UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 16, 2010

Instructions: The examination has two parts consisting of six problems each. You are to work three problems from part A and three problems from part B. If you work more than the required umber of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 72.

A. Ordinary differential Equations: Do three problems for full credit

A1. Suppose $f(x,t)$ is continuous for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and let $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$ be a point. Assume that there is a constant $L < \infty$ so that

$$|f(x_1,t) - f(x_2,t)| \le L|x_1 - x_2|$$

for all x_1, x_2, t. Show that there is and $\epsilon > 0$ and a unique function $y \in \mathcal{C}^1([t_0, t_0+\epsilon], \mathbb{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases}$$

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\frac{dx}{dt} = A(t)\,x \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Let $\zeta(t) \in \mathcal{C}^1([0,\infty), \mathbb{R}^n)$ be a solution of (1). Define: $\zeta(t)$ is a *Stable* solution. (This notion is also sometimes called *Liapunov Stable.*)

(b) Show that the zero solution of (1) is stable if and only if every eigenvalue ω of $\Phi(T)$ satisfies either $|\omega| < 1$, or $|\omega| = 1$ but in this case ω appears only in matrices J_i (from the Jordan Canonical Form for $\Phi(T)$) such that J_i is a 1×1 matrix.

A3. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathrm{e}\,\lambda_i < 0$ for all $i = 1, \ldots, n$. Let $B(t)$ be a smooth real $n \times n$ matrix function and $\beta > 0$ such that $|B(t)| \le e^{-\beta t}$ for all $t \ge 0$. Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \frac{dx}{dt} &= [A + B(t)]x, \\ x(0) &= x_0 \end{aligned} \tag{2}$$

(a) Show that the solution to (2) exists for all time and $\lim\limits_{t\to\infty} x(t) = 0$.

[Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \ge 0$ is a constant that satisfy $u(t) \le c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \ge 0$. Then Gronwall's Inequality implies $u(t) \le c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \ge 0$.]

(b) Let $z(t) = 0$ for all t be the zero solution. Define what it means for z to be *Asymptotically Stable.* Show that z is asymptotically stable for this equation.

A4. Suppose that $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a smooth function and $\Gamma_0 \subset \mathbb{R}^2$ is a nonconstant periodic orbit of the parameter dependent autonomous system for the parameter value $\epsilon = \epsilon_0$,

$$\frac{dx}{dt} = f(x, \epsilon_0). \tag{3}$$

(a) Define the following: Σ is a Poincaré section for for Γ_0 at the point $x_0 \in \Gamma_0$ and $P : \Sigma \to \Sigma$ is the Poincaré first return map.

(b) Define what it means for the orbit Γ_0 to be *hyperbolic.* Your definition should make the following theorem true.

Theorem 1. *Suppose that the Γ_0 is a hypererbolic nonconstant periodic orbit of the system (3). Then there is an $\eta > 0$ depending on f and Γ_0 and a continuous one-parameter family of closed curves Γ_ϵ such that $\Gamma_0 = \Gamma_{\epsilon_0}$ and that Γ_ϵ are nonconstant periodic orbits of the perturbed systems*

$$\frac{dx}{dt} = f(x, \epsilon)$$

for all ϵ such that $|\epsilon - \epsilon_0| < \eta$.

(c) Sketch a proof of Theorem 1.

(d) Show that the $\epsilon = 1$ solution ($x = \cos t$) can be continued to a one parameter family of solutions in a neighborhood of $\epsilon = 1$ by checking that the conditions of the theorem are satisfied:

$$\ddot{x} + (x^2 + \dot{x}^2 - \epsilon)\dot{x} + x = 0$$

A5. Show that the planar system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \left(1 - 3x^2 - y^2\right) y - x \end{aligned}$$

has at least one nonzero periodic solution.

A6. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Construct an approximation to the center manifold at the origin and use it to determine the local behavior of solutions.

$$\begin{aligned} \dot{x} &= xy + x^3 \\ \dot{y} &= -y - x^2 y. \end{aligned}$$

B. Partial Differential Equations. Do three problems to get full credit

B1. Consider the scalar first order PDE

$$\begin{aligned} u\, u_x + u_y &= 0, \\ u(x, 0) &= h(x). \end{aligned} \tag{4}$$

(a) Define what it means to be a weak solution of (4).

(b) Suppose that $u_0 > 0$ and

$$h(x) = \begin{cases} u_0, & \text{if } x \le 0; \\ u_0(1-x), & \text{if } 0 < x < 1; \\ 0, & \text{if } x \ge 1. \end{cases}$$

Show that a shock develops at a finite time and describe the global weak solution.

B2. Traffic flow may be modeled by the equation

$$\frac{\partial}{\partial x}\Big(G(\rho)\Big) + \frac{\partial}{\partial t}\rho = 0$$

where $\rho(x,t)$ is the density of traffic flow along a highway. If $\rho_{\max}$ is the maximum density of cars in a bumper to bumper situation, then

$$G(\rho) = c\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

where $c > 0$ is the free speed constant. Suppose that the initial concentration is given by

$$\rho(x,o) = \begin{cases} \frac{1}{2}\rho_{\max}, & \text{if } x < 0; \\ 0, & \text{if } x \ge 0. \end{cases}$$

Describe the weak solution. Does a shock form? Interpret your result in terms of traffic flow.

B3. Let $D \in \mathbb{R}^n$ be a bounded, connected domain with smooth boundary. Let ϕ be an eigenfuncion of the Laplacian on D with Dirichlet boundary conditions ($\phi = 0$ on ∂D) and with eigenvalue λ.

$$\Delta\phi + \lambda\phi = 0. \tag{5}$$

(a) Show that there is a finite constant C so that for every nonzero continuously differentiable $u \in C^1(D)$ such that $u = 0$ on ∂D satisfies

$$\int_D u^2 \le C \int_D |Du|^2.$$

(b) By integrating eigenfunctions of (5), conclude that eigenvalues are positive $\lambda > 0$.

(c) Denote by

$$\lambda_1(D) = \inf_{u \in C_0^1(D),\ u \not\equiv 0} \frac{\int_D |Du|^2}{\int_D u^2}$$

the least value of the Rayleigh quotient (which is at least $1/C$). Assuming that $\lambda_1(D)$ is an eigenvalue of D with Dirichlet boundary conditions, show that if two such domains satisfy $D_1 \subset D_2$, then $\lambda_1(D_1) \ge \lambda_1(D_2)$.

B4. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x \in \mathbb{R}.$$

along with initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi.$$

(b) Briefly explain the concepts of *domain of dependence*, and *finite propogation speed*, and prove that these properties follow d'Alembert's Formula.

(c) Suppose that we restrict the spatial domain to $x > 0$ and impose the boundary condition $u(0,t) = 0$ for $t \geq 0$. If the initial conditions satisfy $g(0) = h(0) = 0$, using d'Alembert's Formula as a starting point, find a closed form solution for $u(x,t)$ with $x, t > 0$.

B5. Let $D \in \mathbb{R}^d$ be a bounded domain with smooth boundary ∂D. Assume that that a complete orthonormnal set of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ for the Laplacian on D with Dirichlet boundary conditions are known, in other words

$$-\Delta\phi_n = \lambda_n\, \phi_n, \qquad \text{in } D,\ n = 1, 2, 3\ldots, \tag{6}$$

with $\phi_n(x) = 0$ for $x \in \partial D$, and $\int_D \phi_n^2\, dx = 1$.

(a) Given $f \in L^2(D)$, find a closed form series solution $u(x,t)$ for the initial, boundary value problem for the heat equation

$$\begin{aligned} -u_t + \Delta u &= 0, && t > 0,\ x \in D \\ u(x,t) &= 0, && x \in \partial D, \\ u(x,0) &= 0, && x \in D. \end{aligned}$$

(b) Show that there is a constant c such that

$$\left| \int_D u(x,t)\, \phi_n(x)\, dx \right| \leq c\, e^{-\lambda_n t}$$

for all n. Find $\lim_{t\to\infty} u(x,t)$.

(c) Give an example (the case $d = 1$ will suffice) which shows that solutions of the heat equation do not exhibit finite propagation speed.

B6. Suppose that $u(x_1, x_2)$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbb{R}^2$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $y = (y_1, y_2) \in \Omega$ and $r > 0$ is chosen such that $B_r(y) \subset \Omega$ (ball of radius r centered at y) then

$$u(y_1, y_2) = \frac{1}{2\pi}\int_0^{2\pi} u(y_1 + r\cos\theta, y_2 + r\sin\theta)\, d\theta.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.