UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 18, 2014.

Instructions: This examination has two parts consisting of six problems in
part A and five in part B. You are to work three problems from
part A and three problems from part B. If you work more than the
required number of problems, then state which problems you wish
to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be
clearly and carefully presented and should be detailed as possible.
All problems are worth 20 points.   A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $f(t,x) \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R^n}, \mathbf{R^n})$ be a continuously differentiable function. Show that there is a $\varepsilon > 0$ and a unique function $y \in \mathcal{C}^1([0,\varepsilon], \mathbf{R}^n)$ that satisfies the initial value problem for $0 \le t \le \epsilon$

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ x(0) = x_0. \end{cases}$$

A2. Let A be an $n \times n$ real matrix and let $\varphi(t, x_0)$ denote the solution of the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0. \end{cases}$$

Suppose that k not necessarily distinct eigenvalues of A have non-positive real part and the rest have positive real part

$$\Re e\,\lambda_1 \le \cdots \le \Re e\,\lambda_k \le 0 < \Re e\,\lambda_{k+1} \le \cdots \le \Re e\,\lambda_n.$$

Show that there is a k-dimensional subspace $E \subset \mathbf{R}^n$ such that

(a) E is forward invariant: for every $x_0 \in E$ we have $\varphi(t, x_0) \in E$ for all $t \ge 0$;

(b) If $x_0 \notin E$ then the distance from $\varphi(t, x_0)$ to E tends to infinity as $t \to \infty$.

A3. Let A be an $n \times n$ real matrix such that all eigenvalues λ_i have negative real parts $\Re e\,\lambda_i < 0$. Let $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ be a smooth function such that f and its derivative df vanish at zero ($f(0) = 0$ and $d_0 f = 0$.). Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(x) \\ x(0) = x_0 \end{cases} \tag{1}$$

(a) Define: the zero solution of (1) is *asymptotically stable*.

(b) Prove that the zero solution of (1) is asymptotically stable.

[Don't just quote a theorem. Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \ge 0$ is a constant that satisfies $u(t) \le c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \ge 0$. Then Gronwall's Inequality implies $u(t) \le c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \ge 0$.]

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A4. Let $p(t) > 0$ be a smooth function that is periodic of least period $T > 0$. Show that every solution approaches zero as $t \to \infty$.

$$x'' + p(t)\,x' + x = 0$$

A5. Consider Selkov's model of glycolysis, the process whereby living cells break down sugar, where x and y are concentrations of ADP and F6P. Show that it has a nonconstant periodic solution.

$$\dot{x} = -x + \frac{y}{10} + x^2 y$$

$$\dot{y} = \frac{1}{2} - \frac{y}{10} - x^2 y$$

[Hint: show the polygon with vertices $(0,0)$, $(0,5)$, $(\frac{1}{2},5)$ and $(\frac{11}{2},0)$ is forward invariant.]

A6. Show that for small $|\varepsilon|$, there is a 2π-periodic solution near zero to the inhomogeneous equation

$$\ddot{y} + 5y = \varepsilon \sin^3 t.$$

B. Partial Differential Equations. Do three problems to get full credit

B1. Suppose that $\rho(x,t)$ is the number density of cars evolving according to the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

with u the car speed. Take $u = 1 - \rho$. A queue is building up at a traffic light $x = 1$ so that, when the light turns green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0, & x < 0 \text{ and } x > 1 \\ x, & 0 < x < 1. \end{cases}$$

(a) Use the method of characteristics to solve the Cauchy problem. In particular, construct the characteristic projections and identify any shocks or rarefaction waves.

(b) Hence, show that a collision first occurs at $x = 1/2$ when $t = 1/2$ and, thereafter, there is a shock $S(t)$. Use the Rankine-Hugoniot condition to derive the following expression for the speed of the shock:
$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

B2. Consider the global Cauchy problem

$$u_t = D u_{xx} + b u_x + cu, \quad x \in \mathbb{R}, t > 0, \quad u(x,0) = g(x),$$

where $D > 0$, b and c are constant coefficients.

(a) Convert to a diffusion equation by setting $v(x,t) = u(x,t)\exp(hx + kt)$ for appropriately chosen h, k.

(b) Solve the diffusion equation for v using a transform method.

(c) Hence, show that if $c < 0$ and g is bounded, then $u(x,t) \to 0$ as $t \to \infty$.

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B3. A problem in the dynamics of the overhead power wire for an electric locomotive leads to the model
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x \neq X(t),\ t > 0,$$
where X is a prescribed smooth function with $0 < X'(t) < 1$. Here $X(t)$ is the locomotive position and u is the displacement of the wire. Across $x = X$ there are prescribed discontinuities
$$\left[\frac{\partial u}{\partial x}\right]_{X^-}^{X^+} = -V(X(t), t), \quad [u]_{X^-}^{X^+} = 0.$$
Suppose that $u = u_t = 0$ at $t = 0$. By constructing a weak solution that takes into account the discontinuities across X, show that
$$u(x, t) = \int_0^{\tau_0} V(X(\tau), \tau)\left(1 - X'(\tau)^2\right) d\tau,$$
where the range of τ is taken so that $y = X(\tau)$ lies within the range
$$\tau - t < y - x < t - \tau, \quad \tau > 0.$$
[Hint: you will need to derive a relationship between discontinuities in $\partial_t u$ and in $\partial_x u$ along $X(t)$. You will also need to consider the domain of dependence in the (x, t)-plane].

B4. A function $u \in C^2(\Omega)$, $\Omega \subset \mathbf{R}^n$, is subharmonic (superharmonic) in Ω if $\nabla^2 u \geq 0$ ($\nabla^2 u \leq 0$) in Ω.

(a) Show that if u is subharmonic then for every ball $B_R(\mathbf{x}) \subset\subset \mathbf{R}$
$$u(\mathbf{x}) \leq \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\mathbf{x})} u(\sigma)\, d\sigma.$$
Hence show that
$$u(\mathbf{x}) \leq \frac{n}{\omega_n R^n} \int_{B_R(\mathbf{x})} u(\mathbf{y})\, d\mathbf{y}.$$
How are the inequalities changed if u is superharmonic?

(b) Show that if $u \in C(\overline{\Omega})$ is subharmonic (superharmonic), the maximum (minimum) of u is attained only on $\partial\Omega$, unless u is constant.

(c) Show that if u is harmonic in Ω then u^2 is subharmonic.

(d) Let u be subharmonic in Ω and $F : \mathbf{R} \to \mathbf{R}$ smooth. Under what conditions on F is $F(u)$ subharmonic?

B5. Suppose
$$\nabla^2 u - cu = f \ \text{in}\ D,$$
with
$$u = g \ \text{on}\ \partial D,$$
where D is bounded and ∂D is smooth.

(a) Assuming that a solution u exists, use an energy argument to show that it is unique when $c > 0$.

(b) If $c < 0$ and $f = g = 0$, and D is the region $r^2 = x^2 + y^2 < 1$, show that

$$u = a_0 J_0 \left(r\sqrt{-c} \right), \quad a_0 \text{ constant,}$$

is a solution as long as $\sqrt{-c}$ is a zero of the Bessel function $J_0(x)$, which satisfies

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x} \frac{d J_0}{dx} + J_0 = 0.$$

(c) If $c < 0$ and $f = g = 0$, and D is the region $1 < x^2 + y^2 + z^2 < 4$, show that there are non-trivial solutions if $c = -n^2 \pi^2, n = 1, 2, 3 \ldots$.