit circle, then $U(z) = U(e^{i\theta}) = f(\theta)$.

a). Find the relation between the z transform and the Fourier series representation of $f(\theta)$.

b). Use the z-transform to solve the system of equations

$$\frac{du_n}{dt} = \frac{u_{n+1} - u_{n-1}}{2h}, \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

with initial condition $u_n(0) = \delta_{n0}$ (h is a positive constant).

Department of Mathematics
University of Utah

Written Qualifying Examination in

APPLIED MATHEMATICS

September 23, 1997

Instructions: The examination has three parts consisting of four problems each. You are to work two problems from Part A, two problems from Part B and two problems from Part C. If you work on more than six problems then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the qualifying examination your overall score must be at least 60%.

Applied Math Qualifying Exam Part A

Do two out of the four problems for full credit on Part A.

Problem A.1 Let x and y be vectors in an inner product space.

(a) Verify that the choice

$$\alpha = \frac{< x, y >}{||y||^2}$$

makes $||x - \alpha y||^2$ as small as possible.

(b) Show that $| < x, y > |^2 = ||x||^2 ||y||^2$ if and only if x and y are linearly dependent.

Problem A.2 Suppose $A \in I\!R^{n \times m}$ and $b \in I\!R^n$.

(a) Show that the equation

$$A^* A x = A^* b$$

has a solution. Under what conditions on A is the solution unique?

(b) Show that the eigenvalues of A^*A are non-negative and that its positive eigenvalues are the same as the positive eigenvalues of AA^*.

Problem A.3 Let H be a Hilbert space and $L : H \to H$.

(a) Find and prove the condition that ensures

$$Lu = \int_0^1 k(x, y)\, u(y)\, dy$$

is a bounded linear operator in $L^2[0, 1]$.

(b) Under what condition does the adjoint operator L^* exist? Assuming that this condition is satisfied, give L^*.

Problem A.4 Let $\{\phi_i\}$ be a (finite or infinite) orthonormal set in a Hilbert space. Prove that for every vector f in the space

$$||f||^2 \leq \sum_i | < f, \phi_i > |^2.$$

If inequality in above equation is replaced by strict equality, what can be said about the set $\{\phi_i\}$?

Applied Math Qualifying Exam Part B

Do two out of the four problems for full credit on Part B.

Problem B.1 Use a Green's function to find the representation of a possible solution to

$$y'' + \omega^2 y = f(x), \quad y(0) = y'(1) = 0$$

for arbitrary values of the parameter ω. For what values of ω does the solution exist? If the solution does not exist find the least square solution. Explain the meaning of the least square solution.

Problem B2. Derive the boundary value problem for a minimizer w_0 of the functional

$$\min_{w \in C^2[0,\ 1],\ w(0)=0} \frac{\int_0^1 \left(\phi(x)\left(w''\right)^2 + w^2\right) d\,x}{\int_0^1 \phi(x)\left(w'\right)^2 d\,x + \beta w(1)^2}$$

where $\phi(x) > 0,\ \forall\, x \in [0,\ 1]$, β is a real parameter.

Problem B.3 Evaluate the integral

$$I = \int_0^\infty \frac{x^2 \cos x}{x^4 + a^4} d\,x$$

by contour integration. Use the Jordan lemma.

Problem B.4 Find the spectral representation of the delta function for the operator

$$Lu = -\frac{d^2 u}{d^2 x}, \quad x \in [0, \infty), \quad u(0) = u'(0).$$

Classify the spectrum of this operator.

Applied Math Qualifying Exam Part C

Do two out of the four problems for full credit on Part C.

Problem C.1 Define and find the Green's function for the discrete operator Lu $= -\Delta u - \lambda u$ defined on the space of doubly infinite sequences $u = \{u_n\}, -\infty < n < \infty$, with $\sum_{n=-\infty}^{\infty} u_n^2 < \infty$, and with $(\Delta u)_n = u_{n+1} - 2u_n + u_{n-1}$.

Problem C.2 Prove that the Hermite polynomials form a complete set on $-\infty < x < \infty$ with the inner product $\langle u, v\rangle = \int_{-\infty}^{\infty} u(x)v(x)e^{-x^2/2}dx$.

Hint: The following steps will prove useful:

(a). Show that if the Hermite-Bessel coefficients of $f(x)$ are all zero, then $\int_{-\infty}^{\infty} e^{-x^2/2}x^n f(x)dx = 0$ for all n.

(b). Show that the Fourier transform $F(z)$ of the function $f(x)e^{-x^2/2}$ has no singularities in the complex plane and is therefore an entire function.

(c). Calculate the Taylor series of $F(z)$ and show that it is zero. What conclusion can you draw about $f(x)$ and what does this say about Hermite polynomials?

Problem C.3 Approximate the integral

$$I(k) = \int_C \frac{e^{k(z-1)^2}}{z - 1/2}dz \tag{1}$$

where C is the vertical imaginery axis from $z = -i\infty$ to $z = i\infty$.

Problem C.4 Find the first four eigenvalues for a "half drum", that is, for the Laplacian on a semicircular domain, $0 \le r \le R, 0 \le \theta \le \pi$, with homogeneous Dirichlet boundary conditions. Sketch the nodal lines for these eigenfunctions.

Hint: The zeros of Bessel functions are λ_{nm} where $J_n(\lambda_{nm}) = 0$. They are ordered as $\lambda_{01} < \lambda_{11} < \lambda_{21} < \lambda_{02} < \lambda_{31} < \lambda_{12} < \lambda_{41}$.

The Laplacian in polar coordinates is $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

Department of Mathematics
University of Utah

Written Qualifying Examination in

APPLIED MATHEMATICS

September 9, 1996

Instructions: The examination has three parts consisting of four problems each. You are to work two problems from Part A, two problems from Part B and two problems from Part C. If you work on more than six problems then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the qualifying examination your overall score must be at least 60%.

Applied Math Qualifying Exam Part A

Do two out of the four problems for full credit on Part A.

Problem A.1 State and prove the Fredholm Alternative Theorem for linear operators $L : I\!R^n \to I\!R^m$.

Problem A.2 Given a set of linearly independent vectors $u_1, u_2, \ldots, u_n$

(a) Describe the Gram-Schmidt procedure to produce an orthogonal set of vectors from the original set.

(b) Suppose U is a matrix with linearly independent columns. Describe the QR factorization of U using Householder transformations.

(c) In what sense are these two procedures equivalent? Prove your answer.

Problem A.3 Let A be an $m \times n$ matrix and let the $n \times m$ matrix A' denote the Least Square Pseudo-Inverse of A.

(a) State the conditions that define A'.

(b) Show that A' always exists.

(c) Construct A' explicitly for the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0.5 & 0.5 \end{pmatrix}.$$

Problem A.4 Let $Lu(x) = \int_0^x (t - x)u(t)\, dt$.

(a) Find the resolvent operator associated with the integral equation $u = f(x) + Lu$.

(b) Solve the integral equation $u = 1 + Lu$.

(c) Show that the integral equation $u = 1 + Lu$ is equivalent to a differential equation and give the equation.

Applied Math Qualifying Exam Part B

Do two out of the four problems for full credit on Part B.

Problem B.1

(a) Find the extended operator for the boundary value problem (BVP):

$$y'' - y = f(x), \quad ay(0) + y'(0) = 0, \;\; y'(1) = a. \tag{1}$$

(b) Solve the differential equation in the sense of distributions:

$$xu' = \delta(4x^2 - 1), \quad x \in [-1, 1], \;\; x(-1) = 1, \;\; x(1) = 0. \tag{2}$$

Discuss why the first order differential equation satisfies two boundary conditions.

Problem B.2

(a) Find the Green's function for the BVP:

$$-u'' = f; \quad 0 < x < 1, \;\; u(0) = 0, \;\; \int_0^1 xu'(x)\, dx = \gamma. \tag{3}$$

(b) Find the modified Green's function for the BVP:

$$-u'' = f; \quad 0 < x < 1, \;\; u(0) = u(1), \;\; u'(0) = u'(1). \tag{4}$$

Problem B.3 Compute the Euler equation for the minimizer of the following variational problem. Check Legendre's necessary condition. Discuss the dependence of the minimizer on the parameter a.

$$\min_{w \in W} \int_0^1 \left((xw')^2 + aw^2 w' - x \ln w \right) dx, \tag{5}$$
$$W = \{ w : \;\; w \in H_2(0, 1), \; w(0) = 1 \}.$$

Problem B.4 Compute the following integral. Use the Jordan lemma.

$$\int_0^\infty \frac{x \sin x}{x^2 + 1} dx. \tag{6}$$

Applied Math Qualifying Exam Part C

Do two out of the four problems for full credit on Part C.

Problem C.1 Suppose $f(t)$ is a band limited function with Fourier Transform $F(\mu) = 0$ for $|\mu| > \frac{\pi}{h}$. Define the sinc function $S_0(t)$ by

$$S_0(t) = \frac{\sin(\pi t/h)}{\pi t/h}. \tag{7}$$

Show that

$$f(t) = \sum_{k=-\infty}^{k=\infty} f(kh) S_0(t - kh). \tag{8}$$

Problem C.2 Suppose $f(\theta)$ is a 2π periodic function. Define its Hilbert Transform as follows: Let $u(r,\theta)$ be the solution of Laplace's equation on the unit circle satisfying Neumann data $u_r(1,\theta) = f(\theta)$. The Hilbert transform of f is $H(f) = u_\theta(1,\theta)$.

Use separation of variables to find the Hilbert Transform of f in terms of its Fourier coefficients.

Problem C.3 Find the leading order approximation to the integral

$$I = \int_0^\infty \cos x(t^3/3 - t)dt \tag{9}$$

for large positive x.
Hint: Use that $\int_{-\infty}^{\infty} \cos at^2 dt = \int_{-\infty}^{\infty} \sin at^2 dt = \sqrt{\frac{\pi}{2a}}$.

Problem C.4 A uniform spherical potato at temperature T_0 is placed in the center of a large oven that is kept at a uniform temperature T_1. Estimate how long it takes for the temperature at the center of the potato to reach $\frac{1}{2}(T_0 + T_1)$.

Hint: The Laplacian in three dimensions with spherical symmetry is $\nabla^2 u = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{du}{dr})$.

Written Qualifying Examination in

APPLIED MATHEMATICS

September 11, 1995

Instructions: The examination has three parts consisting of four problems each. You are to work two problems from part A, two problems from part B and two problems from part C. If you work on more than six problems then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the Qualifying Examination your overall score must be at least 60%.

Potentially Useful Information

Legendre Polynomials

$$P_0(x) = 1, P_1(x) = x, P_2(x) = (3x^2 - 1)/2$$

$$\int_{-1}^{1} P_j^2(x)dx = 2/(2j+1)$$

Fresnel Integrals

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$$

Bessel Functions

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin(\theta)-n\theta)}d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\sin(\theta)-n\theta)d\theta$$

Part A

Do two of the four problems.

Problem A1. Let A be a real m × n matrix with adjoint A^*. (a) Prove that if $R(A)$ and $N(A)$ denote the range and nullspace of A respectively, then

$$R(A) \oplus N(A^*) = R^m.$$

($\oplus$ means orthogonal direct sum.) (b) Using the result of (a), state and prove Fredholm's Alternative Theorem.

Problem A2. Let A be a real m × n matrix. (a) Define the Least Square Pseudo-Inverse A'. (b) Show how Gaussian Elimination can be used to compute A'.

Problem A3. Let $S = \{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ be a countable orthonormal set in a Hilbert space $\mathcal{H}$. Prove that S is a complete orthonormal set if and only if it is a maximal orthonormal set: i.e., there exists no unit vector ψ such that $\{\psi, \phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ is an orthonormal set.

Problem A4. Consider the integral equation

$$u(x) - \lambda \int_{-1}^{1} \sum_{j=0}^{2} P_j(x)P_j(t)u(t)dt = f(x), \text{ for } -1 \leq x \leq 1,$$

where $P_j(x)$ is the j^{th} Legendre function. (a) For which values of λ does the integral equation have a unique solution? (b) Calculate this solution for the case that $f(x) = x^2 + 1$.

Part B

Do two of the four problems.

Problem B1. Let $f(x)$ be in the Hilbert space $L^2(-\infty, \infty)$. Construct and use a Green's function to find the solution in $L^2(-\infty, \infty)$ of the equation $u'' - \alpha^2 u = f(x)$, $(\alpha > 0)$.

Problem B2. Liouville's theorem states that a bounded entire function $f(z)$ must be a constant. (a) Prove Liouville's theorem. (b) Use Liouville's theorem to prove the Fundamental Theorem of Algebra: every polynomial $P_n(z)$of degree $n \geq 1$ has a root.

Problem B3. (a) State and prove Jordan's Lemma of complex function theory. (b) Use Jordan's Lemma and the residue theorem to prove the equation

$$\int_0^\infty \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi e^{-a}}{2a}$$

where $a > 0$.

Problem B4. Find the spectral representation of the delta function for the selfadjoint operator L in the Hilbert space $L^2(0, 1)$ defined by

$$Lu = -u'', \text{ for } 0 \leq x \leq 1$$

$$u(0) = 0, \text{ and } u(1) + u'(1) = 0.$$

Part C

Do two of the four problems.

Problem C1. The (generalized) Hilbert transform H of a periodic function f can be defined as follows: For any simply connected two-dimensional domain Ω, let ϕ satisfy $\nabla^2\phi = 0$ with $\nabla\phi\cdot\mathbf{n} = f$ on $\partial\Omega$. Then $Hf = \nabla\phi\cdot\mathbf{t}$ on $\partial\Omega$. ($\mathbf{n}$ is the exterior unit normal, $\mathbf{t}$ is the unit tangent vector.)

a) Prove that $H(Hf) = -f$.

b) Find a relationship between the Fourier transform and the Hilbert transform, using a unit circle as the domain Ω.

c) What is the Hilbert transform of $f(x) = a\sin(x) + b\cos(2x)$?

Problem C2. Using an integral representation for the n-th order Bessel function $J_n(x)$, find the leading order asymptotic representation of $J_n(x)$ for large positive x.

Problem C3. The z-transform is defined on sequences in ℓ^2 by

$$U(z) = \sum_{-\infty}^{\infty} u_n z^n,$$

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{-n-1}dz$$

where the contour C is a closed contour enclosing the origin, and integration is counterclockwise. If z is restricted to lie on the unit circle then $U(z) = U(e^{i\theta}) = f(\theta)$

a) Find the relationship between the z-transform and the Fourier series representation of $f(\theta)$.

b) Use the z-transform to solve the system of equations

$$\frac{du_n}{dt} = \frac{1}{2h}(u_{n+1} - u_{n-1})$$

with $u_n(0) = \delta_{n0}$.

Problem C4. The flux q of dilute chemicals in a solute is usually assumed to be governed by Fick's law, $q = -D\nabla c$, where c is the concentration. In an electric field, if the chemical species is ionic, then the flux is given by the Nernst-Planck equation $q = -D\nabla c - \mu c\nabla\phi$ where ϕ is the potential of the electric field.

a) Assuming the field $\nabla\phi$ is constant, and is independent of the concentration of the ionic species, find an evolution equation governing the concentration c as a function of time.

b) Suppose the concentrations at the two ends of a one dimensional domain are held fixed. Find the concentration as a function of time, starting from arbitrary initial conditions.

Caution: The spatial operator is not self-adjoint! A transformation of the form $c(x,t) = e^{ax}u(x,t)$ is helpful.

Written Qualifying Examination in

APPLIED MATHEMATICS

September 12, 1994

Instructions: The examination has three parts which will be assigned separate scores. A perfect score (100%) requires complete solutions of two problems from part A, two problems from part B and two problems from part C. If you work on more than the required number of problems then state which problems you wish to be graded.

In order to pass the Qualifying Examination you must score at least 60% on part A, at least 60% on part B and at least 60% on part C.

Part A

Do two of the four problems for full credit on part A.

Problem A1. Prove that an orthonormal sequence $\{\phi_1, \phi_2, \cdots, \phi_n\}$ in a Hilbert space $\mathcal{H}$ is complete if and only if Parseval's relation

$$\|f\|^2 = \sum_{n=1}^{\infty} |< f, \phi_n >|^2$$

holds for all vectors $f \in \mathcal{H}$.

Problem A2. Let A be an $m \times n$ matrix and let the $n \times m$ matrix A' denote the Least Square Pseudo-Inverse of A. (a) State the conditions that define A'. (b) Show that A' always exists. (c) Construct A' explicitly for the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem A3. Consider the boundary value problem $u'' = f(x), u'(0) = 0, u(1) = 0$ where $f(x)$ is a prescribed function. (a) Show that the problem has at most one solution. (b) Show that the solution can be written as

$$u(x) = \int_0^1 k(x,y)f(y)dy$$

for all continuous functions $f(x)$ where

$$k(x,y) = \begin{cases} x-1 & \text{for } 0 \le y \le x \le 1, \\ y-1 & \text{for } 0 \le x \le y \le 1. \end{cases}$$

Problem A4. Consider the integral equation

$$u(x) = f(x) + 1/2 \int_0^1 xtu(t)dt, \text{ for } 0 \le x \le 1$$

where $f(x)$ is a prescribed function. (a) Show that the equation has at most one solution. (b) Derive a formula for the solution that is valid for any continuous function $f(x)$. (c) Calculate the solution for the special case $f(x) = 5x/6$.

Part B

Do two of the four problems for full credit on part B.

Problem B1. Find the general solution, in the sense of distribution theory, of the differential equation

$$x^2 u'(x) = \delta(x^2 - 1).$$

Problem B2. Construct the Green's function for the equation $u''(x) = f(x)$ on the interval $0 \leq x \leq 1$ if u satisfies the auxiliary conditions $\int_0^1 xu(x)dx = 0$ and $u(0) + u'(1) = 0$.

Problem B3. Find the adjoint operator, its domain and the least squares solution of the problem $\mathcal{L}u \equiv 4u'' + u = e^x$ for $0 \leq x \leq \pi/2$ if $u(0) = \alpha$ and $u(\pi/2) = \beta$.

Problem B4. Find the Euler equations and boundary conditions for the minimization problems of the functionals

$$I_1 = \min_u \int_0^1 [(u'')^2 + x(u')^2]dx,$$

and

$$I_2 = \min_u \int_0^1 [(u'')^2 - uu'' + (x-1)(u')^2]dx,$$

if $u(0) = 1$ and $u'(1) = 2$ for both problems. Why do their Euler equations coincide?

Part C

Do two of the four problems for full credit on part C.

Problem C1. Find the spectral representation of the delta function for the operator $\mathcal{L}$ defined by $\mathcal{L}u = -u''$ on $x \in (-\infty, 0]$ with boundary condition $u(0) = 0$. Then solve the problem $\mathcal{L}u = f$ if $f(x) = 1$ for $-2 \le x \le -1$ and $f(x) = 0$ elsewhere.

Problem C2. Find the Green's function for the Poisson equation $\Delta u = f$ in the infinite strip $0 \le y \le 1$, $-\infty < x < \infty$ with boundary data $u(x,0) = 0$ and $u(x,1) = 1$. Then solve the problem (in quadrature) if $f(x,y) = y(1-y)e^{-x^2}$.

Problem C3. Describe quantitatively the diffusion of 3 units of a diffusing material which is placed at the initial moment $t = 0$ at the origin of an infinite two-dimensional plane. Assume that the plane has unit conductivity.

Problem C4. Derive the first term in the asymptotic representation, valid for $x \to +\infty$, of the function

$$F(x) = \int_0^1 \cos(t^2\sqrt{x^3 + x})\,dt.$$