UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 16, 2017.

Instructions: This examination has two parts consisting of six problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [20] points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $D \subset \mathbf{R} \times \mathbf{R}^n$ be an open set, $(t_0, x_0) \in D$ a point and $f(t,x) \in \mathcal{C}(D, \mathbf{R}^n)$ be a map.

(a) Let X be a Banach Space with norm $\| \bullet \|$. Let $C \subset X$ be a closed subset. State the Contraction Mapping Theorem (otherwise known as the Banach Fixed Point Theorem) for a map $K : C \to X$.

(b) Define what it means for a map $f(t,x) \in \mathcal{C}(D, \mathbf{R}^n)$ to be *locally Lipschitz with respect to x*

(c) Carefully reformulate the Initial Value Problems as a fixed point problem. Prove that the fixed point problem is equivalent to the local existence problem for the (IVP)

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ x(t_0) = x_0. \end{cases} \tag{IVP}$$

(d) Assume $f(t,x) \in \mathcal{C}(D, \mathbf{R}^n)$ is locally Lipschitz with respect to x. Using the Contraction Mapping Theorem, show that there is a $\varepsilon > 0$ and a unique function $y \in \mathcal{C}^1([t_0, t_0 + \varepsilon], \mathbf{R}^n)$ that satisfies the initial value problem (IVP) for $t_0 \le t \le t_0 + \epsilon$.

A2. (a) Let A be an $n \times n$ real matrix. Find necessary and sufficient conditions on A so that for all $x_0 \in \mathbf{R}^n$, the solution $\varphi(t; x_0)$ of (1) remains bounded for $t \ge 0$. Prove your result.

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0. \end{cases} \tag{1}$$

(b) Suppose that $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ is such that $f(0) = 0$ and the Jacobian matrix $A = df(0)$ satisfies the condition in (a). Do all solutions $\psi(t; x_0)$ of the nonlinear equation (2) still remain bounded for $t \ge 0$, at least if x_0 is close to 0? Explain.

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x_0. \end{cases} \tag{2}$$

A3. (a) Let $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ satisfy $f(0) = 0$. State the definitions: the zero solution of

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ x(0) = x_0, \end{cases}$$

is *stable in the sense of Liapunov* and *asymptotically stable.*

(b) Suppose that all eigenvalues of the real $n \times n$ matrix A have negative real part $\Re e\, \lambda_i < 0$. State an estimate for solutions of

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0, \end{cases}$$

and use it to prove that the zero solution is asymptotically stable.

(c) Let A as in (b) and $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ be a smooth function such that for some $L < \infty$, $|f(x)| \le L|x|^2$ for all x. Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = A\,x + f(x), \\ \qquad x(0) = x_0. \end{cases} \tag{3}$$

Prove that the zero solution of (3) is stable in the sense of Liapunov. [Don't just quote a theorem. Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \ge 0$ is a constant that satisfy $u(t) \le c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \ge 0$, then Gronwall's Inequality implies $u(t) \le c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \ge 0$.]

A4. (a) Let $A(t)$ be a continuous, non-constant, real matrix-valued, $T > 0$ periodic function $A(t+T) = A(t)$. For the system

$$\dot{x} = A(t)x \tag{4}$$

define: *Monodromy Matrix*, *Floquet Multipliers* and *Floquet Exponents.*

(b) If $e^{T\gamma} = \rho$ is a Floquet multiplier, then there is a solution of (4) of the form

$$x(t) = e^{\gamma t} p(t)$$

where $p(t) = \pm p(t+T)$.

(c) Prove that this equation does not have a fundamental set of bounded solutions.

$$\ddot{x} - (\cos^2 t)\dot{u} + (\sin^2 t)u = 0$$

A5. Consider the Brusselator system for an autocatalytic reaction. Show that it has a non-constant periodic solution.

$$\begin{cases} \dot{x} = 2 - 13x + x^2 y, \\ \dot{y} = 12x - x^2 y. \end{cases}$$

[Hint: show the trapezoid with sides $y = 0$, $x = \frac{2}{13}$, $y = 78$ and $x + y = 80$ is forward invariant.]

A6. Let $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ have a non-constant, $T > 0$ periodic trajectory $\gamma(t)$ satisfying $\dot{\gamma}(t) = f(\gamma(t))$.

(a) Define: the periodic solution $\gamma(t)$ is *orbitally stable.*

(b) Define: the *Poincaré Map* for the orbit γ.

(c) $\gamma(t) = (2\cos 2t, \sin 2t)$ is a periodic solution to

$$\begin{cases} \dot{x} = -4y + x\left(1 - \dfrac{x^2}{4} - y^2\right), \\ \dot{y} = x + y\left(1 - \dfrac{x^2}{4} - y^2\right). \end{cases}$$

Determine the orbital stability of $\gamma(t)$ by computing the derivative of its Poincaré Map.

B. Partial Differential Equations. Do three problems to get full credit

B1. A problem in the dynamics of the overhead power wire for an electric locomotive leads to the model

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x \neq X(t),\, t > 0,$$

where X is a prescribed smooth function with $0 < X'(t) < 1$. Here $X(t)$ is the locomotive position and u is the displacement of the wire. Across $x = X$ there are prescribed discontinuities

$$\left[\frac{\partial u}{\partial x}\right]_{X^-}^{X^+} = -V(X(t), t), \quad [u]_{X^-}^{X^+} = 0.$$

Suppose that $u = u_t = 0$ at $t = 0$.

(a) By differentiating along the curve $x = X(t)$, show that

$$\left[\frac{\partial u_\pm}{\partial t}\right]_{X^-}^{X^+} = -X'(t)\left[\frac{\partial u_\pm}{\partial x}\right]_{X^-}^{X^+}.$$

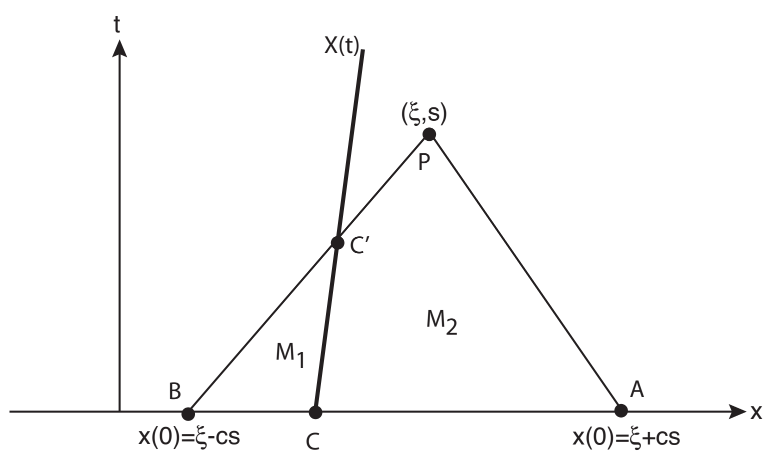
(b) For fixed (ξ, s), consider the domain of dependence $M = \{(x,t)|0 < t < s, \xi - s < x < \xi + s\}$. Partition the domain into the subdomains M_1 and M_2, $M_1 \cup M_2$, as shown in the figure below. By constructing a weak solution that takes into account the discontinuities across X, show that

$$u(x,t) = u(P) = \int_0^{\tau_0} V(X(\tau), \tau)(1 - X'(\tau)^2)d\tau,$$

where the range of τ is taken so that $y = X(\tau)$ lies within the range

$$\tau - t < y - x < t - \tau,\ \tau > 0.$$

[Hint: Within each subdomain the wave equation is satisfied, so apply integration by parts to the subdomains M_1 and M_2 separately. Then combine your results and use part (a).]



B2. Consider the problem of a thin layer of paint of thickness $h(x,t)$ and speed $u(x,y,t)$ flowing down a wall. The paint is assumed to be uniform in the z-direction. The balance between gravity and viscosity (fluid friction) means that the velocity satisfies the equation

$$\frac{\partial^2 u}{\partial y^2} = -c,$$

where c is a positive constant. This is supplemented by the boundary conditions

$$u(x,0,t) = 0, \qquad \left.\frac{\partial u(x,y,t)}{\partial y}\right|_{y=h} = 0.$$

The density of paint per unit length in the x-direction is $\rho_0 h(x,t)$ where ρ_0 is a constant, and the corresponding flux is

$$q(x,t) = \rho_0 \int_0^h u(x,y,t)\,dy.$$

(a) Sketch a figure illustrating the physical problem.

(b) Using conservation of paint, and solving for $u(x,y,t)$ in terms of $h(x,t)$ and y, derive the following PDE for the thickness h:

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0.$$

(c) Set $c = 1$. Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[h^3/3]_-^+}{[h]_-^+}.$$

(d) A stripe of paint is applied at $t = 0$ so that

$$h(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1. \end{cases}$$

Show that, for small enough t,

$$h = \begin{cases} 0, & x < 0 \\ (x/t)^{1/2}, & 0 < x < t \\ 1, & t < x < S(t) \\ 0, & S(t) < x, \end{cases}$$

where the shock is $x = S(t) = 1 + t/3$. Explain why this solution changes at $t = 3/2$, and show that thereafter

$$\frac{dS}{dt} = \frac{S}{3t}.$$

B3. Consider the following inhomogeneous initial-Neumann problem:

$$\begin{aligned} u_t &= Du_{xx} + \alpha t x, && 0 < x < \pi, \quad 0 < t, \\ u(x,0) &= 1, && 0 < x < \pi, \\ u_x(0,t) &= u_x(\pi,t) = 0, && 0 < t; \end{aligned}$$

where α is a constant.

(a) Determine the eigenfunctions of the homogeneous equation

$$u_{xx} = \lambda u, \quad 0 \le x \le \pi, \quad u_x(0) = u_x(\pi) = 0.$$

(b) Solve the inhomogeneous initial-Neumann problem by carrying out an eigenfunction expansion of $u(x,t)$ in terms of the eigenfunctions obtained in part (a). That is, denoting the eigenfunctions by v_k, integer k, set

$$u(x,t) = \sum_{k \ge 0} c_k(t) v_k(x),$$

and determine the time-dependent coefficients $c_k(t)$.

(c) Give a physical interpretation of the solution in the limit $\alpha \to 0$.

B4. (a) Consider the velocity potential ϕ for linear water waves. Let h_0 denote the depth of water in the absence of waves, and g denote the gravitational constant. The corresponding boundary value problem on $\mathbb{R} \times [-h_0, 0]$ is given by Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

with

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = 0,$$

and

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = -h_0.$$

The linear wave solution takes the form

$$\phi(x,y,t) = Y(y) \mathrm{e}^{ikx - i\omega t}.$$

Derive the transcendental dispersion relation

$$\omega^2 = gk \tanh(kh_0).$$

Determine the group and phase velocities for (i) deep water waves (large h_0) and (ii) shallow water waves (small h_0). Which case is non-dispersive?

(b) Suppose that $u(\mathbf{x})$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbf{R}^n$, so $\Delta u = 0$ in Ω. Prove the *mean value property:* if $\mathbf{x} \in \Omega$ and $r > 0$ is chosen such that $B_r(\mathbf{x}) \subset \Omega$ (ball of radius r centered at $\mathbf{x}$) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) d\mathbf{s},$$

where ω_n is the measure of ∂B_1. Hence show that

$$u(\mathbf{x}) \le \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

B5 Consider a string clamped at the end points a and b, say, with $u(a,t) = 0$ and $u(b,t) = 0$, where $u(x,t)$ is the string's deviation from the horizontal rest position. The kinetic energy of the string is

$$E_k(t) = \frac{1}{2} \int_a^b \rho u_t^2 ds,$$

where $\rho(x,t)$ is the mass density and $ds = \sqrt{1+u_x^2}dx$ is an infinitesimal arc length. The potential energy consists of the sum of the energy due to the stretching of the string, and the work done against a load:

$$E_p(t) = \int_a^b \left(d\sqrt{1+u_x^2} - 1) - lu\sqrt{1+u_x^2} \right) dx.$$

Here $d(x,t)$ is the string's elastic coefficient and $l(x,t)$ is the load on the string. The classical action for the continuum model is

$$J = \int_{t_1}^{t_2} [E_k(t) + E_p(t)] dt.$$

(a) Substitute the integral expressions for E_k and E_p into the action J, and consider the variations $u + \epsilon\psi$ such that ψ vanishes at the end points a, b and at the initial and final times t_1, t_2. Calculate the first variation δJ.

(b) After integrating by parts terms of the form $u_t\psi_t$ and $u_x\psi_x$, and using the boundary conditions for the variation ψ, show that Hamilton's principle $\delta J = 0$ yields the following PDE for u:

$$(\rho u_t)_t - (1+u_x^2)^{-1/2}[d(1+u_x^2)^{-1/2}u_x]_x - l = 0.$$

(c) Show that if ρ, d are constants, there is no load ($l = 0$), and the small slope approximation $|u_x| \ll 1$ holds, then the PDE of part (b) reduces to the classical one-dimensional wave equation. Write down an expression for the wave speed.