UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 16, 2018.

Instructions: This examination has two parts consisting of six problems in part A and six in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [20] points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $f : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuous function and $x_0 \in \mathbf{R}^d$. Show that there is $T > 0$ such that the initial value problem

$$\begin{aligned} \dot{x} &= f(t,x) \\ x(0) &= x_0 \end{aligned}$$

has a solution $x(t) \in \mathcal{C}^1([0,T), \mathbf{R}^n)$. Is $x(t)$ unique? [Don't just quote a result. State the theorem and give as complete and detailed a proof as you can.]

A2. (a) Let $f : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuously differentiable function and $x_0 \in \mathbf{R}^d$ be such that $f(t,x_0) = 0$ for all $t \ge 0$. Define: the constant solution $x(t) = x_0$ on $t \ge 0$ is a *stable* (Liapunov stable) solution for of

$$\dot{x} = f(t,x).$$

(b) Let A be an $d \times d$ real matrix. State and prove a theorem that gives necessary and sufficient conditions on A so that the the zero solution $z(t) = 0$ on $t \ge 0$ is stable for

$$\dot{x} = Ax.$$

(c) Let $A(t)$ be a smooth, T-periodic for some $T > 0$, $d \times d$ real matrix function. State but don't prove a theorem that gives necessary and sufficient conditions on $A(t)$ so that the the zero solution $z(t) = 0$ is stable for

$$\dot{x} = A(t)\,x.$$

A3. Let $g : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuously differentiable function and $p > 1$ a constant such that

$$|g(t,x)| \le |x|^p \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbf{R}^d.$$

Let A be a real matrix such that $\Re e\,\lambda < 0$ for all eigenvalues of A.

(a) Show that there is $\delta > 0$ such that if $|x_0| \le \delta$, then the initial value problem

$$\begin{aligned} \dot{x} &= Ax + g(t,x) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

has a bounded solution on $[0,\infty)$. [Provide the complete proof. Do not just quote a theorem.]

(b) Is the zero solution of (1) asymptotically stable? Explain.

[Hint: let $f(t)$, $\varphi(t)$ be nonnegative continuous functions on the interval $J = (\alpha, \beta)$ containing t_0. Let $c_0 \geq 0$. Gronwall's Lemma says that if $f(t) \leq c_0 + \left| \int_{t_0}^{t} \varphi(s)\, f(s)\, ds \right|$ for all $t \in J$ then $f(t) \leq c_0 \exp \left| \int_{t_0}^{t} \varphi(s)\, ds \right|$ for all $t \in J$.]

A4. Consider the torqued pendulum with constant drag $\beta > 0$ and torque $\mu > 1$ large enough to overcome the pull of gravity, no matter what the angle x of the pendulum may be.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu - \beta y - \sin x \end{aligned}$$

(a) Show that the solution of the initial value problem with $x(0) = 0$ and $y(0) = y_0$ exists for all time.

(b) Show that there is a solution such that the pendulum continues to rotate indefinitely in a T-periodic fashion. It would have $x^*(T + t) = x^*(t) + 2\pi$ and $y^*(T + t) = y^*(t)$.

(c) Show that the periodic solution is unique.

(d) Show that the periodic solution is orbitally stable.

A5. Consider the bathtub model for the concentrations $x(t) \geq 0$ and $y(t) \geq 0$

$$\begin{aligned} \dot{x} &= -9x + y + 9x^2 y \\ \dot{y} &= 6 - y - 9x^2 y. \end{aligned}$$

(a) Give as complete a proof as you can that there exists a nonconstant periodic orbit.

(b) State the Hartman-Grobman Theorem. What role does it play in your proof of (a)?

A6. Consider Newton's equation

$$\ddot{x} + \frac{x}{2 + x^2} = \epsilon \cos t$$

Prove that there is an $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ then there is a 2π- periodic solution.

B. Partial Differential Equations. Do three problems to get full credit

B1. Suppose that the initial value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0, \qquad x \in \mathbf{R}^n, \qquad t > 0,$$

$$\begin{cases} u(0, x) = g(x), \\ \dfrac{\partial u}{\partial t}(0, x) = 0, \end{cases} \qquad x \in \mathbf{R}^n.$$

has a solution u_g. Find (and verify) the solution u_h of the same equation subject to the initial conditions

$$\begin{aligned} u(0, x) &= g(x), \\ \frac{\partial u}{\partial t}(0, x) &= h(x), \end{aligned} \qquad x \in \mathbf{R}^n$$

in terms of u_g, g and h.

B2. Find and sketch the solution of the differential equation

$$u_t + uu_x = 0$$

on the domain $-\infty < x < \infty$, $t > 0$ subject to the initial condition $u(0, x) = u_0(x)$, where
(a)

$$u_0(x) = \begin{cases} 0; & \text{if } x < 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x > 1; \end{cases}$$

(b)

$$u_0(x) = \begin{cases} 1; & \text{if } x < 0, \\ 1 - x, & \text{if } 0 < x < 1, \\ 0, & \text{if } x > 1. \end{cases}$$

For which (if any) of these initial profiles does the solution of the problem

$$u_t + uu_x = \varepsilon u_{xx},$$

$\varepsilon > 0$, converges to a traveling wave profile? What is the speed of the travelling wave solution?

B3. The small longitudinal free vibrations of an elastic bar are governed by the following equation:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)\frac{\partial u}{\partial x}\right].$$

Here u is the longitudinal displacement ρ is the linear density of the material, and E is its Young's modulus. Assume that the bar is constructed by welding together two bars of different (constant) Young's moduli E_1, E_2 and densities ρ_1, ρ_2, respectively. The displacement u is continuous across the junction, which is located at $x = 0$.

(a) Give a weak formulation of the global initial value problem, and use this to derive the following jump condition:

$$E_1 u_x(0-, t) = E_2 u_x(0+, t), \quad t > 0.$$

(b) Let $c_j^2 = E_j/\rho_j$. A left incoming wave $u_I(x, t) = \mathrm{e}^{i(t-x/c_1)}$ produces at the junction a reflected wave $u_R(x, t) = R\,\mathrm{e}^{i(t+x/c_1)}$ and a transmitted wave $u_T = T\,\mathrm{e}^{i(t-x/c_2)}$. Determine the reflection and transmission coefficients R, T, and interpret the result.

B4. Suppose that $u(x) \in \mathcal{C}^2(D)$ is a harmonic function on the smooth bounded domain $D \subset \mathbf{R}^n$ so that $\nabla^2 u = 0$ in D.

(a) prove the *mean value property:* if $x \in D$ and $r > 0$ is chosen so that the ball of radius r and center x satisfies $B_r(x) \subset D$ then

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, dy$$

where ω_n is the $n-1$ dimensional area of ∂B_1. Hence show that

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(z)\, dz.$$

(b) Assuming that D is connected, prove that u can attain its maximum value at an interior point $x \in D$ only if u is constant.

B5. Consider the global Cauchy problem

$$u_y = Du_{xx} + bu_x + cu, x \in \mathbf{R}, t > 0; \qquad u(0,x) = g(x),$$

where b, c D are constants such that $D > 0$.

(a) Convert to the standard heat equation by setting $v(t,x) = u(t,x)e^{hx+kt}$ for appropriately chosen h and k.

(b) Solve the heat equation for v using a transform method.

(c) Hence show that if $c < 0$ and g is bounded, then $y(t,x) \to 0$ as $t \to \infty$.

B6. Let $D \subset \mathbf{R}^n$ be a bounded, connected domain with smooth boundary. Let ϕ be an eigenfunction for the Laplacian on D with Dirichlet Boundary conditions ($\phi = 0$ on ∂D) and with eigenvalue λ

$$\Delta u + \lambda u = 0. \tag{2}$$

(a) Show that there is a finite constant C so that every nonzero continuously differentiable function $u \in \mathcal{C}^1(D)$ such that $u = 0$ on ∂D satisfies

$$\int_D u^2 \le C \int_D |\nabla u|^2.$$

(b) By integrating eigenfunctions of (2), conclude that eigenvalues are positive $\lambda > 0$.

(c) Denote the least value of the Rayleigh Quotient

$$\lambda_1(D) = \inf_{u \in \mathcal{C}_0^1(D),\ u \not\equiv 0} \frac{\int_D |\nabla u|^2}{\int_D u^2}.$$

which satisfies $\lambda \ge \frac{1}{C}$ by (a). Assuming that $\lambda_1(D)$ is an eigenvalue of D with Dirichlet boundary conditions, show that if two such domains satisfy $D_1 \subset D_2$ then $\lambda_1(D_1) \ge \lambda_1(D_2)$.