UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 5, 2018.

Instructions: This examination has two parts consisting of six problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [20] points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $p(t)$, $q(t)$ and $r(t)$ be continuous functions defined for $t \in \mathbf{R}$ and $t_0, x_0, x_1 \in \mathbf{R}$. Consider the initial value problem

$$\begin{cases} \ddot{x} + p(t)\,\dot{x} + q(t)\,x = r(t) \\ x(t_0) = x_0, \quad \dot{x}(t_0) = x_1. \end{cases} \tag{1}$$

(a) Let X be a Banach Space with norm $\| \bullet \|$. Let $C \subset X$ be a closed subset. State the Contraction Mapping Theorem (otherwise known as the Banach Fixed Point Theorem) for a map $K : C \to X$.

(b) Carefully reformulate (1) as a fixed point problem. Prove that the fixed point problem is equivalent to the local existence problem for (1).

(c) Using the Contraction Mapping Theorem, show that there is a $\delta > 0$ and a unique function $x(t) \in \mathcal{C}^2([t_0, t_0 + \delta])$ that satisfies your fixed point problem, and hence (1), for $t_0 \le t \le t_0 + \delta$.

A2. Assuming the local existence result in Problem A1 for equation (1), show that in fact a unique solution exists globally for all $t \in \mathbf{R}$.

A3. (a) Let $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ satisfy $f(0) = 0$. State the definitions: the zero solution of

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ x(0) = x_0, \end{cases}$$

is *Stable*, *Asymptotically Stable* and *Unstable*.

(b) Show that the zero solution is asymptotically stable.

$$\begin{cases} \dot{x} = xy - x^3 \\ \dot{y} = -y + x^3 y \end{cases}$$

(c) Suppose that the real $n \times n$ matrix A has an eigenvalue with $\Re e\, \lambda > 0$. Prove that the zero solution of $\dot{x} = Ax$ is unstable.

A4. (a) Let $A(t)$ be a continuous, non-constant, real matrix-valued, $T > 0$ periodic function $A(t + T) = A(t)$. For the system

$$\dot{x} = A(t)x \tag{2}$$

define: *Monodromy Matrix*, *Floquet Multipliers* and *Floquet Exponents.*

(b) If $e^{T\gamma} = \rho$ is a Floquet multiplier, then there is a solution of (2) of the form

$$x(t) = e^{\gamma t} p(t)$$

where $p(t) = \pm p(t+T)$.

(c) Let $a, b \in \mathbf{R}$. Suppose that a Floquet multiplier is $\mu_1 = -1$. What can you say about the boundedness of all solutions on $[0, \infty)$?

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & a \\ b & \sin(\pi t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

A5. Consider Rayleigh's equation

$$\ddot{x} + \dot{x}^3 - 2\mu\dot{x} + x = 0.$$

What kind of bifurcations occur to the equilibrium point of system as the parameter μ is varied? Determine the bifurcation values and describe how the solutions change near the bifurcation point.

A6. Let $f(x) \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ have a non-constant, $T > 0$ periodic trajectory $\gamma(t)$ satisfying $\dot{\gamma}(t) = f(\gamma(t))$.

(a) Define: the periodic solution $\gamma(t)$ is *orbitally stable.*

(b) Define: the *Poincaré Map* for the orbit γ.

(c) Find a periodic solution $\gamma(t)$ to

$$\begin{cases} \dot{x} = x - y - x\left(x^2 + y^2\right), \\ \dot{y} = x + y - y\left(x^2 + y^2\right). \end{cases}$$

Determine the orbital stability of $\gamma(t)$ by computing the derivative of its Poincaré Map.

B. Partial Differential Equations. Do three problems to get full credit

B1. The small longitudinal free vibrations of an elastic bar are governed by the following equation:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)\frac{\partial u}{\partial x}\right].$$

Here u is the longitudinal displacement, ρ is the linear density of the material and E is its Young's modulus. Assume that the bar is constructed by welding together two bars of different (constant) Young's moduli E_1, E_2 and densities ρ_1, ρ_2, respectively. The displacement u is continuous across the junction, which is located at $x = 0$.

(a) Give a weak formulation of the global initial value problem, and use this to derive the following jump condition:

$$E_1 u_x(0-, t) = E_2 u_x(0+, t), \quad t > 0.$$

(b) Let $c_j^2 = E_j/\rho_j$. A left incoming wave $u_I(x,t) = \mathrm{e}^{i(t - x/c_1)}$ produces at the junction a reflected wave $u_R(x,t) = R\,\mathrm{e}^{i(t + x/c_1)}$ and a transmitted wave $u_T = T\,\mathrm{e}^{i(t - x/c_2)}$. Determine the reflection and transmission coefficients R, T, and interpret the result.

B2. Consider the problem of a thin layer of paint of thickness $h(x, t)$ and speed $u(x, y, t)$ flowing down a wall, see Fig. 1. The paint is assumed to be uniform in the z-direction. The balance between gravity and viscosity (fluid friction) means that the velocity satisfies the equation

$$\frac{\partial^2 u}{\partial y^2} = -c,$$

where c is a positive constant. This is supplemented by the boundary conditions

$$u(x, 0, t) = 0, \qquad \left.\frac{\partial u(x, y, t)}{\partial y}\right|_{y=h} = 0.$$

The density of paint per unit length in the x-direction is $\rho_0 h(x, t)$ where ρ_0 is a constant, and the corresponding flux is

$$q(x, t) = \rho_0 \int_0^h u(x, y, t)\, dy.$$

(a) Using conservation of paint, and solving for $u(x, y, t)$ in terms of $h(x, t)$ and y, derive the following PDE for the thickness h:

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0.$$

(b) Set $c = 1$. Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[h^3/3]_-^+}{[h]_-^+}.$$

(c) A stripe of paint is applied at $t = 0$ so that

$$h(x, 0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1. \end{cases}$$

Show that, for small enough t,

$$h = \begin{cases} 0, & x < 0 \\ (x/t)^{1/2}, & 0 < x < t \\ 1, & t < x < S(t) \\ 0, & S(t) < x, \end{cases}$$

where the shock is $x = S(t) = 1 + t/3$.

(d) Explain why this solution changes at $t = 3/2$, and show that thereafter

$$\frac{dS}{dt} = \frac{S}{3t}.$$

B3. Consider the following inhomogeneous initial-Neumann problem:

$$\begin{aligned} u_t &= Du_{xx} + \alpha t x, && 0 < x < \pi, \quad 0 < t, \\ u(x,0) &= 1, && 0 < x < \pi, \\ u_x(0,t) &= u_x(\pi,t) = 0, && 0 < t; \end{aligned}$$

where α is a constant.

(a) Determine the eigenfunctions of the homogeneous equation

$$u_{xx} = \lambda u, \quad 0 \le x \le \pi, \quad u_x(0) = u_x(\pi) = 0.$$

(b) Solve the inhomogeneous initial-Neumann problem by carrying out an eigenfunction expansion of $u(x,t)$ in terms of the eigenfunctions obtained in part (a). That is, denoting the eigenfunctions by v_k, integer k, set

$$u(x,t) = \sum_{k \ge 0} c_k(t) v_k(x),$$

and determine the time-dependent coefficients $c_k(t)$.

(c) Give a physical interpretation of the solution in the limit $\alpha \to 0$.

B4. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, \quad x \in \mathbf{R}.$$

along with initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi.$$

(b) Determine the solution for the initial data $u(x,0) = 1$ if $|x| < a$, $u(x,0) = 0$ if $x > |a|$; $u_t(x,0) = 0$

(c) Determine the solution for the initial data $u(x,0) = 0$; $u_t(x,0) = 1$ if $|x| < a$, $u_t(x,0) = 0$ if $x > |a|$

B5. Suppose that $u(\mathbf{x})$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbf{R}^n$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $\mathbf{x} \in \Omega$ and $r > 0$ is chosen such that $B_r(\mathbf{x}) \subset \Omega$ (ball of radius r centered at $\mathbf{x}$) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) d\mathbf{s},$$

where ω_n is the measure of ∂B_1. Hence show that

$$u(\mathbf{x}) \le \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.