2005 PH.D PRELIM
UU-MATH
DIFFERENTIAL EQUATIONS

INSTRUCTIONS

This examination consists of two parts which are problems from ordinary differentialk equations and partial differential equations, respectively. The examinee should should attempt work on 75% of the problems from each part. All problems are weighted equally and a passing score shall be 70%. Good Luck!

1. Ordinary Differential Equations

1.1. **Problem.** (a) Let D be an open set in $\mathbb{R} \times \mathbb{R}^N$ and let

$$f : D \longrightarrow \mathbb{R}^N$$

be a continuous function such that $f(t,x)$ satisfies a local Lipschitz condition with respect to the x variable. Consider the intial value problem

$$x' = f(t,x), \ x(t_0) = x_0, \ (t_0, x_0) \in D.$$

State what is known about the existence and continuability of solutions of this problem. Be particularly detailed about the behavior of solutions as the time variable approaches the endpoints of existence intervals.

(b)Apply the above results by sketching the solution picture for the scalar differential equation

$$x' = \frac{1}{tx}.$$

(I.e. find D and then consider typical cases.)

1.2. **Problem.** Consider the nonlinear system

$$\begin{aligned} x' &= 2\sin y \\ y' &= 3\sin x - \sin y. \end{aligned}$$

Provide a complete phase plane analysis of this system and give justifications for your reasoning.

1.3. **Problem.** Let A be an $N \times N$ matrix, none of whose eigenvalues have zero real part. Show that if $b : (-\infty, \infty) \to \mathbb{R}^N$ is a continuous function which is periodic of period 1, ($b(t+1) = b(t), \ t \in \mathbb{R}$), then the equation

$$x' = Ax + b(t),$$

has a unique periodic solution of period 1.

1.4. **Problem.** In an isolated region of the Canadian Northwest Territories, a population of white wolves, x, and one of silver foxes, y, compete for survival (for each population one unit represents 100 individuals). They have a common, limited food supply, which consists mainly of mice. A mathematical biologist models the dynamics of the competing species by the nonlinear system

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy. \end{aligned}$$

Can the two species survive together according to this model?

2. Partial Differential Equations

2.1. **Problem.** Let Ω be an open subset of $\mathbb{R}^N$.

(1) Define what is meant by a distribution on Ω.
(2) Show that all functions in $L^1_{loc}(\Omega)$, are, by an appropriate identification, distributions.
(3) Give an example of a distribution which is not of this type.
(4) Give the derivative formula for differentiating distributions.
(5) Give a definition of the L^2 Sobolev spaces $H^m(\Omega)$ and $H_0^m(\Omega)$. (Other notations used are the L^2 Sobolev spaces $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$.)

2.2. **Problem.** State and proof Poincaré's inequality for $H_0^1(\Omega)$ and use it to obtain two equivalent inner products for this space.

2.3. **Problem.** State the Lax-Milgram theorem and give a brief sketch of a proof. Use it to establish the existence and uniqueness of solutions (in the sense of distributions) of the boundary value problem

$$\begin{aligned} -\Delta u &= h, \ in\ \Omega \\ u &= 0, \ on\ \partial\Omega, \end{aligned}$$

where $h \in L^2(\Omega)$. Also, in your discussion, tell, in what sense the boundary data are assumed.

2.4. **Problem.** State the Phillips version of the Hille-Yosida theorem and illustrate it by a discussion of the heat equation

$$u_t - \Delta u = h(t, x).$$

DEPARTMENT OF MATHEMATICS
University of Utah

Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
Winter 2005

Instructions: The examination has two parts. You are to *work a total of seven problems from part A and part B.* At least three of these problems must be from part A, and at least three must be from part B. *Clearly indicate which problems you wish to be graded.*

To receive maximum credit, solutions must be clearly, carefully, and concisely presented and should contain an appropriate level of detail. Each problem is worth 20 points. A passing score is 84.

A. Ordinary Differential Equations

A1. Contraction Mapping Theorem. Show that the problem

$$\dot{x}(t) = f(x(t)) + \frac{1}{a}\int_0^a x(s)\,ds, \quad t \in (0,a)$$
$$x(0) = 0,$$

admits a unique solution $x \in C([0,a],\mathbb{R}^n)$ for $a > 0$ sufficiently small. Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}^n$.

A2. Limit Sets. Let $\varphi_t(x_0)$ be the flow of the differential equation

$$\dot{x} = f(x),$$
$$x(0) = x_0$$

where $f \in C^1(\mathbb{R}^n)$ satisfies a global Lipschitz condition, and $x_0 \in \mathbb{R}^n$.

(a) Given a trajectory Γ of φ_t, define the ω-limit set $\omega(\Gamma)$.

(b) Prove that $\omega(\Gamma)$ is closed and invariant with respect to the flow φ_t.

A3. Periodic Orbits. Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2),\\ \dot{y} &= x + y(1 - x^2 - y^2),\\ \dot{z} &= z.\end{aligned}$$

(a) Find the invariant manifolds for this system.

(b) Find the unique periodic orbit Γ for this system.

(c) Find the global stable manifold $W^s(\Gamma)$ and global unstable manifold $W^u(\Gamma)$ *with respect to the periodic orbit* Γ.

A4. Linear Systems. Consider the linear initial value problem

$$\begin{aligned}\dot{x} &= Ax,\\ x(0) &= x_0,\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$ is constant, and $x_0 \in \mathbb{R}^n$ is given.

(a) Prove that the matrix exponential e^{At} exists and satisfies $\|e^{At}\| \leq e^{\|A\||t|}$, where $\|\cdot\|$ is the operator norm.

(b) Prove *directly* that $x(t) = e^{At}x_0$ is the unique solution to the initial value problem above.

A5. Floquet Theory. Consider the system

$$\dot{x}(t) = A(t)x(t),$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and T-periodic, i.e., $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Let $\Phi(t)$ be the fundamental matrix solution for this system. Assume that there exists a complex-valued matrix B such that $e^{BT} = \Phi(T)$.

(a) Explain in detail how solutions $x(t)$ to the periodic system above are related to solutions $y(t)$ of the constant-coefficient problem $\dot{y} = By$, and prove your assertions.

(b) Explain what the eigenvalues of $A(t)$ or B have to do with stability of the solutions $x(t)$.

A6. Dependence on Initial Conditions. Consider the scalar initial value problem

$$y'' = y - y^3,$$
$$y(0) = a,$$
$$y'(0) = b.$$

Denote the dependence of $y(t)$ on its initial conditions by $y(t; a, b)$. Without explicitly solving for $y(t; a, b)$, find $\frac{\partial y}{\partial a}(t; 0, 0)$ and $\frac{\partial y}{\partial b}(t; 0, 0)$, then write down a first-order approximation for $y(t; a, b)$ which is valid for small $|a|, |b|$.

B. Partial Differential Equations.

B1. Variational Principles. Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and unit outward normal η. Consider the problem

$$\Delta u - \epsilon u = f, \qquad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \partial\Omega,$$

where $f \in L^2(\Omega)$ satisfies $\int_\Omega f = 0$.

(a) Prove that for each real constant $\epsilon > 0$, there exists a unique weak solution $u_\epsilon \in H^1(\Omega)$.

(b) In the case $\epsilon = 0$, find an additional condition on u which guarantees a unique weak solution u_0, such that the solutions u_ϵ from part (a) converge to u_0 as $\epsilon \to 0^+$. Prove existence and uniqueness of u_0 (with the extra condition), and prove $u_\epsilon \to u_0$ in $H^1(\Omega)$ as $\epsilon \to 0^+$.

B2. Heat Equation. Given the constant $\beta \in \mathbb{R}$, consider the initial value problem

$$u_t = \beta \Delta u, \quad x \in \mathbb{R}^n, \quad t > 0,$$
$$u(x,0) = g(x), \quad x \in \mathbb{R}^n.$$

(a) Assume $\beta > 0$ and let $g \in C(\mathbb{R}^n)$ be bounded. Prove that there exists a bounded solution $u(x,t)$ which is C^∞ for all $t > 0$. Under what condition are solutions unique?

(b) Assume $\beta < 0$ and let $g \in C^\infty(\mathbb{R}^n)$ be bounded. Are solutions $u(x,t)$ necessarily bounded and continuous for $t > 0$? Prove, or provide a counterexample.

B3. Laplace Equation. Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $u \in C(\Omega)$ satisfies the *mean value property*: for every ball $B_r(\xi) = \{x \in \mathbb{R}^n : |x - \xi| < r\}$ with $\overline{B_r(\xi)} \subset \Omega$,

$$u(\xi) = \frac{1}{\omega_n} \int_{|x|=1} u(\xi + rx)\, dS_x,$$

where ω_n is the measure of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$. Prove that $\Delta u = 0$ in Ω.

B4. Conservation Laws. Consider the first-order equation

$$(G(u))_x + u_y = 0,$$

where $G \in C^\infty$, with initial condition $u(x,0) = h(x)$.

(a) Using the method of characteristics, derive the general solution of the initial value problem. Assuming $h \in C^\infty$, find a condition on $G'(h(x))$ which implies the solution $u(x,y)$ will develop a singularity in finite time $y > 0$.

(b) Set $G(u) = \frac{1}{2}u^2$. Given $u_0 \in \mathbb{R}$, find explicit weak solution(s) for

$$h(x) = \begin{cases} u_0, & x < 0, \\ 0, & x > 0, \end{cases}$$

Show that your solutions are weak solutions. Which properties do your solutions exhibit: shocks, rarefaction, nonuniqueness?

B5. Distributions. Let $\Omega \subset \mathbb{R}^n$ be an open set.

(a) Let $\{f_n\} \subset C(\Omega)$, with f_n converging to $f \in C(\Omega)$ pointwise, i.e., $f_n(x) \to f(x)$ for all $x \in \Omega$. Does it follow that f_n converges to f in the sense of distributions? Prove or give a counterexample.

(b) Let $\{\mu_n\} \subset \mathcal{D}'(\Omega)$ with μ_n converging to $\mu \in \mathcal{D}'(\Omega)$ in the sense of distributions. Let α be a fixed multi-index. Does it follow that the derivatives $\partial^\alpha \mu_n$ converge to $\partial^\alpha \mu$ in the sense of distributions? Prove or give a counterexample.

(c) Find a sequence of $L^1(\mathbb{R})$ functions which do not converge in $L^1(\mathbb{R})$ but which do converge in the sense of distributions.

B6. Wave Equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the initial boundary value problem

$$\begin{aligned} u_{tt} &= \Delta u, \quad x \in \Omega,\ t > 0, \\ u(x,0) &= g(x), \quad x \in \Omega \\ u_t(x,0) &= 0, \quad x \in \Omega, \\ u(x,t) &= 0, \quad x \in \partial\Omega,\ t > 0. \end{aligned}$$

Show that there exists a sequence of initial conditions $\{g_j\} \subset C_0^2(\overline{\Omega})$ and a sequence of *frequencies* $\{\omega_j\} \subset \mathbb{R}$ with $\omega_j^2 \to \infty$, such that each solution $u^{(j)}(x,t)$ corresponding to initial condition g_j satisfies

$$u_{tt}^{(j)}(x,t) = -\omega_j^2 u^{(j)}(x,t), \quad x \in \Omega,\ t > 0.$$

Moreover, prove that there is at most a countable set of ω for which $u_{tt} = -\omega^2 u$, no matter what initial conditions g are chosen.

DEPARTMENT OF MATHEMATICS
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Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
Autumn 2004

Instructions: The examination has two parts. You are to *work a total of seven problems from part A and part B.* At least three of these problems must be from part A, and at least three must be from part B. If you work more than the required number of problems, **clearly indicate which problems you wish to be graded.**

To receive maximum credit, solutions must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 84.

A. Ordinary Differential Equations

A1. Contraction Mapping. Consider the initial value problem

$$\dot{x} = f(x),$$
$$x(0) = x_0,$$

where f satisfies for all $x, y \in \mathbb{R}^n$

$$|f(x) - f(y)| \leq (1 + |x| + |y|)|x - y|.$$

Use the contraction mapping principle to find a condition on $|x_0|$ which guarantees local existence and uniqueness of solutions. How does the local time interval upon which existence is obtained depend on $|x_0|$?

A2. Global Existence. Let $\varphi_t(x_0)$ be the flow of the differential equation

$$\dot{x} = f(x),$$
$$x(0) = x_0$$

where $f \in C^1(E)$, $E \subset \mathbb{R}^n$ is open, and $x_0 \in E$. Prove that if the system has a Liapunov function V with $\dot{V}(x) = \frac{d}{dt}V(\varphi_t(x))|_{t=0} \leq 0$ for all $x \in E$, and the set $S = \{x \in E : V(x) \leq 1\}$ is compact, then solutions $\varphi_t(x_0)$ with $x_0 \in S$ exist for all positive times $t > 0$.

A3. Periodic Orbits. A model of an autocatalytic chemical reaction is given by the Brusselator system

$$\begin{aligned} \frac{dx}{dt} &= 1 - 4x + x^2 y, \\ \frac{dy}{dt} &= 3x - x^2 y. \end{aligned} \tag{1}$$

Show that the trapezoidal region with vertices $(\frac{1}{4}, 0)$, $(13, 0)$, $(1, 12)$, $(\frac{1}{4}, 12)$, is an invariant set for (1). Show that the system has a nonconstant periodic trajectory in $\mathbb{R}^2$.

A4. Linear Systems. Suppose A is a real $n \times n$ matrix whose eigenvalues λ_i satisfy $\Re\mathfrak{e}\, \lambda_i \le 0$. Show that there are constants $C < \infty$ and $k \le n - 1$ so that any solution $\mathbf{x}(t) \in \mathbb{R}^n$ of the initial value problem

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= A\mathbf{x}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned}$$

satisfies

$$|\mathbf{x}(t)| \le C\left(1 + |t|^k\right) |\mathbf{x}_0| \qquad \text{for all } t \in \mathbb{R} \text{ and } \mathbf{x}_0 \in \mathbb{R}^n.$$

A5. Linearization. Assume $g \in C^\infty(\mathbb{R}^n)$ has a strict local minimum at the origin. Consider the second-order system

$$\ddot{x} + B\dot{x} + \nabla g(x) = 0,$$

where $x(t) \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

(a) Rewrite the system as a first-order system.

(b) Write the linearization of the system at the origin.

(c) Prove that the origin is an asymptotically stable fixed point.

A6. Stable and Unstable Manifolds. Find the stable and unstable manufolds in the neighborhood of the origin for the system

$$\begin{aligned} x' &= -x \\ y' &= y + x^2 \\ z' &= z + y^2. \end{aligned}$$

How are the stable and unstable manifolds for the linearization of this system related to those of the nonlinear system?

B. Partial Differential Equations.

B1. Variational Principles. Suppose $\Omega \in \mathbb{R}^n$ is a smooth, bounded domain. Show that there is an $\varepsilon > 0$ depending on n and Ω such that if $f, g \in C(\overline{\Omega})$ are continuous functions such that $f \leq \varepsilon$ then the equation

$$\begin{aligned} \Delta u + f(x)\, u &= g(x), && \text{for } x \in \Omega, \\ u(x) &= 0, && \text{for } x \in \partial\Omega. \end{aligned}$$

has a weak solution $u \in H_0^1(\Omega)$.

B2. Heat Equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let $f \in C^2(\overline{\Omega})$, $g \in C^\infty(\partial\Omega)$, and assume $f = g$ on $\partial\Omega$. Let $u(x,t)$ solve

$$\begin{aligned} u_t &= \Delta u, && x \in \Omega,\ t > 0 \\ u(x,0) &= f(x), && x \in \overline{\Omega} \\ u(x,t) &= g(x), && x \in \partial\Omega,\ t > 0. \end{aligned}$$

Prove that $\lim_{t\to\infty} \int_\Omega |u(x,t) - v(x)|^2\, dx = 0$ where v solves

$$\begin{aligned} \Delta v &= 0, && \text{in } \Omega, \\ v &= g, && \text{on } \partial\Omega. \end{aligned}$$

B3. Laplace Equation. Let $\Omega \subset \mathbb{R}^n$ be a domain. Let Ω_1 be a bounded subdomain with $\overline{\Omega}_1 \subset \Omega$.

(a.) State the Harnack Inequality for harmonic functions.

(b.) Let $\{u_n\}$ be a monotone increasing sequence of nonnegative harmonic functions on Ω. Suppose that for some point $\xi \in \Omega_1$, the sequence of real numbers $\{u_n(\xi)\}$ is bounded. Prove that $\{u_n\}$ converges uniformly on Ω_1 to a harmonic function. You may use standard theorems about convergence of sequences of harmonic functions without proof.

B4. Conservation Laws. A model for flow of fluid in an infinite pipe says that the density $\rho(x,t)$ satisfies

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = -\rho^2, \qquad \rho(x,0) = \begin{cases} 1, & \text{if } x \le 0, \\ \frac{1}{2}, & \text{if } x > 0. \end{cases}$$

(a) Find a solution ρ.

(b) Define what it means for your solution ρ from part (a) to be a weak solution of the system. (Do not carry out the calculations to prove that ρ is a weak solution.)

B5. Distributions. Suppose $f(\xi), g(\xi) \in C_0^2(\mathbb{R})$ are compactly supported twice continuously differentiable functions. Then

$$u(x,t) = f(x+t) + g(x-t) \tag{2}$$

is a classical solution of the wave equation

$$u_{tt} - u_{xx} = 0 \qquad \text{for } -\infty < x, t < \infty. \tag{3}$$

Show that if $f(\xi), g(\xi) \in C_0(\mathbb{R})$ are merely compactly supported continuous functions, then (2) still defines a solution to (3) *in the sense of distributions.*

B6. Wave Propagation. Consider the equation

$$u_{tt} - u_{xx} + \lambda u = 0, \quad x \in \mathbb{R},\ t > 0,$$

where $\lambda > 0$ is fixed.

(a) Explain what is meant by a *dispersive* solution, and find a *dispersion relation* for the equation.

(b) Define an "energy" functional $E(t)$ for this equation and show that each solution $u(x,t)$ with finite initial energy *conserves energy.* State conditions under which uniqueness of solutions holds.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
Winter 2004

Instructions: The examination has two parts. You are to work four problems from part A and four problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 96.

A. Ordinary differential equations: Do four problems for full credit.

A1. Picard's Existence Theorem.

(a.) Suppose $f(x,t) \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Show that there is an $\epsilon > 0$ so that the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases}$$

has a unique solution $y \in C^1([t_0 - \epsilon, t_0 + \epsilon], \mathbb{R}^n)$.

(b.) Suppose that the hypothesis on f was replaced by $f(x,t) \in C^0(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$. Would all or part of the conclusions in (a.) continue to hold?

A2. Constant Coefficients. Suppose that A is a real $n \times n$ matrix all of whose eigenvalues λ have strictly negative real part $\Re e\, \lambda < 0$. Suppose

(1.) $$\frac{dx}{dt} = Ax, \qquad x(0) = x_0.$$

(a.) Show that there is a decreasing function $\delta(t) > 0$ such that $\delta(t) \to 0$ as $t \to \infty$ and such that every solution of (1.) satisfies

$$|x(t)| \le \delta(t)|x_0| \qquad \text{as } t \to \infty.$$

(b.) Define: z is *Liapunov Stable.* Define: z is *Asymptotically Stable.* Show that the zero solution $\mathbf{z}(t) = 0$ of the autonomous system is asymptotically stable.

A3. Periodic Orbits. Show that the following translated Fitzhugh-Nagumo system modelling a nerve axon has a nonconstant periodic trajectory in $\mathbb{R}^2$.

$$\begin{cases} \dfrac{dx}{dt} = 0.9x(1-x^2) - y \\ \dfrac{dy}{dt} = 0.5x - 0.5y \end{cases}$$

A4. Nonhomogeneous Equations. Let $v_0, x_0 \in \mathbb{R}^n$ and A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathfrak{e}\, \lambda_i \le -\varepsilon_0$ for all $i = 1, \dots, n$ and for some $\epsilon_0 > 0$. Show that solution of the initial value problem (2.) is bounded for $t \ge 0$ as long as it exists. Argue that therefore it exists for all time.

$$(2.) \qquad \begin{cases} \dfrac{dx}{dt} = Ax + e^{-t}(x + v_0) \\ \quad x(0) = x_0. \end{cases}$$

A5. Periodic Coefficients. Prove that if $|\varepsilon| \ne 0$ is small enough, then all solutions are bounded

$$\ddot{u} + [1 + \varepsilon \sin(3t)]\, u = 0$$

[Convert to a system of equations. Give a condition for the boundedness of all solutions. Verify that the condition is satisfied in this case.]

A6. Three Species Predator-Prey Model. Consider the following Lotka-Volterra model for populations of three species $x, y, z \ge 0$.

$$\begin{cases} \dot{x} = x(4 - y - z), \\ \dot{y} = y(1 - z), \\ \dot{z} = z(x + y - 5); \\ x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0. \end{cases}$$

The only stationary point for this system with all positive coordinates is at $P = (2, 3, 1)$. Show that there is an open $\mathcal{U} \subset \mathbb{R}^2$ such that $(2, 3) \in \mathcal{U}$, a function $f(x, y) \in C^1(\mathcal{U})$ and $\varepsilon > 0$ so that for initial points which are ε-close to P, that is for $|(x_0, y_0, z_0) - P| < \varepsilon$, if $z_0 = f(x_0, y_0)$ then the solution converges to P as $t \to \infty$, but if $z_0 \ne f(x_0, y_0)$ then the solution eventually goes away from P, that is for some $t_0 > 0$, $|(x(t_0), y(t_0), z(t_0)) - P| \ge \varepsilon$.

B. Partial Differential Equations. Do four problems to get full credit.

B1. Elliptic Equations. Suppose $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Let $a^i(x) \in C^\infty(\overline{\Omega})$ be smooth functions. Give conditions on $a^i(x)$ so that for all $f \in L^2(\Omega)$

there exists a weak solution to the boundary value problem

$$\begin{cases} \Delta u + \sum_{i=1}^{n} a^i(x) D_i u = f(x), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \partial\Omega. \end{cases}$$

[Here $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $D_i = \frac{\partial}{\partial x_i}$ is the partial derivative.]

B2. Heat Equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and $0 < T < \infty$, $c \in \mathbb{R}$ be constants. Let $\psi(x) \in C_0^\infty(\overline{\Omega})$. Let $u \in C^{2,1}(\overline{\Omega} \times [0,T])$ be a solution to

$$(3.) \qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + cu, & \text{if } (x,t) \in \Omega \times [0,T], \\ u(x,0) = \psi(x), & \text{if } x \in \Omega, \\ u(x,t) = 0, & \text{if } (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

(a.) Show that the solution $u(x,t)$ is bounded on $\bar{\Omega} \times [0,T]$ in terms of c, $\sup_{x\in\Omega} |\psi(x)|$ and t. [You may assume $c \le 0$ but it is true for any c.]

(b.) Show that such solutions of (3.) depends continuously on ψ.

B3. Vibrating Plate. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $\varphi, \psi \in C_0^\infty(\Omega)$. Suppose that there exists a solution $u(x,t) \in C^4(\bar{\Omega} \times [0,T])$ to the initial-boundary value problem for the vibrating clamped plate equation

$$(4.) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta\Delta u = 0, & \text{if } x \in \Omega \text{ and } t \ge 0; \\ u(x,0) = \varphi(x), & \\ & \text{for } x \in \Omega. \\ \frac{\partial u}{\partial t}(x,0) = \psi(x), & \\ u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0, & \text{if } x \in \partial\Omega \text{ and } 0 \le t \le T. \end{cases}$$

(a.) Show that the total energy

$$\mathcal{E}(t) = \int_\Omega \left\{ \left(\frac{\partial u}{\partial t}(x,t) \right)^2 + \left(\Delta u(x,t) \right)^2 \right\} dx$$

is bounded in terms of φ, ψ and t.

(b.) Show that such a solution of (4) is unique.

B4. Conservation Laws. Find a solution to (5.) and prove that it is a weak solution.

$$(5.) \qquad \begin{cases} \frac{\partial u}{\partial t} + (1+u^2)\frac{\partial u}{\partial x} = 0, & \text{for } x \in \mathbb{R} \text{ and } t \ge 0. \\ u(x,0) = \begin{cases} 1, & \text{if } x < 0, \\ 0, & \text{if } x \ge 0. \end{cases}, & \text{for } x \in \mathbb{R}. \end{cases}$$

B5. Distributions. Let $k(x,y) = \dfrac{1}{2\pi} \log|x-y|$, where $x, y \in \mathbb{R}^2$.

(a.) Show that $k(\cdot, y)$ is a distribution on $\mathbb{R}^2$ for each $y \in \mathbb{R}^2$.

(b.) Show that as distributions, $\Delta_x k(x,y) = \delta_y(x)$ for all $y \in \mathbb{R}^2$.
[$\delta_y(x) = \delta(x-y)$ and δ is the Dirac delta function.]

B6. Fundamental Elliptic Estimate. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $H^k(\Omega)$ denote the Sobolev space of functions whose distributional derivatives up to the k-th order are in $L^2(\Omega)$. Show that if for some constant λ, $u \in H^1(\Omega)$ is a weak solution to

$$\Delta u + \lambda u = 0$$

then $u \in C^\infty(\Omega)$. [You may assume the fundamental elliptic interior estimate: If $\Omega' \subset\subset \Omega$ is a compactly contained subdomain, then there is a finite constant $C = C(\Omega, \Omega', k, n)$ so that

$$||u||_{k+2,\Omega'} \le C\,(||\,\Delta u||_{k;\Omega} + ||u||_{0;\Omega})$$

for all $u \in H^{k+2}(\Omega)$. Moreover, if $f \in H^k(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution to $\Delta u = f$, then $u \in H^{k+2}(\Omega')$.]

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
Autumn 2003

Instructions: The examination has two parts consisting of six problems each. You are to work four problems from part A and four problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 96.

A. Ordinary differential equations: Do four problems for full credit.

A1. Peano Existence Theorem. Suppose $f(x,t) \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Consider the system of ODE's

$$(\clubsuit) \qquad \begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases}$$

(a.) Find an example that shows that the solution of ($\clubsuit$) may not be unique. Give another example that shows that the solution may not exist for all time.

(b.) Prove the Peano Existence theorem: Suppose $f(x,t) \in C(\mathbb{R}^n \times \mathbb{R})$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Then there is an $\varepsilon > 0$ and a function $y \in C^1([t_0, t_0 + \varepsilon], \mathbb{R}^n)$ which solves ($\clubsuit$).

A2. Picard's Existence Theorem. Suppose $f(x,t) \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Suppose that there is a continuous function $m(t) \in C(\mathbb{R})$ such that $|f(x,t)| \le m(t)(1 + |x|)$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Show that the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases}$$

has a unique solution $y \in C^1(\mathbb{R}, \mathbb{R}^n)$ which exists for all time.

A3. Periodic Orbits. Show that the following system has a nonconstant periodic trajectory.

$$\begin{cases} \dfrac{dx}{dt} = \left[3 - \sin\left(x^2 + y^2\right)\right] y \\ \dfrac{dy}{dt} = -3x + \cos\left(x^2 + y^2\right) y \end{cases}$$

A4. Stability of Solutions. The following questions have to do with the stability of the zero solution $z(t) = 0$ to a system of ODE's.

$$y' = f(t, y),$$

where $f(t,x) \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ such that $f(t,0) = 0$ for all t.

(a.) Define: z is *Liapunov Stable.* Define: z is *asymptotically stable.*

(b.) Give an example of an ODE system such that the zero solution is Liapunov stable but not asymptotically stable.

(c.) Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathfrak{e}\, \lambda_i < 0$ for all $i = 1, \ldots, n$. Let $g(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that there are constants $1 < \beta$, $c < \infty$ so that $|g(x)| \le c|x|^\beta$ for all $x \in \mathbb{R}^n$. Show that the zero solution $z(t)$ is asymptotically stable in the equation

$$y' = Ay + \sin(t) g(y).$$

A5. Periodic Coefficients. Prove that the equation

$$\ddot{u} + \left(\sin^2 t\right) \dot{u} + (1 + \sin 2t) u = 0$$

does not have a fundamental set of periodic solutions.

A6. Two Species Competition Model. Consider the simplified competition model for two populations $x(t) \ge 0$ and $y(t) \ge 0$,

$$\begin{cases} \dot{x} = x(1 - x - 3y), \\ \dot{y} = 3y(1 - y - 2x). \end{cases}$$

Its equilibrium points are $(0,0)$, $(0,1)$, $(1,0)$ and $(.4, .2)$. The equilibria $(0,1)$ and $(1,0)$ are stable, but $(0,0)$ is unstable. This system has a seperatrix, a curve which divides the first quadrant of the plane into two basins of attraction corresponding to the two stable equilibria. Thus one species or the other wins out, depending which has the starting advantage. As a first step in showing that $(.4, .2)$ lies on the seperatrix, prove that in an open neighborhood $\mathcal{O}$ of $(.4, .2)$ there is a C^1 curve σ through $(.4, .2)$ with the property that trajectories starting on one side of the σ in $\mathcal{O}$, stay on that side of σ while in $\mathcal{O}$.

B. Partial Differential Equations. Do four problems to get full credit.

B1. Conservation Laws. Consider the first order equation

$$(\diamondsuit) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (\tan u)\,\dfrac{\partial u}{\partial x} = 0, & \text{for } x \in \mathbb{R} \text{ and } t \geq 0. \\ u(x,0) = \varphi(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

(a.) Find a solution $u(x,t)$ to $(\diamondsuit)$ where $\varphi(x) = \operatorname{Arctan}(x)$.

(b.) For

$$\varphi(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{\pi}{4}, & \text{if } x > 0, \end{cases}$$

find two weak solutions to $(\diamondsuit)$. Prove that one of them is a weak solution.

(c.) State the entropy (*Rankine-Hugoniot*) condition. Which weak solution of part (b.) satisfies this condition?

B2. Distributions. Let $\Omega \subset \mathbb{R}^n$ be an open set and $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

(a.) Let $\delta \in \mathcal{D}'(\mathbb{R}^n)$ be the Dirac delta function. Fix a multiindex β. Show that the derivative $D^\beta \delta$ is a distribution.

(b.) Let $C_0^\infty(\Omega)$ denote the space of compactly supported smooth functions on Ω. Briefly explain why there exist nonzero $\varphi \in C_0^\infty(\Omega)$.

(c.) Suppose $\rho \in C_0^\infty(\mathbb{R}^n)$ is such that $\rho \geq 0$, the support is contained in the unit ball around zero, $\mathbf{spt}\,\rho \subset B_1(0)$, and $\int_{\mathbb{R}^n} \rho\,dx = 1$. Let $\eta_h = h^{-n}\rho(x/h)$. Show that as distributions, $\eta_h \to \delta$ as $h \to 0$.

B3. Heat Equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $0 < T < \infty$ be a constant, $c(x) \in C^\infty(\overline{\Omega})$ such that $c(x) \leq 0$ for all x and $\psi(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$. Let $u \in C^{2,1}(\overline{\Omega} \times [0,T])$ be a solution to

$$(\heartsuit) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + c(x)u, & \text{if } (x,t) \in \Omega \times [0,T], \\ u(x,t) = \psi(x,t), & \text{if } (x,t) \in \Omega \times \{0\} \text{ or if } (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

(a.) Prove the **Maximum Principle:** If $\psi(x,t) \geq 0$ then the solution satisfies $u(x,t) \geq 0$.

(b.) Show that such solutions of $(\heartsuit)$ are unique.

B4. Wave Equation. Let $f(x,t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ and $R > 0$ be fixed. Suppose that the support of f is contained in the cylinder $\mathbf{spt}\,f \subset \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq R \text{ and } t \geq 0\}$. Show that there exists a solution $u(x,t)$ to the inhomogeneous wave equation (♠). Show that the solution satisfies $u(x,t) = 0$ whenever $|x| \geq ct + R$.

$$(\spadesuit)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\,\Delta u + f(x,t), & \text{if } x \in \mathbb{R}^3 \text{ and } t \geq 0 \\ u(x,0) = 0, & \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & \text{for } x \in \mathbb{R}^3. \end{cases}$$

B5. Laplace's Equation. Let $\varphi(y_1,y_2) \in C_0\left(\mathbb{R}^2\right)$ be continuous with compact support. Let

$$u(x) = \frac{x_3}{2\pi}\int_{\mathbb{R}^2} \frac{\varphi(y_1,y_2)\,dy_1\,dy_2}{\left[(y_1-x_1)^2+(y_2-x_2)^2+x_3^2\right]^{3/2}},$$

for $x \in \mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in R^3 : x_3 > 0\}$, the open upper three space. Show that $u \in C^2\left(\mathbb{R}^3_+\right) \cap C\left(\overline{\mathbb{R}^3_+}\right)$ and solves the boundary value problem for Laplace's equation

$$\begin{cases} \Delta u = 0, & \text{if } x \in \mathbb{R}^3_+, \\ u(x_1,x_2,0) = \varphi(x_1,x_2), & \text{if } (x_1,x_2) \in \mathbb{R}^2. \end{cases}$$

$$\left[\text{You may assume } \int_{\mathbb{R}^2} \frac{x_3\,dy_1\,dy_2}{\left[(y_1-x_1)^2+(y_2-x_2)^2+x_3^2\right]^{3/2}} = 2\pi.\right]$$

B6. Fundamental Elliptic Estimate. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be the n-dimensional torus. Let $u \in C^\infty(\mathbb{T}^n)$, which is the same as saying u is a complex valued smooth function on $\mathbb{R}^n$ which is 2π-periodic in each coordinate. Then for $k \in \mathbb{R}$, the Sobolev norm is given by

$$\|u\|_k^2 = \sum_\alpha \left(1+|\alpha|^2\right)^k |u_\alpha|^2$$

where $u \sim \sum_\alpha u_\alpha e^{i\alpha\cdot x}$ is the Fourier series expansion of u such that the sum runs through all multiindices $\alpha \in \mathbb{Z}^n$. Let $H^k(\mathbb{T}^n)$ be the $\|\cdot\|_k$-completion of the trigonometric polynomials on $\mathbb{T}^n$ (finite sums of the form $p(x) = \sum_{|\alpha|\leq N} c_\alpha e^{i\alpha\cdot x}$.)

(a.) Show that for $k = 2$, there is a constant $c(k,n) < \infty$ so that whenever $u \in H^{k+2}(\mathbb{T}^n)$ then

$$(\dagger)\qquad \|u\|_{k+2} \leq c\big(\|\Delta u\|_k + \|u\|_0\big).$$

(b.) Supposing that (†) holds for all k, prove that if $f \in C^\infty(\mathbb{T}^n)$ such that $\int_{\mathbb{T}^n} f\,dx = 0$ then there is $u \in C^\infty(\mathbb{T}^n)$ such that

$$\Delta u = f \qquad \text{on } \mathbb{T}^n.$$

Differential Equations Preliminary Examination

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

August 2002

Instructions: The examination consists of two parts. **Part A** consists of exercises concerning *Ordinary differential Equations* and **Part B** consists of exercises concerning *Partial Differential Equations*.

To obtain full credit, please complete three exercises from part A and three exercises from part B, a total of six (6) exercises. All exercises are equally weighted and partial credit applies to each. A passing score is 60% of the total possible score.

Sound and detailed solutions are expected, but bear in mind that too many details are time-consuming. Judgement of what is essential will be an important factor in determining the final score.

Date: ____________________ **UofU ID No:** ____________________

Part A

Ordinary Differential Equations

Do three (3) exercises from Part A for full credit.

Exercise A-1. Consider the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0$$

on the set X: $|t - t_0| \leq a$, $|x - x_0| \leq b$. Below, *outline the proof* means a sequence of statements and lemmas and very brief details. In the outlines, you may freely use the statement and details of proof from the Picard–Lindelöf existence and uniqueness theorem, which says that there exists a unique solution $x(t)$ defined on the interval $|t - t_0| \leq \alpha = \min(a, b/m)$, $m = \max\{|f(t, x)| : (t, x) \in X\}$, provided f satisfies some special conditions. Select with a check–mark $\boxed{\surd}$ and solve either A-1.I or A-1.II:

☐ A-1.I. Let $f(t, x)$ be measurable in t for each fixed x, continuous in x for fixed t a.e. and for some $m \in L^1(t_0 - a, t_0 + a)$, $|f(t, x)| \leq m(t)$ whenerever $(t, x) \in X$. Outline the proof of Carathéodory's existence theorem: There exists an absolutely continuous function $x(t)$ defined on an interval $|t - t_0| \leq h$ with $h \leq a$ such that $x'(t) = f(t, x(t))$ a.e., and $x(t_0) = x_0$.

☐ A-1.II. Assume that f is continuous on X. Outline the proof of the Peano existence theorem: There exists for some $h > 0$ at least one solution $x(t)$ defined on the interval $|t - t_0| < h$.

Exercise A-2. Assume that the eigenvalues of a real $n \times n$ matrix A have negative real part. Select with a check–mark $\boxed{\surd}$ and solve either A-2.I or A-2.II:

☐ A-2.I.

(a) Prove that positive constants M and α exist such that for all $x \in R^n$ and $t \geq 0$

$$\|e^{At}x\| \leq M\|x\|e^{-\alpha t}.$$

(b) Prove that the zero solution of $u' = Au$ is asymptotically stable.

☐ A-2.II. Prove that for $h(t)$ continuous and T-periodic, the equation $u' = Au + h(t)$ has a unique T-periodic solution $u(t)$.

Exercise A-3. Let the system $x' = f(x)$ define a C^1 flow ϕ_t on the open set E contained in R^n. Prove that the positive limit set $\Gamma^+(v)$ of a trajectory $x(t)$ with $x(0) = v$ is closed. Then, select with a check–mark $\boxed{\surd}$ and solve either A-3.I or A-3.II:

☐ A-3.I. Consider the autonomous planar dynamical system

$$x' = 6x - 2xy - 6x^2, \quad y' = -7y + 2xy - y^2.$$

(a) Compute the four rest points (=equilibrium or stationary points) of the system and the linearization about each rest point.

(b) Make a table in which each row contains a rest point, the classification stable or unstable, and the geometric classification node, spiral, center or hyperbolic point.

(c) Sketch the phase diagram showing the rest points and the local behavior of solution curves (rough and brief!).

☐ A-3.II. Let $r^2 = x^2 + y^2$, $w = (r^2 - 1)(r^2 - 4)$ and consider the planar system $x' = -y + xw$, $y' = x + yw$ ($r' = rw$, $\theta' = 1$ in polar coordinates). Apply the Poincaré-Bendixson theorem to prove that $r = 1$ and $r = 2$ are limit cycles (a periodic orbit γ with $\gamma = \Gamma^+(v)$ or $\gamma = \Gamma^-(v)$ for nearby v).

Exercise A-4. Select with a check–mark [√] and solve either A-4.I or A-4.II:

[ ] A-4.I: Assume $f : D \to R^N$ is continuous and f is bounded by a constant m on a subdomain $D_0 \subset D$. Let $u(t)$ be a solution of $u' = f(t, u)$ with $(t, u(t)) \in D_0$ on $a < t < b$.

(a) Prove that $u(t)$ satisfies a Lipschitz condition $|u(t_1) - u(t_2)| \leq m|t_1 - t_2|$.

(b) Prove that $u(t)$ has one-sided limits at $t = a$ and $t = b$: $\lim_{t \to a+} u(t)$ and $\lim_{t \to b-} u(t)$ exist and are finite.

(c) Explain the connection between (b) and the extension of solutions of initial value problems to a maximal interval of existence.

[ ] A-4.II: Let $f : [a, b] \to R^1$ be continuous and assume $f(a)f(b) \neq 0$. Verify the following properties of topological degree:
(a) If $f(b) > 0 > f(a)$, then $d(f, (a, b), 0) = 1$
(b) If $f(b) < 0 < f(a)$, then $d(f, (a, b), 0) = -1$
(c) If $f(a)f(b) > 0$, then $d(f, (a, b), 0) = 0$.

Part B

Partial Differential Equations

Do three (3) problems from Part B for full credit.

Exercise B-1. Consider the Sturm–Liouville problem $x^2(x^2u')' + \lambda u = 0$ on $1/2 \leq x \leq 1$ with boundary conditions $u(1/2) = u(1) = 0$.

(a) State without proof the main theorem on eigenfunction expansions which applies to this example.

(b) Use the change of variables $w(t) = u(1/t)$ to transform the differential equation into $d^2w/dt^2 + \lambda w = 0$. Then calculate the eigenvalues λ_n and eigenfunctions u_n, by citing without proof a result for the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(a) = 0 = y(b)$.

(c) Sturm oscillation theory and the Prüfer transformation are used in the general theory to produce the candidate eigenvalues and eigenfunctions. Sketch briefly how this is accomplished, without proofs.

Exercise B-2. Select with a check–mark [✓] and solve either B-2.I or B-2.II:

[ ] B-2.I: Define the Sobolev space $H^m(\Omega)$ for open $\Omega \subset R^n$. Then

(a) Prove that $H^m(\Omega)$ is a Hilbert space.

(b) Give an example of a sequence which shows that the subspace $C([0,1])$ in $L^1(0,1)$ is not complete in the L^1-norm.

(c) Compute the distributional derivatives ∂f and $\partial^2 f$ for $f(x) = |x|$ in $H^2(-\infty, \infty)$. Assume results for the Heaviside unit step and Delta.

[ ] B-2.II: Define what it means for H to be a Hilbert space. Then:

(a) Explain the meaning of the formula $H = M \oplus M^\perp$ and give conditions on M for when it is true (do not give proofs).

(b) State the Riesz representation theorem and use (a) to prove it.

Exercise B-3. Select with a check–mark [√] and solve either B-3.I or B-3.II:

[ ] B-3.I: Let u, h denote elements of some Sobolev space and consider the distributional differential equation $-u'' + u = h$ with Dirichlet boundary conditions $u(0) = u(1) = 0$.

(a) Formulate an abstract boundary value problem $a(u, v) = (h, v)$, by defining the sequilinear form $a(u, v)$, the Hilbert space H and the inner product $(\cdot, \cdot)$.

(b) Discuss in detail how the Riesz theorem applies to solve the abstract boundary value problem.

(c) Is enough known about the Hilbert space solution u for it to be a solution of the distributional differential equation? Explain.

[ ] B-3.II: Let $\phi \in H_0^1(G)$ and denote by $\|\cdot\|$ the usual norm in $L^2(G)$, G open in R^n. Assume $|x_1| \leq K$ for all $x \in G$.

(a) Prove the Poincaré inequality $\|\phi\| \leq 2K\|\partial_{x_1}\phi\|$.

(b) Explain, without proof, how to use generalizations of the Poincaré inequality to solve the abstract boundary value problem for distributional differential equations of the form $\Delta u = f$.

Exercise B-4. Sobolev proved an imbedding inequality of the form $\|f\|_B \le M\|f\|_A$, where $A = H^m(G)$ and B is the set of functions u such that $D^\alpha u$ is uniformly continuous on G for $|\alpha| \le k$. Give, without proof, the conditions on the open set $G \subset R^n$ and the integers m and k.

Select with a check–mark [√] and solve either B-4.I or B-4.II:

[ ] B-4.I: Under Sobolev's conditions on G, m and k, each $f \in H^m(G)$ satisfies $\partial^\alpha f = D^\alpha g$ a.e. for some k-times continuously differentiable function g, $|\alpha| \le k$.

(a) Prove this, assuming the imbedding inequality above.

(b) Determine for $n = 2$ the least m such that an element in $H^m(G)$ has 4 continuous derivatives (G as above).

[ ] B-4.II: Regularity theory implies that certain abstract boundary value problems $a(u, v) = (F, v)$ can be solved for $u \in H^{2+s}(G)$ provided $F \in H^s(G)$. Consider the Dirichlet problem for $\Delta u = F$, $x \in G$.

(a) Assume G is a disk in R^2. Give without proof a smoothness condition on F for the existence of C^2 solutions u.

(b) Assume G is bounded and open in R^2. Give without proof conditions on G and ∂G, and a smoothness condition on F, for the existence of C^2 solutions u.

Differential Equations Preliminary Examination

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

August 2001

Instructions

This examination consists of two parts, called **Part A** and **Part B**. Part A consists of exercises concerning *Ordinary Differential Equations* and Part B consists of exercises concerning *Partial Differential Equations.*

To obtain full credit you should fully complete **three** exercises from each part. All problems are weighted equally and a passing score will be 60% of the total possible score.

It is important that your arguments and discussion will be sound and detailed, bearing in mind that too many details are time consuming. Thus your judgement of what is essential will be an important factor in determining the final score.

Good Luck!

Part A - ODE

Exercise 1

Consider the system of ordinary differential equations

$$u' = -|u|^2 u + h(t), \tag{1}$$

where

$$h : [0,1] \to \mathbb{R}^N$$

is a continuous function and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. Use Brouwer's fixed point theorem to show that (1) has a solution $u \in C^1[0,1]$ such that

$$u(0) = u(1).$$

Use this result to conclude that if

$$h : \mathbb{R} \to \mathbb{R}^N$$

is continuous and periodic of period 1, then (1) has a solution of period 1. (Hint: Consider equation (1) subject to initial data $u_0 : |u_0| \leq r$, where r is large (in relation to h) and examine solution curves by computing $\frac{d}{dt}|u|^2$ for solutions u.) In relation to the above, what can you say about the equation

$$u' = |u|^2 u + h(t). \tag{2}$$

Exercise 2

Let A be a real $N \times N$ matrix, all of whose eigenvalues have negative real parts. Let

$$h : \mathbb{R} \to \mathbb{R}^N$$

be a continuous function which is periodic of period T. Let

$$\begin{array}{ccc} (t, v) & \mapsto & u(t, v) \\ \mathbb{R} \times \mathbb{R}^N & \to & \mathbb{R}^N \end{array}$$

denote the unique solution of

$$u' = Au + h(t), \; u(0) = v. \tag{3}$$

For given $T > 0$, define

$$S_T : \mathbb{R}^N \to \mathbb{R}^N$$

by

$$S_T(v) = u(T, v).$$

1. Use the variation of constants formula (and what you know about stability theory for the linear system $u' = Au$) to show that S_{nT} is a contraction mapping, for some positive integer n, sufficiently large.

2. Show that
$$S_{nT} = \underbrace{S_T \circ \cdots \circ S_T}_{n}.$$
the n-fold composition of S_T with itself.

3. Show that if S_{nT} has a unique fixed point, then so does S_T.

4. Apply this to show that if h is a continuous periodic function of period T, then (3) has a unique $T-$periodic solution.

Exercise 3

Let
$$f : \mathbb{R}^N \to \mathbb{R}^N$$
be a C^1 function and consider the system of differential equations
$$u' = f(u). \tag{4}$$

Let there exist, for a given $v \in \mathbb{R}^N$, a compact subset K of $\mathbb{R}^N$, such that
$$u(t, v) \in K,\ \ t \in I_v,$$
where I_v is the maximal interval of existence of the solution $u(t, v)$ of (4) which satisfies
$$u(0, v) = v.$$

Sketch proofs of the following:

1.
$$I_v = (-\infty, \infty).$$

2. The ω limit set $\Gamma(v)$ (define this term) is a nonempty, compact, connected, and invariant set.

What more can you say about the set $\Gamma(v)$ in case of dimension $N = 2$? Are there similar statements in higher dimensions?

Exercise 4

Consider the system in the plane

$$\begin{aligned} u' &= -v + uw \\ v' &= u + vw, \end{aligned} \tag{5}$$

where

$$r^2 = u^2 + v^2, \; w = r^4 - 6r^2 + 8.$$

Show that (5) has precisely two nontrivial periodic orbits and provide a "complete" phase plane analysis of the system.

Exercise 5

Use the implicit function theorem for mappings between Banach spaces to discuss the unique solvability of the scalar boundary value problem

$$u'' + 2u + u^2 = h(t), \; u(0) = u(\pi), \tag{6}$$

for continuous functions h of small maximum norm. Results needed from Sturm-Liouville theory may be assumed but should be quoted.

Part B - PDE

Exercise 6

Let $G \subset \mathbb{R}^N$ be an open set. Define what is meant by a distribution on G. Also define the concept of distributional derivatives of any order α, where α is a multiindex.

Give a definition of the Sobolev spaces $H^m(G)$, $H_0^m(G)$, where m is a nonnegative integer. Use properties of distributions to prove the generalized integration by parts formula

$$\langle \partial^\alpha f, g \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha g \rangle, \;\; f \in H^m(G), g \in H_0^m(G), \;\; |\alpha| \leq m,$$

and $\langle \cdot, \cdot \rangle$ is the $L^2(G)$ inner product.

Exercise 7

Consider the space of real-valued functions $H^2(0,1)$ and define the quadratic form

$$a(u,v) = \int_0^1 u''v''dx + b\int_0^1 u'v'dx + c\int_0^1 uvdx,$$

where b and c are positive constants. Show that a is a form to which the Lax-Milgram theorem may be applied and state the result thus obtained. Let $f \in L^2(0,1)$, and restrict the form a to the space $H_0^2(0,1)$. Show that f may be thought of as a continuous linear functional on $H_0^2(0,1)$, and apply the Lax-Milgram theorem to the pair a, f. Determine a boundary value problem which is solved in a distributional sense by the Lax-Milgram solution and show that this solution is at least three times continuously differentiable.

Exercise 8

Let G be a bounded open set in $\mathbb{R}^N$ with smooth boundary. State and indicate a proof of Poincaré's inequality for functions $u \in H_0^1(G)$. Using the inequality show that

$$\|u\|_0^2 = \int_G |\nabla u|^2 dx,$$

defines a norm equivalent to the H^1 norm. Consider the form

$$a(u,v) = \int_G \nabla u \cdot \overline{\nabla v} + \int_G c(x) \cdot \nabla u \bar{v} dx, \tag{7}$$

where the components of the vector function c belong to $L^\infty(G)$.

Provide a condition on the function c in order that a is a sesquilinear form satisfying the conditions of the Lax-Milgram theorem. Determine a boundary value problem solved by the Lax-Milgram solution.

Exercise 9

1. Let H be a Hilbert space and let

$$A : D(A) \to H$$

be a linear operator with domain $D(A)$. State the Hille-Yosida-Phillips theorem giving necessary and sufficient conditions for the unique solvability of the problem

$$\begin{aligned} &u' + Au = 0 \\ &u(0) = u_0 \in D(A). \end{aligned}$$

Be sure to define pertinent terms.

2. Apply the above result to the initial boundary value problem

$$\begin{aligned} &u_t - u_x = 0, \ t > 0, \ x \in \mathbb{R} \\ &u(0,x) = g(x), \ x \in \mathbb{R}. \end{aligned}$$

Solve this problem explicitly.

Also state a result about the initial value problem

$$\begin{aligned} &u_t - u_x = f(t,x), \ t > 0, \ x \in \mathbb{R} \\ &u(0,x) = g(x), \ x \in \mathbb{R}. \end{aligned}$$

Exercise 10

1. State the theorem of Rellich-Kondrachov for bounded open sets $G \subset \mathbb{R}^N$.

2. Let

$$f : \overline{G} \times \mathbb{R} \to \mathbb{R}$$

be a continuous bounded function such that

$$|f(x,u)| \leq M, \ (x,u) \in \overline{G} \times \mathbb{R}$$

where M is a constant. Show that the boundary value problem:

$$-\Delta u = f(x, u), \ \ u \in H_0^1(G). \tag{8}$$

has a solution by converting (8) into an equivalent fixed point problem in $L^2(G)$ and by applying Schauder's fixed point theorem (give reasons why its hypotheses hold!).

Differential Equations Preliminary Examination

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

August 2000

Instructions

This examination consists of Parts A and B. To obtain full credit you should fully complete three exercises from each part. If you attempt more than three questions from a part, indicate which three questions you would like graded. All problems are weighted equally and a passing score will be 60% of the total possible score. Please show all your work and give details where needed.

Part A

1. Let E be an open subset of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f}$ is a continuously differentiable function mapping E into $\mathbb{R}^n$.

   (a) Use the method of successive approximations to prove that the initial value problem $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution on some interval $[-a, a]$, and give an estimate for a.

   (b) Indicate how the result in (a) can be used to prove a similar result for the nonautonomous case.

2. Let E be an open subset of $\mathbb{R}^n$ containing the origin and $\mathbf{x}_0$. Assume that $\mathbf{f}$ is a continuously differentiable function mapping E into $\mathbb{R}^n$. Suppose $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $A = D\mathbf{f}(\mathbf{0})$. Let U and V be subsets of E containing the origin.

   (a) Define the maximal interval of existence $I(\mathbf{x}_0)$.

   (b) Define the flow $\phi_t(\mathbf{x}_0)$.

   (c) Define a homeomorphism H of U onto V.

   (d) Suppose $\mathbf{x}_0 \in U$. State the Hartman-Grobman theorem in terms of ϕ_t, A, H and $\mathbf{x}_0$. Be sure to include all necessary conditions.

   (e) Consider $\mathbf{x} = (x_1, x_2)$ with

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= x_2 - x_1^2, \end{aligned}$$

   and $\mathbf{x}(0) = \mathbf{x}_0$. Write down the explicit solution to this equation. Give the linearized system $\dot{\mathbf{y}} = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ and its explicit solution. Give the homeomorphism H which relates the two solutions.

3. (a) Prove Bendixson's Criterion: Let $\mathbf{f}$ be of class C^1 in a simply connected region E in $\mathbb{R}^2$. Assume the divergence of $\mathbf{f}$ is not identically zero and is either nonnegative in E or nonpositive in E. Then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbit which is entirely in E.

   (b) Extend the above proof to cover Dulac's Criterion: Let $B(x_1, x_2)$ and $\mathbf{f}$ be of class C^1 in a simply connected region E in $\mathbb{R}^2$. Assume the divergence of $B\mathbf{f}$ is not identically zero and is either nonnegative in E or nonpositive in E. Then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has no closed orbit which is entirely in E.

(c) Consider the nondimensionalized Lotka-Volterra competition equations

$$\dot{u} = ru(1 - u - \alpha_1 v)$$
$$\dot{v} = v(1 - v - \alpha_2 u)$$

where r, α_1 and α_2 are positive parameters. Apply Dulac's criterion to this system for $u, v > 0$. (HINT: try $B(u,v) = 1/uv$.) Does Bendixson's criteria give a similar result?

(d) Draw a biological conclusion.

4. Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in I\!R^n$ and $\mathbf{f} \in C^1(I\!R^n, I\!R^n)$. Suppose that the solution through any point $\mathbf{x}_0 \in I\!R^n$ exists for all $t \in I\!R$.

(a) Define the α- and ω-limit sets of a point $\mathbf{x}_0 \in I\!R^n$.

(b) Consider the nonlinear spring equation

$$x'' + \mu x' + g(x) = 0$$

where $g(x)$ is an odd function of x, $g'(x) > 0$ and $\mu \geq 0$. Translate this equation to the $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ form and linearize about the trivial equilibrium. What conclusions can be made about the stability of this equilibrium?

(c) Define what is meant by a Lyapunov function for the above equation; derive one and verify that it is a Lyapunov function. Describe the α- and ω-limit sets of a point $\mathbf{x}_0 \in I\!R^n$ for the case $\mu = 0$ and for the case $\mu > 0$. Make a conclusion about the stability of the trivial equilibrium for each of these two cases.

5. Consider the system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_1} &= -x_1 + x_2(1 - x_1^2 - 2x_2^2). \end{aligned}$$

Prove, by means of the Poincaré-Bendixson theorem, that there exists at least one periodic orbit in the annulus $1/2 < x_1^2 + x_2^2 < 1$.

Part B

1. Consider the equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

where a, b and c are suitably smooth

(a) Define what is meant by an integral surface for the above equation.

(b) Write down the characteristic equations and show how they can be used to construct an integral surface that contains the curve $x = \phi(s)$, $y = \psi(s)$, $u = \eta(s)$, where ϕ, ψ and η are C^1.

(c) What additional conditions, if any, must be placed on the data to guarantee a locally unique solution? Explain carefully.

(d) Demonstrate the above method by solving

$$\begin{aligned} u_x + 2u_y &= -u \\ u(0,y) &= f(y) \end{aligned}$$

where f is of class C^1.

2. Let Ω be a bounded open set. Define the Sobolev space $H_0^{1,2}(\Omega)$. Indicate why this is a Hilbert space. State and prove Poincaré's inequality for functions $u \in H_0^{1,2}(\Omega)$ and thus verify that $H_0^{1,2}(\Omega)$ is also a Hilbert space with respect to the inner product

$$(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx.$$

3. Let Ω be a bounded domain in $I\!R^n$. Consider

$$\begin{aligned} \Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $f \in L^2(\Omega)$.

(a) Define the concept of a weak $H_0^{1,2}(\Omega)$ solution to the above problem.

(b) Use Poincaré's inequality to prove existence of a weak $H_0^{1,2}(\Omega)$ solution.

(c) Assume that Ω is smooth. Prove that any two solutions in $C^2(\Omega) \cap C^1(\bar{\Omega})$ must agree. How would you prove this result for any two solutions in $C^2(\Omega) \cap C(\bar{\Omega})$?

4. Give the heat equation on a semi-infinite rod with initial temperature $g(x)$ and Dirichlet boundary conditions. Apply the method of odd extension of the heat kernel formula for $(-\infty, \infty)$ to derive a formula for the solution.

5. (a) State d'Alambert's formula for the solutions u of the wave equation $u_{tt} = u_{xx}$.

   (b) Consider the initial boundary value problem

$$\begin{array}{rlll} & u_{tt} = u_{xx} & \text{for} & 0 < x < \pi, \ t > 0 \\ u(x,0) = f(x), & u_t(x,0) = 0 & \text{for} & 0 < x < \pi \\ u(0,t) = 0, & u(\pi,t) = 0 & \text{for} & t \geq 0. \end{array}$$

   Use d'Alambert's formula and the parallelogram rule to define a weak solution solution to this problem. HINT: divide the region $R = \{(x,t) : 0 < x < \pi, t > 0\}$ into subregions. What compatibility conditions are needed upon f to ensure that the above solution is C, is C^1 and is C^2?

   (c) Show that the energy

$$E(t) = \frac{1}{2} \int_0^{\pi} (u_t^2 + u_x^2)\, dx$$

   is conserved, and use this to prove that a C^2 solution to the above initial boundary value problem is unique.

Differential Equations Preliminary Examination

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

August 1999

1 Instructions

This examination consists of two parts, called **Part A** and **Part B**. Part A consists of exercises concerning *Ordinary Differential Equations* and Part B consists of exercises concerning *Partial Differential Equations*.

To obtain full credit you should fully complete **three** exercises from each part. All problems are weighted equally and a passing score will be 60% of the total possible score.

It is important that your arguments and discussion will be sound and detailed, bearing in mind that too many details are time consuming. Thus your judgement of what is essential will be an important factor in determining the final score.

Good Luck!

2 Part A - ODE

2.1 Exercise 1

Let

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

be a continuous mapping which satisfies

$$|f(t,x) - f(t,y)| \leq L(t)|x-y|, \ \forall t \in \mathbb{R}, \ \forall x, y \in \mathbb{R}^N, \tag{2.1}$$

where

$$L : \mathbb{R} \to [0, \infty)$$

is a continuous function and where $|\cdot|$ is a norm in $\mathbb{R}^N$. Consider the initial value problem

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \ t_0 \in \mathbb{R}, \ x_0 \in \mathbb{R}^N. \end{aligned} \tag{2.2}$$

1. State the fundamental existence uniqueness theorem for solutions of (2.2) and sketch a proof.
2. State a general principle about the existence intervals for solutions of (2.2) and the behavior of solutions at the endpoints of these intervals.
3. Use the above and a Gronwall inequality argument (note: $|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)|$) to show that solutions of (2.2) are defined on all of $\mathbb{R}$.

2.2 Exercise 2

Let again $A : \mathbb{R} \to \mathbb{R}^{N \times N}$ be a continuous $N \times N$ matrix. Consider the systems

$$x' = A(t)x \tag{2.3}$$

$$x' = -A^T(t)x, \tag{2.4}$$

where A^T is the transpose of the matrix A.

1. Define what is meant by a fundamental matrix solution $\Phi(t)$ of (2.3).

2. Use general existence uniqueness principles to give the form of all fundamental matrix solutions of (2.3).

3. Let $\Psi(t)$ be a fundamental matrix solution of (2.4). Prove that

$$\Psi^T(t)\Phi(t)$$

is a nonsingular constant matrix.

4. Derive the *variation of constants* formula (using the fundamental solution matrix Φ) for solutions of

$$x' = A(t)x + h(t) \tag{2.5}$$

5. Let $\Phi(0) = \text{identity} = \Psi(0)$. Give the variation of constants formula derived above in terms of Φ and Ψ. Do you see any computational advantages in this form of the variation of constants formula.

2.3 Exercise 3

Use an indirect argument (and basic existence uniqueness principles) to prove the following:

If $u : \mathbb{R} \to \mathbb{R}$ belonging to $C^2(-\infty, \infty)$ is a one-signed solution (i.e. it either is everywhere nonnegative or everywhere nonpositve) of the equation

$$-u'' + V(t)u - (u^2)''u = |u|^3u,$$

where $V : \mathbb{R} \to \mathbb{R}$ is continuous, then u is either identically zero or never zero.

2.4 Exercise 4

Let

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

be a C^1 mapping and consider the system

$$x' = f(x). \tag{2.6}$$

Assume there exists a C^1 functional

$$\phi : \mathbb{R}^N \to \mathbb{R}$$

such that

$$\phi'(x) := \nabla\phi(x) \cdot f(x) \leq 0, \; x \in \mathbb{R}^N.$$

Prove that the $\omega-$limit set of any bounded semiorbit of (2.6) is an invariant set for the flow associated with (2.6) and is contained in the set

$$E = \{x : \phi'(x) = 0\}.$$

Apply this result to the system

$$\begin{aligned} u' &= -v \\ v' &= -v + u^3 \\ x &= \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

using

$$\phi(u, v) = \frac{v^2}{2} + \frac{u^4}{4}.$$

What is your conclusion?

2.5 Exercise 5

Let A be a $N \times N$ matrix all of whose eigenvalues have negative real parts and consider the system

$$x' = Ax + h(t), \tag{2.7}$$

where $h : \mathbb{R} \to \mathbb{R}^N$ is a continuous function which satisfies

$$h(t + T) = h(t), \; \forall t \in \mathbb{R}.$$

Show that for all such h (2.7) has a unique solution x such that

$$x(t + T) = x(t), \; \forall t \in \mathbb{R}.$$

What stability properties does this periodic orbit have? How may the requirement stipulated above on the eigenvalues of A be changed and still maintain the existence uniqueness result you just proved?

3 Part B - PDE

3.1 Exercise 6

1. Let $G \subset \mathbb{R}^N$ be an open set. Define what is meant by a distribution on G. Let $f \in L^2(G)$ be a real function. Show that f may be identified with a distribution T_f.

2. Let T be a distribution on G and let $\alpha = (\alpha_1, \cdots, \alpha_N)$ be a multiindex. Define the concept of the $\alpha-$th distributional derivative of T.

3. Let
$$r(t) = \begin{cases} 0, & t \leq -1 \\ 1, & -1 < t < 1 \\ 0, & t \geq 1 \end{cases}$$
Think of r as a distribution on $\mathbb{R}$ and compute all distributional derivatives of r.

4. Provide the higher dimensional analogue of the previous exercise, i.e., let
$$r : \mathbb{R}^N \to \mathbb{R}$$
be defined by
$$r(x) = 1, \quad |x| < 1$$
$$r(x) = 0, \quad |x| \geq 1.$$
Compute the distributional gradient ∇r of r.

3.2 Exercise 7

1. Let G be an open set. Define the Sobolev spaces $H^m(G)$, $m = 1, 2, \cdots$, and $H_0^m(G)$, $m = 1, 2, \cdots$. Indicate why these spaces are Hilbert spaces.

2. Let G be a bounded open set. State and prove Poincaré's inequality for functions $u \in H_0^1(G)$ and thus verify that $H_0^1(G)$ is also a Hilbert space with respect to the inner product
$$(u, v)_{H_0^1(G)} = \int_G \nabla u \cdot \overline{\nabla v} dx.$$

3.3 Exercise 8

1. Let G be an open subset of $\mathbb{R}^N$ and let $f \in L^2(G)$. Show that f defines a continuous linear functional on $H^1(G)$.

2. State the Lax-Milgram theorem for subspaces $V \subset H^1(G)$.

3. Show that in the sense of distributions there exists a unique solution $u \in H_0^1(G)$ of the equation

$$-\Delta u + u = f.$$

4. Characterize the orthogonal complement of $H_0^1(0,\infty)$ in the space $H^1(0,\infty)$.

3.4 Exercise 9

1. Let H be a Hilbert space and let

$$A : D(A) \to H$$

be a linear operator with domain $D(A)$. State the Hille-Yosida-Phillips theorem giving necessary and sufficient conditions for the unique solvability of the problem

$$\begin{aligned} &u' + Au = 0 \\ &u(0) = u_0 \in D(A). \end{aligned}$$

Be sure to define pertinent terms.

2. Apply the above result to the initial boundary value problem

$$\begin{aligned} &u_t - u_x = 0,\ t > 0,\ x \in \mathbb{R} \\ &u(0,x) = g(x),\ x \in \mathbb{R}. \end{aligned}$$

Solve this problem explicitly.

Also state a result about the initial value problem

$$\begin{aligned} &u_t - u_x = f(t,x),\ t > 0,\ x \in \mathbb{R} \\ &u(0,x) = g(x),\ x \in \mathbb{R}. \end{aligned}$$

3.5 Exercise 10

State the minimum principle for superharmonic functions and use it to show that if $u \in C^2(G)$ is a nontrivial solution of

$$\Delta u + \lambda e^u = 0, \; u = 0, \; \text{on } \partial G, \tag{3.1}$$

where $\lambda > 0$ is a real parameter and G is a bounded open set, then $u(x) > 0, \; x \in G$.

Also show that the set of $\lambda > 0$, for which (3.1) has a solution is bounded above. (Hint: Let $\lambda_0 > 0$ be the smallest value for which

$$\Delta \phi + \lambda_0 \phi = 0, \; \phi = 0, \; \text{on } \partial G, \tag{3.2}$$

has a nontrivial solution (recall the spectral theorem and properties of the eigenfunction ϕ).)

Department of Mathematics
University of Utah
Ph. D. Preliminary Exam in Differential Equations
Autumn 1998

Instructions The exam has two parts, each comprising five questions. You must answer **three** questions from Part I, and **three** questions from Part II, and all questions carry equal weight. If you choose to answer more than three questions in either part, indicate clearly which three are to be graded.

To receive full credit your solutions must be presented clearly and as completely as possible, and your work must be readable. If you find yourself short of time but can sketch the solution to a problem, state how to do it as precisely as possible and you will receive partial credit. You might find it useful to write down the major steps in any case to help organize your thoughts, but this is optional if the detailed work is correct.

Part I.

(1) Consider the problem

$$\begin{aligned} \frac{dx}{dt} &= f(x,t) \\ x(t_0) &= x_0 \end{aligned} \tag{1.1}$$

where $f \in C(D, R^n)$ and $D \subset R^n \times R$ is open. Suppose that f is Lipschitz continuous in x for $(x,t) \in D$.

(a) Prove the existence of a local (in t) solution to (1.1) using successive approximations.

(b) Prove the uniqueness of this solution and show that it can be extended until it meets ∂D.

(c) Prove that the solution is continuous in (x_0, t_0).

(2) Consider

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x) \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

where $x \in R^n$ and A is a constant $n \times n$ matrix. Suppose that A is semisimple, that $f \in C^2$ and that $f(0) = 0$, $D_x f(0) = 0$.

(a) Define what is meant by the stable manifold, the center manifold and the unstable manifold at the origin for (2.1).

(b) Give a representation of the solution of (2.1) when $f \equiv 0$, in terms of the eigenvalues of A and the associated projections and nilpotents. Indicate how to compute the projections.

(c) Discuss the asymptotic behavior of the solution in (b) for $t \to \infty$. Characterize the manifolds defined in (a) as precisely as possible.

(d) Give a representation of the solution of (2.1) for $f \not\equiv 0$. Under what conditions, if any, on A and the initial data can one conclude that this solution exists for all $t \in R^+$. You need not prove all the assertions for the last part, but should at least state your resaons.

(e) State the Hartman-Grobman theorem for

$$x' = F(x)$$

and state clearly what further conditions, if any, are needed on (2.1) to apply the Hartman-Grobman theorem to (2.1).

(3) Consider

$$x' = f(x) \tag{3.1}$$

where $x \in R^n$ and $f \in C^1(R^n, R^n)$. Suppose that the solution through any point $x_0 \in R^n$ exists for all $t \in R$.

(a) Define the positive and negative semiorbits of (3.1) through $x_0 \in R^n$.

(b) Define the $\alpha-$ and $\omega-$limit sets of a point $x_0 \in R^n$.

(c) Consider the nonlinear spring equation

$$x'' + g(x) = 0 \tag{3.2}$$

where $g(x)$ is an odd function of x and $g'(x) > 0$. Describe the $\alpha-$ and ω-limit set for any point on the positive x-axis.

(d) Consider

$$x'' + x' + g(x) = 0 \tag{3.3}$$

where g is as in (c). Define what is meant by a Liapunov function for this equation and derive one (verify that it is a Liapunov function). Describe the ω-limit set for any point in the plane.

(4) Consider the system

$$x' = f(x) \tag{4.1}$$

where $f \in C^1(R^n, R^n)$, and suppose that (4.1) has a periodic solution $\gamma(t)$ of least period T.

(a) Define what it means to say that $\gamma(t)$ is orbitally asymptotically stable with asymptotic phase.

(b) Let

$$\xi' = A(t)\xi \tag{4.2}$$

be the variational equation (the linearization of (4.1)) relative to $\gamma(t)$. State the Floquet representation for the fundamental matrix solution of (4.2) and discuss the stability of $\gamma(t)$ in terms of properties of the fundamental matrix solution.

(c) Consider the system

$$\begin{aligned} x' &= x - y - x(x^2+y^2) \\ y' &= x + y - y(x^2+y^2) \end{aligned}$$

Find a nontrivial periodic solution of this equation, and prove (in the easiest way possible) that this periodic solution is orbitally asymptotically stable with asymptotic phase. Compute the asymptotic phase explicitly.

(d) Compute the Poincare map associated with the periodic orbit you found in (c).

(5) Consider the problem

$$\begin{aligned} Lu \equiv -\frac{d}{dx}(p\frac{du}{dx}) + qu &= \lambda u \\ \alpha u(0) + \beta\frac{du}{dx}(0) &= 0 \\ \gamma u(1) + \delta\frac{du}{dx}(1) &= 0 \end{aligned} \tag{5.1}$$

where $x \in [0,1], p \in C^1[0,1], qeC[0,1]$, and $\alpha, \beta, \gamma, \delta$ are constants.

(a) Show that (5.1) is self-adjoint if and only if p and q are real, $\gamma\bar{\delta} = \bar{\gamma}\delta$, and $\alpha\bar{\beta} = \bar{\alpha}\beta$.

(b) Suppose hereafter that $p = q = 1, \alpha = 1 \quad \beta = 0 \quad \gamma = 0 \quad \delta = 1$. Compute the eigenvalues and eigenfunctions for L with the given boundary conditions.

(c) Compute the Green's function for L (with coefficients and boundary conditions as in (b)).

(d) Show how to solve the nonhomogeneous problem

$$Lu = f(x)$$

with L and the boundary conditions as in (c). What conditions on f suffice to make u a classical solution?

Part II.

(6) Consider the linear partial differential equation

$$Lu \equiv \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha u = f \tag{6.1}$$

where $\alpha \equiv (\alpha_1, \cdots, \alpha_n)$ is a multi-index. Let $\Omega \subset R^n$ be a bounded domain with piecewise smooth boundary and suppose throughout this problem that the coefficients in the equations are smooth.

(a) Define what is meant by a weak solution of (6.1).

(b) Define the principal part of L, the characteristic variety of L, and a characteristic surface for L.

(c) Obtain conditions on a, b and c under which

$$Lu = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = 0$$

is

(i) hyperbolic
(ii) elliptic
(iii) parabolic.

(d) Define what is meant by a well-posed problem for a partial differential equation with boundary and/or initial data specified. Give an example of a well-posed problem for one of the three types of equation in (c) and briefly justify your assertions.

(7) Consider the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \\ u(x,0) &= F(x) \\ u_t(x,0) &= 0 \\ u(0,t) = u(\pi,t) &= 0 \end{aligned} \tag{7.1}$$

(a) Use separation of variables to formally solve (7.1) when $F \in C[0,\pi]$.

(b) Suppose that $F \in C^4[0,\pi]$ and that $F''(0) = F''(\pi) = 0$. Prove that the series you obtain in (a) defines a classical solution of (7.1) under the foregoing conditions on F.

(8) Consider the equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{8.1}$$

where a, b and c are suitably smooth.

(a) Define what is meant by an integral surface for (8.1).

(b) Write down the characteristic equations for (8.1) and show how they can be used to construct an integral surface of (8.1) that contains the curve

$$\chi = \varphi(s) \qquad y = \psi(s) \qquad u = \eta(s),$$

where φ, ψ and η are C^1.

(c) What additional conditions, if any, must be placed on the data to guarantee a locally unique solution? Explain carefully.

(d) Consider the equation

$$u_t + uu_x = 0 \tag{8.2}$$

with initial data given by

$$u(x,0) \equiv u_0(x) = \begin{cases} 1 & x \le 0 \\ 1-x & x \in (0,1) \\ 0 & x \ge 1 \end{cases} \tag{8.3}$$

Does (8.2)–(8.3) have a unique classical solution for $t \ge 0$? A unique weak solution for $t \ge 0$? Explain.

(9) (a) Let $\Omega \subset R^n$ be open and let $u \in C(\Omega)$. Define what it means to say that u is subharmonic in Ω.

(b) Let Ω be open, bounded and connected and let $u \in C^2(\Omega) \cap C^o(\bar{\Omega})$ be subharmonic. State the weak form of the maximum principle for u.

(c) Consider the Dirichlet problem

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{9.1}$$

where $f \in C^0(\Omega)$ and Ω is as in (b). Describe in as much detail as you can the major steps in proving the existence of a classical solution of (9.1) via Perron's Method.

(10) Let $\Omega \subset R$ be open, and let $C_0^\infty(\Omega)$ be the space of C^∞ functions with support in Ω. Define convergence of a sequence φ_n as follows: $\varphi_n \to 0$ iff $\exists$ a compact set $K \subset \Omega$ such that

(i) Support $\varphi_n \subset K$ for all n

(ii) $\lim_{n\to\infty} \sup_{x\in K} |\varphi^{(p)}(x)| = 0$ for $p \in Z^+$.

(a) Define what is meant by a distribution on $C_0^\infty(\Omega)$.

(b) Define

$$\mu_f(\varphi) = \int_{-\infty}^{\infty} f(x)\bar{\varphi}(x)dx$$

where $f \in L_1^{loc}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Does this define a distribution on $C_0^\infty(\Omega)$? Give reasons.

(c) Give a concrete example of a regular distribution and verify that is is a distribution.

(d) Show how to define the derivatives of a regular distribution.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
Autumn 1997

Instructions: The examination has two parts consisting of six problems each. You are to work four problems from part A and four problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 96.

A. Ordinary differential equations: Do four problems for full credit.

A1. Suppose that $f \in C^1(\mathbb{R}^2)$ and satisfies the condition $|f_y(x,y)| \leq c_1$ for all $(x,y) \in \mathbb{R}^2$ where $c_1 < \infty$ is constant. Consider the initial value problem

$$(1) \qquad \begin{cases} \frac{dy}{dt} &= f(t, y(t)), \\ y(0) &= \xi, \end{cases}$$

where $\xi \in \mathbb{R}$. Let $\varphi(t;\xi)$ denote the unique solution of (1) defined for $0 \leq t < \infty$.

(a.) Show that $\dfrac{\partial \varphi}{\partial \xi}(t;\xi)$ exists for all $t \geq 0$, $\xi \in \mathbb{R}$.

(b.) Let $f(t,y) = \dfrac{y-3}{1+y^2+t^2}$. Find $\dfrac{\partial \varphi}{\partial \xi}(t;3)$.

A2. Show that the following system

$$\begin{cases} \frac{dx}{dt} &= 2x + yz \\ \frac{dy}{dt} &= y + xz \\ \frac{dz}{dt} &= -z + xy \end{cases}$$

has trajectories which come in arbitrarily close to the origin and then escape from it. That is, show that there is an open set $V \subset \mathbb{R}^3$ containing the origin so that given any other open neighborhood $0 \in U \subset V$, there is a solution $\gamma \in C^1(\mathbb{R}, \mathbb{R}^3)$ of the system and three real numbers $\tau_1 < \tau_2 < \tau_3$ such that $\gamma(\tau_1) \notin V$, $\gamma(\tau_2) \in U$ and $\gamma(\tau_3) \notin V$.

A3. Show that the following system

$$\begin{cases} \frac{dx}{dt} &= \ 4\sin(x)\cos(y) + \cos(3x), \\ \frac{dy}{dt} &= -4\cos(x)\sin(y) + \cos(3y). \end{cases}$$

has a nontrivial limit cycle. You may assume that the set of all stationary points is $\{(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + m\pi) \ : \ k, \ m \text{ are integers}\}$.

A4. Let $f \in C^{0,1-}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Consider the initial value problem

$$(2) \qquad \begin{cases} \frac{dx}{dt} &= f(t,x), \\ x(\tau) &= \xi \qquad \text{for } (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n. \end{cases}$$

Assume $f(t,0) = 0$ so that $z(t) = 0$ is a global solution of (2).

(a.) Define: z is a *uniformly Liapunov stable* solution of (2).

(b.) Suppose $f(t,x) = Ax$ where A is a constant matrix. State the precise conditions on A under which $z(t)$ is uniformly Liapunov stable in equation (2).

(c.) Assume that A is a real matrix whose eigenvalues have negative real part ($\Re e \, \sigma(A) < 0$) and $B(t)$ is continuous for all $t \in \mathbb{R}$. Show that there is an $\varepsilon > 0$ depending on A so that if $\|B\| \le \varepsilon$ then the zero solution of (2) with $f(x,t) = Ax + B(t)x$ is uniformly Liapunov stable.

A5. Determine if there are any nontrivial periodic solutions to the system

$$\begin{cases} \frac{dx}{dt} = & 2x + y + x\cos t - y\sin t \\ \frac{dy}{dt} = & -x + 2y - x\cos t + y\sin t \end{cases}$$

One solution is $(e^{2t}\cos t, e^{2t}\sin t)$.

A6. Let A and be a complex $n \times n$ matrix whose eigenvalues $\lambda \in \sigma(A)$ all satisfy $\Re e \, \lambda < 0$. Show that the flows $\varphi^t(x) = e^{tA}x$ and $\psi^t(x) = e^{-t}x$ are flow equivalent, *i.e.*, there is a homeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$ so that $h(\varphi^t(x)) = \psi^t(h(x))$ for all $t \in \mathbb{R}$ and $x \in \mathbb{C}^n$.

B. Partial Differential Equations. Do four problems to get full credit.

B1. Consider the initial value problem

$$(3) \qquad \begin{aligned} u_t + u\,u_x &= f(x,t) \qquad \text{for all } x, t \in \mathbb{R}, \ t \ge 0; \\ u(x,0) &= \varphi(x) \qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

where $\varphi \in C^1(\mathbb{R})$

(a.) Define what is meant by a weak solution of (3).

(b.) Using the method of characteristics, solve the initial value problem (3) assuming $f(x,t) = x$ and $\varphi(s) = s$.

(c.) Assume $f(x,t) = 0$. Show by example that if

$$\varphi(x) = \begin{cases} \alpha, & \text{if } x < 0, \\ \beta, & \text{if } x \geq 0. \end{cases}$$

where $\alpha < \beta$ are constants, then the IVP (3) may have more that one weak solution. (Check that your examples are weak solutions.)

(d.) Under what additional condition are the weak solutions unique?

B2. Consider the wave equation with dispersion in halfspace

$$(4) \qquad \begin{aligned} u_{tt} - \Delta u + u &= 0 && \text{for } x \in \mathbb{R}^n,\ t \in \mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\} \\ u(x,0) &= \varphi(x) && \\ u_t(x,0) &= \psi(x) && \text{for } x \in \mathbb{R}^n \end{aligned}$$

where the initial functions φ, ψ are smooth with compact support. Show that signals propogate at finite speed by obtaining a domain of dependence result.

B3. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and consider the boundary value problem for harmonic functions

$$(5) \qquad \begin{aligned} \Delta u &= 0 && \text{for } x \in \Omega, \\ u(x) &= \varphi(x) && \text{for } x \in \partial\Omega. \end{aligned}$$

where $\varphi \in C^0(\partial\Omega)$ is just continuous. Let $B_{2r}(\xi) \subset \Omega$ be a ball of radius $2r > 0$. Suppose that we are proving the existence of a solution of the boundary value problem (5) using the Perron Process. Namely, let

$$S_\varphi = \{u \in C^0(\overline{\Omega}) : u \text{ subharmonic and } u(x) \leq \varphi(x) \text{ for all } x \in \partial\Omega\}$$

and let $w_\varphi(x) := \sup\{u(x) : u \in S_\varphi\}$.

(a.) Show that there is a constant $c < \infty$ depending only on n and r so that

$$\sup\{|Du(x)| : x \in \overline{B_r(0)}\} \leq c \sup\{|u(x)| : x \in \partial B_{2r}(0)\}.$$

for all harmonic functions u on $B_{2r}(0)$.

(b.) Show that there is a sequence of functions $u_i \in S_\varphi$ so that $u_i \to w_\varphi$ uniformly in $\overline{B_r(\xi)}$.

(c.) Show that w_φ is harmonic in $B_r(\xi)$.

B4. For $v \in C_0^\infty(\mathbb{R})$, define F by the formula $\langle F, v\rangle = v'(0)$.

(a.) Show that F is a distribution on $\mathbb{R}$.

(b.) Let $\zeta \in C_0(\mathbb{R})$ be continuous with compact support such that the integral $\int_{-\infty}^{\infty} \zeta(x)\,dx = 1$. Show that as distributions, the convolutions $\zeta_\varepsilon \star F \to F$ as $\varepsilon \to 0$, where $\zeta_\varepsilon(x) := \frac{1}{\varepsilon}\zeta\left(\frac{x}{\varepsilon}\right)$.

(c.) Find $\frac{d^k}{dx^k} F$ for all integers $k > 0$.

B5. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $f(x) \in C^0(\overline{\Omega})$ and $V(x) \in C^0(\overline{\Omega}, \mathbb{R}^n)$ a continuous functions. Consider the boundary value problem

$$\begin{aligned} \Delta u + V(x) \cdot \nabla u &= f(x) \qquad x \in \Omega \\ u(x) &= 0 \qquad x \in \partial\Omega \end{aligned} \tag{6}$$

(a.) State the Poincaré inequality for functions $u \in H_0^1(\Omega)$.

(b.) Find a condition relating Ω and V which implies there exists a unique weak solution $u \in H_0^1(\Omega)$ for problem (6). Prove the existence and uniqueness under your condition.

B6. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ denote the n-dimensional torus, the n-fold product of unit circles. Saying $u(x) \in C^\infty(\mathbb{T}^n)$ is equivalent to saying $u(x)$ is a smooth function on $\mathbb{R}^n$ which is 2π-periodic in each variable. For a constant $a \le 0$ let the operator

$$Lu := \Delta u + au.$$

(a.) Define the Sobolev Space $H^k(\mathbb{T}^n) = H^{k,2}(\mathbb{T}^n)$ for k a nonnegative integer. Explain how to make sense of $H^s(\mathbb{T}^n)$ for $s \in \mathbb{R}$.

(b.) Show that for all nonnegative integers k there is a constant $c(k,n) < \infty$ so that for all $u \in H^{k+2}(\mathbb{T}^n)$ we have

$$|u|_{H^{k+2}(\mathbb{T}^n)} \le c\left(|Lu|_{H^k(\mathbb{T}^n)} + |u|_{H^0(\mathbb{T}^n)}\right).$$

(c.) Assume that $u \in H^1(\mathbb{T}^n)$ is a weak solution to $Lu = f(x)$ on $\mathbb{T}^n$, where $f \in C^\infty(\mathbb{T}^n)$. Show that $u \in C^\infty(\mathbb{T}^n)$.

Differential Equations Preliminary Examination

Department of Mathematics
University of Utah
Salt Lake City, Utah 84112

September 1996

1 Instructions

This examination consists of two parts, called **Part A** and **Part B**. Part A consists of exercises concerning *Ordinary Differential Equations* and Part B consists of exercises concerning *Partial Differential Equations.*

To obtain full credit you should fully complete **three** exercises from each part. All problems are weighted equally and a passing score will be 60% of the total possible score.

It is important that your arguments and discussion will be sound and detailed, bearing in mind that too many details are time consuming. Thus your judgement of what is essential will be an important factor in determining the final score.

Good Luck!

2 Part A - ODE

2.1 Exercise 1 *Use of the implicit function theorem*

Let E denote the normed linear space

$$E = C^2([0,\pi],\mathbb{R})$$

with norm

$$\|u\|_E = \sum_{i=0}^{2} \max_{x\in[0,\pi]} |u^{(i)}(x)|$$

and X the space

$$X = C([0,\pi],\mathbb{R}).$$

with norm

$$\|u\|_X = \max_{x\in[0,\pi]} |u(x)|.$$

1. Give brief arguments that E and X are Banach spaces.

2. Define

$$f : E \to X$$

by

$$u \mapsto f(u) = u'' + \lambda u + u^3, \tag{2.1}$$

where $\lambda \in \mathbb{R}$. Show that f is a C^1 mapping.

3. Let

$$E_0 = E \cap \{u : u(0) = u(\pi) = 0\},$$

and restrict f to E_0. Show that, except for a countable set of values of λ (describe this set !), the following holds: There exists $\delta = \delta(\lambda)$ such that for all $h \in X$, $\|h\|_X < \delta$, there exists a unique $u \in E_0$ such that

$$f(u) = h, \tag{2.2}$$

i.e. the problem

$$\begin{cases} u'' + \lambda u + u^3 = h \\ u(0) = 0 = u(\pi) \end{cases} \tag{2.3}$$

has a unique solution.

2.2 Exercise 2 *Use of the Brouwer fixed point theorem*

Let

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

be a C^1 mapping which satisfies

$$x \cdot f(t,x) < 0, \ t \in \mathbb{R}, \ |x| = R, \tag{2.4}$$

where $|\cdot|$ is the Euclidean norm and $\cdot$ is the inner product in $\mathbb{R}^N$ and R is a positive number.

1. Define the mapping

$$y \in \mathbb{R}^N \mapsto F_t(y) \in \mathbb{R}^N \tag{2.5}$$

by

$$F_t(y) = x(t,y), \tag{2.6}$$

where $x(t,y)$ solves

$$\begin{aligned} x' &= f(t,x) \\ x(0,y) &= y. \end{aligned} \tag{2.7}$$

Give reasons that F_t is well-defined for each small t and well-defined for all $t \in \mathbb{R}$, whenever $|y| \leq R$.

2. Show that for each $T \geq 0$ there exists y_T, $|y_T| \leq R$ such that

$$F_T(y_T) = y_T. \tag{2.8}$$

3. What can one say if the condition (2.4) is replaced by

$$x \cdot f(t,x) > 0, \ t \in \mathbb{R}, \ |x| = R. \tag{2.9}$$

4. Let f have the special form

$$f(t,x) = Ax + h(t), \tag{2.10}$$

where $h : \mathbb{R} \to \mathbb{R}^N$ is a continuous function of period T and A is a constant $N \times N$ matrix which is positive (negative) definite (i.e. $Ax \cdot x \geq \mu |x|^2$, resp. $Ax \cdot x \leq -\mu |x|^2$, $\mu > 0$). Use the result proved above to show that the equation

$$x' = Ax + h(t) \tag{2.11}$$

has a **unique** T-periodic solution.

2.3 Exercise 3 *Use of the Leray-Schauder degree*

Let again A be a positive (negative) definite $N \times N$ matrix (see previous exercise) and consider the system

$$x' = Ax + \epsilon h(t, x), \tag{2.12}$$

where ϵ is a real parameter and $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous T−periodic function, i.e.

$$h(t + T, x) = h(t, x), \; t \in \mathbb{R}, \; x \in \mathbb{R}^N.$$

which satisfies

$$|h(t, x)| \leq a + b|x|, \; t \in \mathbb{R}, \; x \in \mathbb{R}^N,$$

where a and b are positive constants. Consider the problem of the existence of T−periodic solutions of (2.12), i.e. solutions of

$$\begin{aligned} x' &= Ax + \epsilon h(t, x) \\ x(0) &= x(T). \end{aligned} \tag{2.13}$$

1. Convert problem (2.13) into an equivalent fixed point problem in an appropriate Banach space.

2. Use Gronwall's inequality (or a similar argument) to establish the existence of a priori bounds on $|x(t)|^2$ for a solution x of (2.13).

3. Use properties of the Leray-Schauder degree to deduce that (2.13) has a T−periodic solution for all ϵ.

2.4 Exercise 4 *Invariant sets - asymptotic stability*

Let

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

be a C^1 mapping and consider the system

$$x' = f(x). \tag{2.14}$$

Assume there exists a C^1 functional

$$\phi : \mathbb{R}^N \to \mathbb{R}$$

such that

$$\phi'(x) := \nabla\phi(x) \cdot f(x) \leq 0, \ x \in \mathbb{R}^N.$$

1. Give a brief sketch of the proof of the following theorem:

   Theorem: Let

   $$E = \{x : \phi'(x) = 0\}$$

   and let M be the largest invariant set of (2.14) in E. Then every bounded (for $t \geq 0$) semiorbit of (2.14) tends to M as $t \to \infty$.

2. Apply this theorem to the system

   $$\begin{aligned} u' &= -v \\ v' &= -v + g(u) \\ x &= \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

   where $g : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that

   $$ug(u) > 0, \ u \neq 0,$$

   using

   $$\phi(u, v) = \frac{v^2}{2} + \int_0^u g(\xi)d\xi.$$

   What is your conclusion?

2.5 Exercise 5 *Lyapunov stability*

Let A be a $N \times N$ matrix all of whose eigenvalues have negative real parts and consider the system

$$x' = Ax + h(t, x), \tag{2.15}$$

where $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 function which satisfies

$$|h(t, x)| \leq b(t)|x|^2, \ t \in \mathbb{R}, \ x \in \mathbb{R}^N,$$

where $b \in L^\infty(0, \infty)$. Show that every solution x of (2.15) with $|x(0)|$ small satisfies

$$\lim_{t \to \infty} x(t) = 0.$$

3 Part B - PDE

3.1 Exercise 6 *Distributions*

1. Let $G \subset \mathbb{R}^N$ be an open set and let $m \geq 0$ be an integer. Let $f \in C^m(G)$ be real or complex valued function. Show that f may be identified with a distribution T_f and compute the distributional derivatives of T_f up to order m.

2. Let
$$r(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x \leq 0 \end{cases}.$$
Compute all distributional derivatives of r.

3.2 Exercise 7 *Sobolev spaces*

1. Let G be an open set. Define the Sobolev spaces $H^m(G)$, $W^{m,2}(G)$, and $W_0^{m,2}(G)$ and show that $H^m(G)$ is a Hilbert space. Give conditions under which any two of the three spaces are equal.

2. Let G be a bounded open set. State and prove Poincaré's inequality for functions $u \in W_0^{1,2}(G)$ and thus verify that $W_0^{1,2}(G)$ is also a Hilbert space with respect to the inner product

$$(u, v)_{W_0^{1,2}(G)} = \int_G \nabla u \cdot \overline{\nabla v} dx.$$

3.3 Exercise 8 *Distributional solutions, Riesz representation theorem*

Let G be an open subset of $\mathbb{R}^N$ and let $f \in L^2(G)$. Show that f defines a continuous linear functional on $W_0^{1,2}(G)$.

1. Show that in the sense of distributions there exists a unique solution $u \in W_0^{1,2}(G)$ of the equation
$$-\Delta u + u = f.$$

2. Let G be bounded. Show that in the sense of distributions there exists a unique solution $u \in W_0^{1,2}(G)$ of the equation
$$-\Delta u = f.$$

3. Characterize the orthogonal complement of $W_0^{1,2}(G)$ in the space $W^{1,2}(G)$.

3.4 Exercise 9 *Lax - Milgram theorem*

State the Lax-Milgram theorem (defining pertinent terms) and provide an application to boundary value problems for elliptic partial differential equations.

3.5 Exercise 10 *Semigroups*

1. Let H be a Hilbert space and let

$$A : D(A) \to H$$

be a linear operator with domain $D(A)$. State the Hille-Yosida-Phillips theorem giving necessary and sufficient conditions for the unique solvability of the problem

$$\begin{aligned} &u' + Au = 0 \\ &u(0) = u_0 \in D(A). \end{aligned}$$

Be sure to define pertinent terms.

2. Apply the above result to the initial boundary value problem

$$\begin{aligned} &u_t - u_{xx} = 0,\ t > 0,\ x \in (0, \pi) \\ &u(0, x) = g(x),\ x \in (0, \pi) \\ &u(t, 0) = 0 = u(t, \pi),\ t \geq 0. \end{aligned}$$

Written Qualifying Examination in

DIFFERENTIAL EQUATIONS

September 14, 1995

Instructions: The examination has two parts consisting of eight and four problems respectively. You are to work four problems from part A and two problems from part B. If you work on more than the required number of problems then state which problems you wish to be graded.

Problems will be assigned equal weight for grading. In order to pass the Qualifying Examination your overall score must be at least 60 %.

Part A

Do four problems for full credit.

Problem A1. Suppose D is the rectangle defined by $|x-x_0| \leq a, |y-y_0| \leq b$ in the plane R^2. Suppose that f is a real valued function defined on D which is continuous, hence bounded by some nonnegative number M on D.

(i) Show that if $\alpha = \min\{a, \frac{b}{M}\}$, then there exists a function $y(x)$, defined on $|x - x_0| \leq \alpha$, such that

$$y'(x) = f(x, y(x)) \quad \text{and} \quad y(x_0) = y_0.$$

(ii) Show that the solutions y guaranteed in part (i) may not be unique; that is, give an example in which (i) holds and more than one solution $y(x)$ exists through the point (x_0, y_0) in your example.

Problem A2.

(i) Suppose A is a constant $n \times n$ matrix and consider the equation

$$\dot{x}(t) = Ax(t).$$

Show that if all eigenvalues have negative real parts then the trivial solution $x(t) \equiv 0$ is asymptotically stable; that is, every solution $x(t) \to 0$ as $t \to \infty$.

(ii) Show that if one eigenvalue of A has positive real part, then the trivial solution of A is not stable.

(iii) Under what circumstances is the trivial solution stable if A has no eigenvalue with positive real part, but does have eigenvalues with zero real part? (State, don't prove)

(iv) Do any of these results carry over if A is not a constant matrix? (State, don't prove.)

Problem A3.

(Concrete cases from 2.) Change to vector-matrix form and discuss the stability of the trivial solution of the following equations.

(i) $u'' - u = 0$

(ii) $u'' + 2ku' + u = 0$ with $k > 0$

(iii) $u^{(4)} + 2u'' + u = 0$

(iv) $y' = \Theta y$ where Θ is the matrix of all zeros.

Problem A4.

(i) Suppose A is an $n \times n$ matrix whose elements $a_{ij}(t)$ are each of period T. You may assume existence and uniqueness of a fundamental matrix $\Phi(t)$ for the equation

$$\dot{x}(t) = A(t)x(t).$$

Show that $\Phi(t+T)$ is also a fundamental matrix for the equation, hence $\Phi(t+T) = \Phi(t) \cdot C$ for some non-singular matrix C.

(ii) You may assume that a non-singular matrix C has a logarithm, i.e. , $C = e^B$ for some $n \times n$ matrix B.

Show that there exists a non-singular matrix $P(t)$ of period T such that

$$\Phi(t) = P(t)e^{Bt} \quad \text{Floquet's Theorem.}$$

Hint: Solve for $P(t)$ and check.

(iii) Make the change of variables $y = P(t)u(t)$ and change the periodic system $y' = A(t)y$ into the system

$$u' = Bu.$$

Problem A5. Write a short essay discussing the basic concepts of autonomous dynamical systems. Describe flows, trajectories, equilibrium points, stability, asymptotic stability and local linearizations, etc.

Problem A6. Write a brief discussion of the method of characteristics and then demonstrate the method by solving

$$u_x + u_y = u^2; \quad u(x,0) = x^2.$$

Problem A7. Suppose $u(x,t)$: $x \in R, t \in [0,\infty)$ is of class C^∞ and that u is a solution of

$$\begin{cases} u_{tt} &= u_{xx} \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x). \end{cases}$$

Suppose g and h are in C_0^∞. (C^∞ functions with compact support.) Define

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx \quad \text{(kinetic energy)}$$

and

$$p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx \quad \text{(potential energy)}$$

Prove that $k(t) + p(t)$ is constant and that for sufficiently large times t, $k(t) = p(t)$. Hint: d'Alembert's solution.

Problem A8. State the "strong maximum" principle for Laplace's equation and sketch a proof of it. Tell why one cares.

Part B

Do two problems for full credit.

Problem B1. Consider the Sturm-Liouville operator

$$\mathcal{L}u = \frac{1}{w(x)}\{-(p(x)u')' + q(x)u\}$$

where $p(x) > 0$, $w(x) > 0$ and $1/p(x)$, $q(x)$ and $w(x)$ are locally Lebesgue integrable on an arbitrary interval I (open or closed, bounded or unbounded).

(a) Define what is meant by a solution of $\mathcal{L}u = \lambda u$ on I.

(b) Show that the initial value problem for the differential equation $\mathcal{L}u = \lambda u$ on I with $u(x_0) = c_1$, $(pu')(x_0) = c_2$ at $x_0 \in I$ is equivalent to an integral equation for the vector $(u_1, u_2) = (u, pu')$. Prove the equivalence.

Problem B2. A bounded linear operator P in a complex Hilbert space $\mathcal{H}$ is a projection if and only if $P^2 = P$. Prove that a projection P is an orthogonal projection (so that $P\mathcal{H}\perp(1-P)\mathcal{H}$) if and only if P is selfadjoint. (You may use the known theorem that a bounded linear operator A in $\mathcal{H}$ is selfadjoint if and only if (Ax, x) is real for all $x \in \mathcal{H}$.)

Problem B3. Define a regular Sturm-Liouville operator by the equation $\mathcal{L}u = -(2x)^{-1}((2x)^{-1}u')'$ on $1 \le x \le \sqrt{2}$ using boundary conditions $u(1) = u(\sqrt{2}) = 0$.

(a) Explain how $\mathcal{L}$ and the boundary conditions can be used to construct a selfadjoint operator S in a suitable Hilbert space $\mathcal{H}$. Do not give proofs.

(b) Calculate the eigenvalues and eigenfunctions of S using the solution basis $u_1(x) = \sin(\sqrt{\lambda}x^2)$, $u_2(x) = \cos(\sqrt{\lambda}x^2)$ of the differential equation $\mathcal{L}u - \lambda u = 0$. Do not derive the basis.

(c) Verify by direct integration that the eigenfunctions are orthogonal in $\mathcal{H}$.

Problem B4. Let S be the selfadjoint realization in the Hilbert space $\mathcal{H} = L_2(0, \infty)$ for the singular Sturm-Liouville operator $\mathcal{L} = -u''$ on $0 < x < \infty$ with boundary condition $u'(0) = 0$. Assume that S is defined by the domain $D(S) = \{u \in \mathcal{H} : u'' \in \mathcal{H} \text{ and } u'(0) = 0\}$ and the relation $Su = -u''$ for all $u \in D(S)$.

(a) Show that the Green's function G for the operator S is given by the equation below, where $\zeta = re^{i\theta}$ and $\sqrt{\zeta} = \sqrt{r}e^{i\theta/2}$ on $0 < \theta < 2\pi$:

$$G(x, y, \zeta) = \begin{cases} ie^{i\sqrt{\zeta}y}\cos(\sqrt{\zeta}x)/\sqrt{\zeta} & \text{for } 0 \le x \le y < \infty, \\ ie^{i\sqrt{\zeta}x}\cos(\sqrt{\zeta}y)/\sqrt{\zeta} & \text{for } 0 \le y \le x < \infty. \end{cases}$$

(b) Prove that the operator S has a pure continuous spectrum consisting of all non-negative real numbers.

Written Qualifying Examination in

DIFFERENTIAL EQUATIONS

September 13, 1994

Instructions: The examination has three parts which will be assigned separate scores. A perfect score (100%) requires complete solutions of three problems from part A, three problems from part B and two problems from part C. If you work on more than the required number of problems then state which problems you wish to be graded.

In order to pass the Qualifying Examination you must score at least 60% on part A, at least 60% on part B and at least 60% on part C.

Part A

Ordinary Differential Equations 641

Do three (3) problems for full credit.

Partial credit applies to all problems.

Problem A1. Assume that the eigenvalues of a real $n \times n$ matrix A have negative real part strictly less than -5. Prove that there exists a positive constant M such that for all $x \in R^n$ and $t \geq 0$

$$\|e^{At}x\| \leq M\|x\|e^{-5t}.$$

Problem A2. Consider the initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0$$

on the set X: $|t - t_0| \leq a$, $|x - x_0| \leq b$. Let $M = \max|f(t,x)|$ on X, $h = \min(a, b/M)$. Assume that f is continuous on X, $L \geq 0$ and f satisfies the Lipschitz condition $|f(t,x) - f(t,y)| \leq L|x-y|$ for $(t,x) \in X$, $(t,y) \in X$.

Prove the Picard–Lindelöf existence and uniqueness theorem: There exists a unique solution $x(t)$ defined on the interval $|t - t_0| \leq h$.

Problem A3. Let P and Q be infinitely differentiable. Prove that $x' = P(x,y)$, $y' = Q(x,y)$ is a Hamiltonian system iff $x' = Q(x,y)$, $y' = -P(x,y)$ is a gradient system.

Problem A4. Let the system $x' = f(x)$ define a C^1 flow ϕ_t on the open set E contained in R^n. Prove that the ω limit set of a trajectory Γ is closed and if it is also bounded, then it is nonvoid, compact and connected.

Problem A5. Let $r^2 = x^2 + y^2$, $u = r^4 - 3r^2 + 1$ and consider the planar system $x' = -y + xu$, $y' = x + yu$. Prove by means of the Poincaré - Bendixson theorem that there are two periodic orbits, one in $0 < r < 1$ and another in $1 < r < 3$.

Problem A6. Prove:
Let f be of class C^1 in a simply connected region E in R^2. Assume the divergence of f is not identically zero and either nonnegative in E or non-positive in E. Then $x' = f(x)$ has no closed orbit which is entirely in E.

Part B

Partial Differential Equations 642

Do three (3) problems for full credit.

Partial credit applies to all problems.

Problem B1. Solve the wave equation $u_{tt} = c^2 u_{xx} + e^{2x}$, $u(x, 0) = 0$, $u_t(x, 0) = 0$ on $-\infty < x < \infty$, $0 \le t < \infty$.

Problem B2. Suppose u is of class C^2 and $\Delta u \ge 0$ in a bounded domain Ω with C^2 boundary. State and prove a maximum principle.

Problem B3. Apply the method of odd extensions on the heat kernel formula for $(-\infty, \infty)$ to derive a formula for the solution of the diffusion equation on $(0, \infty)$ with Dirichlet boundary conditions.

Problem B4. State and prove the Classical Fourier pointwise convergence theorem for the representation of a function by its Fourier series.

Problem B5. Prove by means of the Poisson Formula that harmonic functions are of class C^∞.

Problem B6. Define the Sobolev space $H^k(\Omega)$ and give an example of a set Ω and a function $f \in H^1(\Omega)$ which does not agree a.e. with any C^1 function.

Part C

Hilbert Space Theory 643

Do two (2) problems for full credit.

Partial credit applies to all problems.

Problem C1. Let $\mathcal{H}$ be a Hilbert space with inner product (x, y) and let the set $\{e_n\}_{n=1}^{\infty}$ be orthonormal in $\mathcal{H}$. Complete the following:

(a) Prove that the formal series $g = \sum_{n=1}^{\infty}(f, e_n)e_n$ converges for each $f \in \mathcal{H}$ to an element $g \in \mathcal{H}$.

(b) Let the mapping $\mathcal{P}$ be defined on $\mathcal{H}$ by $\mathcal{P}f = g$, $g = \sum_{n=1}^{\infty}(f, e_n)e_n$. Prove that $\mathcal{P}$ is an orthogonal projection on $\mathcal{H}$.

Problem C2. Define a regular Sturm-Liouville operator by the equation $\mathcal{L}u = -x^2(x^2u')'$ on $1/2 \le x \le 1$ using boundary conditions $u(1/2) = u(1) = 0$. Complete the following:

(a) Explain how $\mathcal{L}$ and the boundary conditions can be used to construct a selfadjoint operator $\mathcal{S}$ in a suitable Hilbert space $\mathcal{H}$. Do not give proofs.

(b) Calculate the eigenvalues and eigenfunctions of the selfadjoint operator $\mathcal{S}$, using the solution basis $u_1(x) = \sin(\lambda/x)$, $u_2(x) = \cos(\lambda/x)$ of the differential equation $x^2(x^2u')' + \lambda u = 0$. Do not derive the basis.

(c) Verify by direct integration that the eigenfunctions are orthogonal in the Hilbert space $\mathcal{H}$.

Problem C3. Let $\mathcal{S}$ be the selfadjoint realization in the Hilbert space $\mathcal{H} = L_2(0, \infty)$ for the singular Sturm-Liouville operator $\mathcal{L} = -u''$ on $0 < x < \infty$ with boundary condition $u(0) = 0$. Assume that $\mathcal{S}$ is defined by the domain $D(\mathcal{S}) = \{u \in \mathcal{H} : u'' \in \mathcal{H} \text{ and } u(0) = 0\}$ and the relation $\mathcal{S}u = -u''$ for all $u \in D(\mathcal{S})$. Complete the following:

(a) Show that the Green's function G for the operator $\mathcal{S}$ is given by the equation below, where $\xi = re^{i\theta}$ and $\sqrt{\xi} = \sqrt{r}e^{i\theta/2}$ on $0 < \theta < 2\pi$.

$$G(x, y, \xi) = \begin{cases} e^{i\sqrt{\xi}y}\sin(\sqrt{\xi}x)/\sqrt{\xi} & \text{for } 0 \le x \le y < \infty, \\ e^{i\sqrt{\xi}x}\sin(\sqrt{\xi}y)/\sqrt{\xi} & \text{for } 0 \le y \le x < \infty, \end{cases}$$

(b) Prove that the operator $\mathcal{S}$ has a pure continuous spectrum consisting of all non-negative real numbers.

Problem C4. Let T be a linear operator in a Hilbert space $\mathcal{H}$ with domain $D(T)$ dense in $\mathcal{H}$. Complete the following:

(a) Define the adjoint operator T^*.

(b) Let $G(T)$ be the graph of T in $\mathcal{H} \oplus \mathcal{H}$. Define the operator V on $\mathcal{H} \oplus \mathcal{H}$ by $V(f, g) = (g, -f)$. Prove that the graph of T^* is the orthogonal complement in $\mathcal{H} \oplus \mathcal{H}$ of $V(G(T))$, that is, $G(T^*) = (V(G(T)))^{\perp}$.

(c) Let $G(\bar{T})$ be the closure in $\mathcal{H} \oplus \mathcal{H}$ of the graph of T. Prove that $\mathcal{H} \oplus \mathcal{H} = V(G(\bar{T})) \oplus G(T^*)$.

1993 Preliminary Examination
Differential Equations

This examination consists of two parts, Part I - ODE and Part II - PDE. The two parts are equally weighted. In order to receive maximum credit solutions to problems should be clearly and carefully presented and should be as detailed as possible.

The total number of points is 120. A passing score shall be a total of at least 78 points.

PART I

O D E

The number of points assigned to the ODE problems is as follows:

1	–	10 points
2	–	20 points
3	–	10 points
4	–	10 points
5	–	10 points
Total		60 points

PART I: ODE

1. Let $f : [0,1] \times \mathbf{R} \longrightarrow \mathbf{R}$ be of class C^1 and let φ be a solution of the equation

$$x'' = f(t,x), \ ' = \frac{d}{dt} \tag{1}$$

satisfying

$$\varphi(0) = a, \ \varphi(1) = b.$$

suppose $\dfrac{\partial f}{\partial x} > 0$ for $t \in [0,1], x \in \mathbf{R}$.
Prove that if β is near b, then there exists a solution ψ of (1) such that

$$\psi(0) = a, \quad \psi(1) = \beta.$$

Also prove that only one such solution exists.
Hint: Let $\Theta(0,\alpha)$ be the solution of (1) satisfying

$$\Theta(0,\alpha) = a,$$
$$\Theta'(0,\alpha) = \alpha.$$

Use dependence upon a parameter results to prove the claim.

2. Consider the linear system

$$x' = A(t)x, \qquad (2)$$

where A is a continuous $n \times n$ matrix defined on $(-\infty, \infty)$, and $x \in \mathbb{R}^n$.

a) Show that solutions of initial value problems for (2) are defined on $(-\infty, \infty)$.

b) Let trace $A(t) \equiv 0$, $\ t \in (-\infty, \infty)$ and let $\underline{X}(t)$ be a matrix solution of (2) show that det $\underline{X}(t)$ = constant, $t \in (-\infty, \infty)$.

c) Let $\underline{X}(t)$ be a matrix solution of (2) with det $\underline{X}(t_o) \neq 0$, some t_o. Characterize all other matrix solutions of (2) and provide a characterization of all solutions of (2).

d) Assume that $\underline{X}(t)$ is a matrix solution of (2) with $\underline{X}(0)$ nonsingular and $\underline{X}(t)$ uniformly bounded on $[0, \infty)$. Also assume

$$\liminf_{t \to \infty} Re \int_o^t \text{trace } A(s)ds > -\infty.$$

Prove that $\underline{X}^{-1}(t)$ is uniformly bounded on $[0, \infty)$ and no solution $\varphi(t)$ of (2), not identically zero, can satisfy $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

e) Assume the conditions of d) and let $B(t)$ be a continuous matrix function with

$$\int_o^\infty ||A(t) - B(t)||dt \quad < +\infty.$$

Let Ψ be a solution of

$$\Psi' = B(t)\Psi.$$

Show that Ψ must be uniformly bounded on $[0, \infty)$.
<u>Hint:</u> Ψ also solves

$$\Psi' = A(t)\Psi + (B(t) - A(t))\Psi.$$

Use variation of constants.

3. Consider the system

$$x' = Ax + f(t, x), \tag{3}$$

where A is an $n \times n$ constant matrix, all of whose eigenvalues have negative real part and $f : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous with

$$f(t, x) = o(|x|)$$

uniformly in $t \geq 0$.
Prove that the identically zero solution of (3) is asymptotically stable.
What can you say about the trivial solution of (3) (stability properties) under others assumptions about the eigenvalues of A?

4. Consider the planar system

$$\begin{cases} \dfrac{dx}{dt} = \lambda x + y - \alpha x(x^2 + y^2) \\ \\ \dfrac{dy}{dt} = -x + \lambda y - \alpha y(x^2 + y^2), \end{cases} \tag{4}$$

where $\alpha \in \mathbb{R}$ is fixed and $\lambda \in \mathbb{R}$ is a parameter. Apply the Hopf bifurcation theorem to deduce the existence of nontrivial periodic orbits bifurcating from the trivial solution. What is the approximate period of the bifurcating solutions?
Use other means to deduce the existence of periodic orbits for various cases of the pair (λ, α).

5. Consider again the equation

$$x' = Ax + f(t,x), \tag{5}$$

where A is a negative definite $n \times n$ matrix, i.e. $\exists\ \mu > 0$ such that

$$x^T Ax \leq -\mu|x|^2, \quad (x^T = \text{ transpose of } x)$$

and $f : \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a Lipschitz continuous function which is periodic with respect to t of period τ, i.e.

$$f(t+\tau, x) = f(t,x),\ t \in \mathbf{R}, x \in \mathbf{R}^n,$$

and is such that

$$x \cdot f(t,x) \leq \frac{\mu}{2}|x|^2, \quad |x| = R > 0$$

where R is fixed.
Show that (5) has a periodic solution $x(t)$ of period τ with $|x(t)| \leq R$.

PART II

P D E

The number of points assigned to the pde problems is as follows:

1	–	15 points
2	–	15 points
3	–	15 points
4	–	15 points
Total		60 points

PART II: PDE

1. Let G be an open subset of $\mathbf{R}^n$ and consider $C_o^\infty(G)$.

   a) Define what is meant by a distribution on G.

   b) Let $f : G \longrightarrow \mathbf{R}$ be a measurable function such that $\int_{G_o} |f| dx < \infty$ where G_o is an open subset of G with $\overline{G}_o$ compact, i.e. $f \in L^1_{loc}(G)$, show that each such f defines a unique distribution on G. Give an example of a distribution which is not defined by such an f.

   c) If T is a distribution on G, define $\partial^\alpha T$, where α is a multi-index. Show that if $f \in C^1(G)$, then f and its partial derivatives $\frac{\partial f}{\partial x_i}$, $1 \le i \le n$, define unique distributions $T, T_i, 1 \le i \le n$, on G and $\partial^\alpha T = T_i$ $\alpha = (0, \ldots, 1, 0, \ldots, 0)$, where the 1 appears in the *ith* place.

   d) Let

$$f = x_+^\lambda = \begin{cases} 0, \ x \le 0 \\ x^\lambda, \ x > 0, \end{cases}$$

   where $-1 < \lambda < 0$. Then $f \in L^1_{loc}(\mathbf{R})$, hence f may be thought of as a distribution.

   Compute ∂f. Note that $\frac{\partial f}{\partial x} \notin L^1_{loc}(\mathbf{R})$.

   e) Let $U : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be given by

$$U(x,t) = f(x+t) + g(x-t),$$

   where $f, g \in L^1_{loc}(\mathbf{R})$. Show that

$$\partial_x^2 U = \partial_t^2 U$$

   in the sense of distributions.

   f) Evaluate $\Delta_3(\frac{1}{r^2})$, $r^2 = x^2+y^2+z^2$ in $\mathbf{R}^3$, where Δ_3 is the distributional Laplacian.

2. Again let G be an open set in $\mathbf{R}^n$.

a) Define the Sobolev spaces $H^m(G)$ and $H^m_o(G)$, where m is a nonnegative integer.

b) Give a characterization of these spaces in terms of $L^2(G)$ functions and their distributional derivatives.

c) Let G be an open set in $\mathbf{R}^n$ with $\sup\{|x_1| : (x_1, x_2, \ldots, x_n) \in G\} = K < +\infty$. Then

$$||\partial\varphi||_{L^2(G)} \leq 2\,K\,||\partial_1\varphi||_{L^2(G)}, \; \forall\varphi \in H^1_0(G). \tag{6}$$

(Hint: For $\varphi \in C^\infty_o(G)$ compute

$$\frac{\partial}{\partial x_1}(x_1|\varphi(x)|^2)$$

and use the divergence theorem.)

Use (6) to obtain Poincarè's inequality for functions $\varphi \in H^1_0(G)$, whenever G is a bounded domain.

3. Let H be a complex Hilbert space and let

$$A : D(A) \text{ (the domain of } A) \subset H \longrightarrow H$$

be a densely defined linear operator which is self-adjoint (give definition!). Then

a) (Ax, x) is real for all $x \in D(A)$, ($(\cdot, \cdot)$ is the inner product of H).

b) All eigenvalues of A are real.

c) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

d) Let $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda \neq 0$. Show that $\lambda - A$ is invertible.

e) Let $G = (0, 1)$, $V = H_0^1(G)$, and $H = L^2(G)$. Define $a(u, v) = \int_0^1 \partial u \partial \bar{v} \, dx$. Use what you know about coercive and elliptic sesquilinear forms and the spectral theorem for operators defined by them to find a basis for $L^2(G)$. Be sure to present a fairly complete discussion.

4. Let H a Hilbert space.

a) Define the concepts "strongly continuous semigroup on H" and "infinitesimal generator" of the semigroup.

b) Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup on H with infinitesimal generator A. Show that

$$T(t) : D(A) \longrightarrow D(A)$$

and that the map

$$t \mapsto T(t)x, \quad t \in [0, \infty).$$

is differentiable and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \; x \in D(A).$$

c) State necessary and sufficient conditions in order that a linear operator be the generator of a strongly continuous contraction semigroup on H.

d) Use the above to establish the existence and uniqueness of solutions to the initial value problem

$$U_t = U_{xx} + f(t,x), t > 0, \quad x \in (0,1)$$

$$U(0,x) = \varphi(x)$$

$$U(t,0) = U(t,1) = 0;$$

be sure to provide the appropriate function spaces for U, f, φ.

Department of Mathematics
University of Utah

Written Qualifying Examination in DIFFERENTIAL EQUATIONS September 18, 1992

Directions: Work two of Problems 1-4, two of Problems 5-8 and two of Problems 9-12. Each problem is worth 20 points. 72 or more points will be a passing score. For maximum credit your work must be legible, coherent and as detailed as possible. If you work on more than 6 problems indicate which 6 are to be graded.

Problem 1. Let E be an open subset of R^n. Let f be a continuously differentiable function mapping E into R^n, and let x_0 belong to E.

a) Use the method of successive approximations to prove that the initial value problem $x' = f(x)$, $x(0) = x_0$ has a unique solution on some interval $[-a, a]$, and give an estimate for a.

b) Suggest how the result in a) can be used to prove a similar result for the nonautomomous case.

Problem 2. Let A be a constant $n \times n$ matrix, and define the matrix exponential by $e^A = \sum_{k=0}^{\infty} A^k/k!$.

a) Prove that e^{At} is differentiable (you may use known theorems about scalar power series), and that it satisfies the initial value problem $X' = AX$, $X(0) = I$ (the identity matrix). Justify your steps.

b) Let A have distinct eigenvalues. Prove that for the origin to be a globally asymptotically stable point for $X' = AX$ it is sufficient that all of the eigenvalues of A have negative real parts.

c) Is the condition in b) that A have distinct eigenvalues required in order to prove the result? (no proof necessary)

d) Is the sufficient condition in b) also necessary? Justify your answer.

Problem 3. a) What does Floquet's Theorem say about solutions of linear $n \times n$ systems $X' = A(t)X$ (A is a continuous $n \times n$ matrix)?

b) Show that, given any $n \times n$ matrices B (constant) and $A(t)$ (continuous), there is a change of variables $Y = P(t)X$ which reduces the system $X' = A(t)X$ to the system $Y' = BY$.

c) Show that there is a particular choice of $P(t)$ and B that is related to Floquet's Theorem. Why might you be interested in that choice?

Problem 4. a) Show that the only equilibrium point of the system $x' = -2y - 4x^3 - 4xy^2$, $y' = x - 2x^2y - 2y^3$ is the origin.

b) Draw what conclusions you can about stability, asymptotic stability, and regions of attraction, using first i) linearization and then ii) Lyapunov functions. Make clear the reasons for your conclusions.

c) What if the signs of the nonlinear terms in a) were changed?

d) What kinds of functions f and g would give similar results for systems of the form $x' = -2y + f(x,y)$, $y' = x + g(x,y)$?

Problem 5. The quasilinear Cauchy problem

$$xu_x + yuu_y + xy = 0, \quad u = 5 \quad \text{when} \quad xy = 1,$$

has the solution

$$u(x,y) = -1 + \sqrt{38 - 2xy}.$$

a) Derive this solution by the method of characteristics. (Hint: Along a characteristic $x(s)$, $y(s)$, $u(s)$ find $d(xy)/ds$ and then du/dz where $z = xy$.)

b) Verify that $u(x,y)$ is a solution in the plane domain defined by $xy < 19$.

Problem 6. Consider the Dirichlet problem in a bounded plane domain Ω:

$$u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega,$$

$$u = f \quad \text{on} \quad \partial\Omega,$$

where f is continuous on $\partial\Omega$.

a) Prove that this Dirichlet problem has at most one solution. Define clearly what you mean by a solution.

b) Prove that if Ω is a bounded domain for which the Dirichlet problem is solvable for all $f \in C(\partial\Omega)$ then the mapping $f \in C(\partial\Omega) \to u \in C(\overline{\Omega})$ is continuous in the uniform norms. ($\overline{\Omega}$ denotes the closure of Ω.)

Problem 7. Consider the Dirichlet problem

$$\begin{aligned} \Delta u = f \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset R^n$ is a bounded open set and $f \in L_2(\Omega)$.

a) Define the concept of a weak $H_0^1(\Omega)$ solution of the above problem.

b) Use the Poincaré inequality,

$$\int_\Omega u^2\,dx \le c \int_\Omega |\nabla u|^2\,dx \quad \text{for all} \quad u \in H_0^1(\Omega),$$

to prove the existence of a weak $H_0^1(\Omega)$ solution.

Problem 8. Consider the integral

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4t}\,d\xi.$$

a) Prove that if $f(x)$ is continuous and bounded for all x then u is a solution of the initial value problem

$$u_t = u_{xx} \quad \text{for all} \quad -\infty < x < \infty,\ \ t > 0.$$

$$u(x,0) = f(x) \quad \text{for all} \quad -\infty < x < \infty.$$

Formulate clearly what you mean by a solution.

b) Show how the results of part a) can be used to solve the initial-boundary value problem

$$u_t = u_{xx} \quad \text{for all} \quad 0 < x < 1,\ \ t > 0,$$

$$u(x,0) = g(x) \quad \text{for} \quad 0 \le x \le 1,$$

$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t \ge 0.$$

Define what you mean by a solution and state what conditions on g guarantee that such a solution exists.

Problem 9. Consider the wave equation $u_{tt} = u_{xx}$.

a) State and derive the D'Alembert formula for the solutions u of this equation.

b) Use the method of separation of variables to solve the initial-boundary value problem

$$u_{tt} = u_{xx} \quad \text{for} \quad 0 < x < 1, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t \ge 0.$$

Define what you mean by a solution.

c) Show how the solution of b) can be written in the D'Alembert form.

Problem 10. Let A be a selfadjoint operator in a Hilbert space $\mathcal{H}$ and let $\{E_\lambda\}$ be the spectral family of A. Prove that if a real number μ is in the resolvent set of A then there exists an interval $I = \{\lambda : |\lambda - \mu| < \delta\}$ such that E_λ =const. for all $\lambda \in I$. (You may use without proof the Stieltjes- Hellinger-Stone theorem: $R_z = (A - z)^{-1}$.

$$([E_{b+} + E_{b-} - E_{a+} - E_{a-}]u, u) = \lim_{\epsilon \to 0+} \frac{1}{\pi i} \int_a^b ([R_{\lambda+i\epsilon} - R_{\lambda-i\epsilon}]u, u)\, d\lambda.$$

Problem 11. Consider the regular Sturm-Liouville operator

$$\mathcal{L}u = -e^{-2x}(e^{2x}u')' - u$$

on $0 \le x \le \pi$ with boundary conditions $u(0) = 0$, $u(\pi) = 0$.

a) Show how $\mathcal{L}u$ and the boundary conditions can be used to construct a selfadjoint operator S in a suitable Hilbert space $\mathcal{H}$.

b) Calculate the eigenvalues and the corresponding eigenfunctions of S.

Problem 12. The singular Sturm-Liouville operator $\mathcal{L}u = -u''$ on $-\infty < x < \infty$ is known to have a unique selfadjoint realization T in the Hilbert space $\mathcal{H} = L_2(-\infty, \infty)$. It is defined by $D(T) = \mathcal{H} \cap \{u : u'' \in \mathcal{H}\}$ and $\mathcal{L}u = -u''$ for all $u \in D(T)$. (Do not prove this). Calculate the Green's function (resolvent kernel) of T.