UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Applied Mathematics

This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three from each part will be graded.

All problems are worth 10 points and a passing score is 40.

January 2nd, 2020.

Part A.

A1. Consider the sequence of linear operators (U_n) where $U_n : H \to H$, H is a Hilbert space and the U_n are unitary, i.e. $U_n^* U_n = I$, where I is the identity operator on H. Show that if $U_n \to U$ in the operator norm, the limit U must also be unitary, i.e. $U^* U = I$.

A2. Let X be a real Banach space, $f : X \to \mathbb{R}$ be a bounded linear functional and $T : X \to X$ a bounded linear operator. In the following $y \in X$ is fixed and $\alpha \in \mathbb{R}$.

(a) Prove that there exists a constant $C > 0$ such that if $|\alpha| < C$, then the non-linear equation
$$x + \alpha f(x) T x = y, \tag{1}$$
has a unique solution x in the ball $B = \{x \in X \mid \|x - y\| \le 1\}$.

(b) Suggest an iterative procedure for approximating the unique solution x to equation (1).

A3. Let $T : X \to X$ be a compact linear operator and $S : X \to X$ be a bounded linear operator, defined on a Banach space X.

(a) Show that TS is a compact linear operator.

(b) Show that ST is a compact linear operator.

A4. Show that a mapping $T : X \to Y$ where X and Y are metric spaces is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

A5. Let $x_+ = xH(x)$, where $x \in \mathbb{R}$ and $H(x)$ is the Heaviside function so that $x_+ = x$ for $x > 0$ and $x_+ = 0$ for $x \le 0$.

(a) Show that differentiating in the sense of distributions we have for any integer $k \ge 0$ that
$$\partial(x_+^k) = \begin{cases} k x_+^{k-1} & \text{if } k \ge 1, \\ \delta & \text{if } k = 0. \end{cases}$$

(b) Let $k \ge 1$ be an integer. Use (a) to find a distribution E such that $\partial^k E = \delta$.

Part B.

B1. Consider the complex function $f(z) = \ln((z-1)(z-2))$. Find the branch points. Choose a branch cut and find the corresponding principal branch of the function.

You must justify the choice of angles in the dual polar coordinate representation.

B2. Consider two entire functions with no zeroes and having a ratio of unity at infinity. Use Liouville's theorem to show that there are in fact the same function.

You must justify all steps in verifying the assumptions of Liouville's theorem.

B3. (a) Show that the transformation $w = 2z + 1/z$ maps the unit circle conformally onto the ellipse

$$\left(\frac{u}{3}\right)^2 + v^2 = 1.$$

Here $w = u + iv$.

(b) Find the Möbius transformation that takes the points $z_1 = 0, z_2 = 1, z_3 = -i$ into $w_1 = 1, w_2 = 0, w_3 = i$.

B4. Evaluate

$$I = 2\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4} dx.$$

You must indicate the contour of integration and treat integration over each contour.

B5. Find the entire asymptotic expansion of

$$I(x) = \int_x^{\infty} e^{-t^2} dt \quad \text{as} \quad x \to 0^+.$$