UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 5, 2011.

---

Instructions: The examination has two parts: part A consisting of six problems, and part B consisting of five problems. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 72.

---

**A. Ordinary differential Equations: Do three problems for full credit**

A1. Let $f(x,t)$ be continuous for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and let $x_0 \in \mathbb{R}^n$ be a point. Show that there is and $\epsilon > 0$ and a function $y \in \mathcal{C}^1([0,\epsilon], \mathbb{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(0) = x_0. \end{cases}$$

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in $t$ and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\, x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: *Floquet Matrix, Floquet Multiplier and Floquet Exponent.* How are these related to $\Phi(t)$?

(b) Show that every solution of (1) tends to zero ($x(t, x_0) \to 0$ as $t \to \infty$) if and only if every Floquet Multiplier $\omega$ satisfies $|\omega| < 1$.

A3. Let $A$ be an $n \times n$ real matrix whose eigenvalues $\lambda_i$ satisfy $\Re\mathrm{e}\, \lambda_i < 0$ for all $i = 1, \ldots, n$. Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{dx}{dt} = Ax + |x|^2 x, \\ x(0) = x_0 \end{cases} \tag{2}$$

(a) Show that if $|x_0|$ is small enough, the solution to (2) exists for all time and

$$\lim_{t\to\infty} x(t) = 0.$$

[Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \geq 0$ is a constant that satisfy $u(t) \leq c_0 + \int_0^t g(s)\, u(s)\, ds$ for all $t \geq 0$. Then Gronwall's Inequality implies $u(t) \leq c_0 \exp\left(\int_0^t g(s)\, ds\right)$ for all $t \geq 0$. ]

(b) Let $z(t) = 0$ for all $t$ be the zero solution. Define what it means for $z$ to be *Asymptotically Stable.* Show that $z$ is asymptotically stable for this equation.

A4. Suppose that $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Consider the Hamiltonian System

$$\begin{aligned}\dot{x} &= H_y(x, y)\\ \dot{y} &= -H_x(x, y).\end{aligned}$$

Suppose that $\Gamma_0$ is a nonconstant periodic orbit for this system. Show that $\Gamma_0$ cannot be Asymptotically Orbitally Stable. (This is sometimes referred to as asymptotic stability of the periodic orbit $\Gamma_0$ or asymptotic Poincarè stability.)

A5. Determine whether the planar system

$$\begin{aligned}\dot{x} &= x - x^3 - xy^2,\\ \dot{y} &= 2y - y^5 - yx^4\end{aligned}$$

has any nonconstant periodic solutions.

A6. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Construct an approximation to the center manifold at the origin and use it to determine the local behavior of solutions.

$$\begin{aligned}\dot{x} &= -y - xz,\\ \dot{y} &= x - y^3,\\ \dot{z} &= -z - 2xy - 2x^4 + x^2.\end{aligned}$$

**B. Partial Differential Equations. Do three problems to get full credit**

B1. Suppose that $u \in C^2$ solves

$$u_{tt} - \operatorname{div}(aDu) = 0, \qquad \text{in } \mathbb{R}^n \times (0, \infty),$$

where $a \in C^1(\mathbb{R}^n)$ satisfies $\frac{1}{2} \le a(x) \le 1$ for all $x$.

(a) Assunimg $\operatorname{supp}_x u(x,t) \subset B(0,R) \subset \mathbb{R}^n$ for all $0 \le t \le T$, show that the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} u_t^2 + a|Du|^2\, dx, \qquad 0 \le t \le T,$$

is conserved.

(b) Using an appropriately defined local energy, establish that solutions exhibit finite propagation speed: if for some $r > 0$, $u(x, 0)$ and $u_t(x, 0)$ are zero for $x \in B(x_0, r) \subset \mathbb{R}^n$, then $u$ is zero in the cone $K = \{(x,t) : 0 \le t \le r, |x - x_0| < r - t\}$.

B2. The time-dependent Schrödinger equation can be written

$$i\,u_t = -\,\Delta\, u + Vu,$$

where $u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, is the complex-valued "wave function," and $V(x) \ge 0$ is a given continuous real-valued potential function.

(a) Using complex exponential plane waves, describe the *dispersion relation* for this equation in case $V \equiv 1$. Is this problem dispersive?

(b) Assuming $V$ is bounded, and the problem is formulated over the smooth bounded connected spatial domain $U \subset \mathbb{R}^n$, with homogeneous Dirichlet boundary conditions, prove that the solutions to the time dependent problem can be expressed as a discrete sum involving solutions to the *time-independent* problem

$$E_k \phi_k = \Delta\, \phi_k + V \phi_k, \qquad \text{in } U$$

where $0 < E_1 \leq E_2 \leq \cdots$.

B3. Consider Burger's equation

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= 0, \qquad && \text{in } \mathbb{R} \times (0, \infty), \\ u &= g, && \text{on } \mathbb{R} \times \{0\}. \end{aligned}$$

(a) Construct an integral solution satisfying the Rankine-Hugoniot condition which exhibits "nonphysical shocks."

(b) Write down the explicit entropy condition for this problem and explain why it rules out the solution in part (a).

(c) Construct another integral solution which satisfies the entropy condition, and explain why this condition does not allow shocks.

B4. Let $A(x)$ be an $n \times n$ matrix-valued function defined on the smooth bounded connected domain $U \subset \mathbb{R}^n$. Assume $A(x) = (a_{i,j}(x))$ is uniformly symmetric positive definite, and $a_{i,j} \in C^\infty(\overline{U})$ for all $1 \leq i, j \leq n$. Consider the problem

$$\begin{aligned} -\operatorname{div}(ADu) &= \lambda u, \qquad && \text{in } U, \\ u &= 0, && \text{on } \partial U. \end{aligned}$$

(a) Characterize the principal (smallest) eigenvalue $\lambda_1$ in terms of the Rayleigh quotient for this problem.

(b) Prove that $\lambda_1 > 0$, and that the associated eigenfunction $u_1$ is a minimizer of the Rayleigh quotient.

B5. Consider the problem

$$\begin{aligned} u_t &= u_{xx}, \qquad && t \in (0,1),\ x \in (0,1), \\ u(0,t) = u(1,t) &= 0, && t \in (0,1), \\ u(x,1) &= f, && x \in (0,1). \end{aligned}$$

(a) What are the least restrictive condition you can find on $f$ which guarantee that the problem admits a solution $u$, bounded in the norm

$$\|u\|_B = \sup_{t \in [0,1]} \left( \int_0^1 u(x,t)^2\, dx \right)^{1/2} ?$$

(b) Under what conditions on $f$ does the stability estimate

$$\|u\|_B \leq C \|f\|_{L^2(0,1)}$$

hold?

(c) Assuming $u \in C^2(\overline{U_1})$, where $U_1 = (0,1) \times (0,1]$, is uniqueness of solutions guaranteed?

(Error corrected 6-28-11.)