PhD Preliminary Qualifying Examination: Applied Mathematics

August, 2010

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. (a) State and prove the Fredholm Alternative Theorem for a linear transformation $A : R^n \to R^m$.

   (b) Find and give a graphical interpretation of the pseudo-inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

2. This problem examines features of the differential operator

$$Lu = u''$$

subject to boundary conditions $u(-1) + u(1) = 0$ and $u'(-1) = u'(1)$.

   (a) What is the adjoint of this operator? Is it self-adjoint?

   (b) What is the nullspace of the operator and what is the nullspace of the adjoint operator?

   (c) Under what conditions on f, a, and b does the boundary value problem

$$u'' = f, \qquad u(-1) + u(1) = a, \qquad u'(-1) - u'(1) = b$$

   have a solution?

   (d) Find the smallest least squares solution for the boundary value problem

$$u'' = 1, \qquad u(-1) + u(1) = 1, \qquad u'(-1) - u'(1) = 1.$$

   What equation does this least squares solution satisfy?

3. (a) Find all the eigenvalues and eigenfunctions for the operator

$$(Lu)(x) = \int_0^1 xyu(y)dy.$$

(b) Do these eigenfunctions form a complete set? If so, how do you know (state a relevant theorem) and on what Hilbert space, and if not, why not?

4. (a) Find the weak formulation for the boundary value problem

$$u'' - \delta(x)u = 1$$

subject to boundary conditions $u(-1) = u(1) = 0$.

(b) Find the solution to this boundary value problem.

5. Let $P_n(x), n = 0, 1, \cdots$, be the Legendre polynomials, $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, which are also eigenfunctions for the linear differential operator $Lu = -\frac{d}{dx}((1-x^2)\frac{du}{dx})$ on the interval $-1 \le x \le 1$.

(a) Determine the eigenvalue associated with P_n

(b) Show that the Legendre polynomials are orthogonal.

(c) Suppose $f(x)$ is square integrable on the interval $-1 < x < 1$. Use that the Legendre polynomials are orthogonal to prove that

$$g(x) = \sum_{n=0}^{\infty} \alpha_n P_n(x)$$

where $\alpha_n = \frac{\int_{-1}^{1} f(x)P_n(x)dx}{\int_{-1}^{1} P_n^2(x)dx}$, is also square integrable.

Part B.

1. For a function $f(z)$ which is analytic within a simple, nonintersecting, closed curve C a corollary of the Cauchy integral formula implies that for any point z inside C,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} d\xi.$$

   Use this to prove Liouville's theorem that any function which is bounded and analytic in the entire complex plane must be a constant, and hence prove the fundamental theorem of algebra that an nth order polynomial $P_n(z)$ has n roots (counting multiplicity) in the complex plane.

2. The fourth quadrant $x \geq 0$, $y \leq 0$ is occupied by an impermeable solid, around which air flows. Treating the flow as incompressible and irrotational use conformal mapping to map to a simpler problem and give one non-trivial solution for the complex potential $f(z)$ in the original coordinates. Find the equation in polar coordinates for the streamline passing through $z = -1$. What happens to the fluid velocity as one approaches the corner?

3. Use antiderivatives to evaluate the contour integral

$$I = \int_C \frac{1}{z + i}$$

   where C is a contour (possibly with loops) starting at $z = 0$ and ending at $z = 1$. Express your answer in the form $I = a + ib$ where a and b are real. The answer is multivalued. Explain the interpretation of the different values, and account for the difference between successive values using the residue theorem.

4. The operator $Lu = -u''$ on $L^2(-\infty, \infty)$ has a Green's function defined by $-G'' - \lambda G = \delta(x - \xi)$ given by

$$G(x, \xi; \lambda) = -\frac{e^{i\sqrt{\lambda}|x-\xi|}}{2i\sqrt{\lambda}}.$$

   Assume the representation

$$-\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = \delta(x - \xi)$$

   and show how it leads to the Fourier Transform Theorem that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} F(\mu) d\mu, \quad F(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} f(x) dx$$

5. Use the method of steepest descent to estimate the integral

$$I(s) = \int_0^1 e^{s(-z^3+iz^3)}dz$$

for large positive real values of s.