PhD Preliminary Qualifying Examination

Applied Mathematics

January 7 2015

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Let M be a subset of a Hilbert space H. The goal of this problem is to show that:

   $\overline{\operatorname{span} M} = H$ if and only if $M^\perp = \{0\}$.

   (a) Assume $\overline{\operatorname{span} M} = H$ and let $x \in M^\perp$. Show that $x = 0$.

   (b) Assume $M^\perp = \{0\}$. Show that $\overline{\operatorname{span} M} = H$.

2. Let (x_n) be a sequence in a Hilbert space H. Show that:

   $x_n \to x$ **strongly** if and only if ($x_n \to x$ **weakly** and $\|x_n\| \to \|x\|$).

3. Let u, v be non-zero elements of a Hilbert space H. Consider the linear operator $T : H \to H$ defined for $x \in H$ by $Tx = u\,\langle x, v\rangle$.

   (a) Briefly explain why the operator T is compact.

   (b) Find a condition on u and v that guarantees that the equation

   $$(I - T)x = y$$

   admits a unique solution x for all $y \in H$.

4. Let $(\lambda_n) \in \ell^2$ and consider the operator $T : \ell^2 \to \ell^2$ defined by $y = Tx$, where $x = (\xi_j) \in \ell^2$, $y = (\eta_j) \in \ell^2$ and

   $$\eta_j = \sum_{k=1}^{\infty} \alpha_{jk}\xi_k, \; j = 1, 2, \dots,$$

   where the α_{jk} satisfy

   $$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty.$$

   Show that T is a compact operator. **Hint:** Approximate T with finite rank operators T_n.

5. Let $T : X \to X$ be a bounded linear operator defined on a Banach space X. Denote by R_λ the resolvent operator associated with T for some $\lambda \in \rho(T)$. The resolvent set of T is denoted by $\rho(T)$.

   (a) Use Neumann series to show that for $\lambda, \mu \in \rho(T)$,

   $$R_\lambda = \sum_{j=0}^{\infty} (\lambda - \mu)^j R_\mu^{j+1},$$

   where the series is absolutely convergent in the operator norm when

   $$|\lambda - \mu| < \|R_\mu\|^{-1}.$$

   (b) Deduce from part (a) whether the spectrum $\sigma(T)$ is open, closed or neither.

Part B.

1. Find the radius of convergence of the Taylor series with the center at the origin ($x_0 = 0$) for the following function

$$(a) \qquad f(x) = \frac{\sqrt{x+3}}{x^2+3}$$

$$(b) \qquad f(x) = \frac{1}{(\cos x + 3)^2}$$

(x is a real variable; you do not need to find the series themselves).

2. Let D_1 and D_2 be two disjoint domains, whose boundaries share a common curve Γ. Let

   - $f(z)$ be analytic in D_1 and continuous in $D_1 \cup \Gamma$
   - $g(z)$ be analytic in D_2 and continuous in $D_2 \cup \Gamma$
   - $f(z) = g(z)$ for $z \in \Gamma$

   Show that the function

$$H(z) = \begin{cases} f(z) & z \in D_1 \\ f(z) = g(z) & z \in \Gamma \\ g(z) & z \in D_2 \end{cases}$$

   is analytic in $D = D_1 \cup \Gamma \cup D_2$.

   [Hint: Use the Morera theorem.]

3. (a) Show that any analytic function (not identically equal to zero) can have only isolated zeros inside its analyticity domain.

   (b) Can an analytic function have a non-isolated singularuty?

   (c) Prove the Uniqueness Theorem: If two functions are analytic in a domain D and equal on some set of points that has a limiting point inside D, then these functions are identically equal in D.

4. Evaluate

   (a) $\int_{-\infty}^{\infty} e^{ix^2}\, dx$

   (b) $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

5. Find the leading behavior, as $s \to +\infty$, of the integral

   (a) $I(s) = \int_0^3 \frac{1}{\sqrt{x^2+2x}} e^{-sx}\, dx$

   (b) $I(s) = \int_0^{\pi/2} e^{is\cos x}\, dx$