DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

August 2007

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let S^1 denote the unit circle in $\mathbb{C}$ and define $f : S^1 \times S^1 \to \mathbb{R}$ by $f(z, w) = Re(z) + Re(w)$. Show that f is a Morse function and compute the indices of the critical points of f.

2. Let $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ be defined by
$$f([x_0 : x_1 : x_2 : x_3]) = [x_0^2 : x_1^2 : x_2^2 : x_3^2]$$
Compute the degree and the Lefschetz number of f.

3. Let ω be an n-form on $\mathbb{R}^n$ with compact support. Show that there exists a compactly supported $(n-1)$ form η with $\omega = d\eta$ if and only if $\int_{\mathbb{R}^n} \omega = 0$. You are allowed to use the standard results about deRham cohomology of $\mathbb{R}^n$ and spheres.

4. Let $f : X \to Y$ be a smooth map between smooth manifolds that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$
$$df_x : T_x(X) \to T_{f(x)}(Y)$$
is an isomorphism. Prove that there is a neighborhood U of Z such that $f|U : U \to f(U)$ is a diffeomorphism.

5. Let X be any subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ a function with the following property. For every $x \in X$ there is a neighborhood U_x of x in $\mathbb{R}^n$ and a smooth function $f_x : U_x \to \mathbb{R}$ such that $f = f_x$ on $X \cap U_x$. Show that there is an open set U in $\mathbb{R}^n$ containing X and a smooth function $g : U \to \mathbb{R}$ such that $g|X = f$.

6. Identify the set of all $n \times n$ matrices with $\mathbb{R}^{n^2}$ and let $O(n)$ be the subset consisting of orthogonal matrices (i.e. those that satisfy $AA^\mathsf{T} = I$). Show that $O(n)$ is a submanifold of $\mathbb{R}^{n^2}$ and compute the tangent space at I.

7. Prove that for any $x, y \in \mathbb{R}^n$ there a compactly supported isotopy taking x to y (i.e. there is an isotopy $f_t : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_0 = id$, $f_1(x) = y$ and $f_t = id$ outside some ball).

8. Suppose X is a smooth submanifold of $\mathbb{R}^3$ and a nonempty closed surface. Prove that there is a plane $P \subset \mathbb{R}^3$ such that $X \cap P$ is a nonempty collection of circles.

B. Answer four of the following questions. Each question is worth ten points.

9. Give an example of an irregular (i.e. not normal) covering space of the Klein bottle (with a proof).

10. Let $f : M \to N$ be a covering map between closed connected nonempty manifolds. Suppose f is homotopic to a constant map. Prove that N is a point.

11. Give an example of a space X such that $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2$ and $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$.

12. Suppose X is a connected 2-dimensional CW complex such that $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2$ and $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$. Compute $H_i(X;\mathbb{Z}_2)$, $H^i(X;\mathbb{Z})$ and $H^i(X;\mathbb{Z}_2)$ for all $i \in \mathbb{Z}$.

13. Prove or disprove:

    a) For every $d \in \mathbb{Z}$ there is a degree d map $f_d : T^2 \to S^2$.

    b) For every $d \in \mathbb{Z}$ there is a degree d map $g_d : S^2 \to T^2$.

14. Let $f, g : S^2 \to S^2$ be two maps. Show that at least one of the following holds:

    - there exists $x \in S^2$ such that $f(x) = g(x)$,
    - there exists $x \in S^2$ such that $f(x) = -g(x)$, or
    - both f and g are homotopic to constant maps.

15. Prove that $\mathbb{C}P^3$ and $S^2 \times S^4$ have cell complex structures with isomorphic cellular chain complexes, and therefore $H^i(\mathbb{C}P^3;\mathbb{Z}) \cong H^i(S^2 \times S^4;\mathbb{Z})$ for all i. Also prove that cohomology rings $H^*(\mathbb{C}P^3;\mathbb{Z})$ and $H^*(S^2 \times S^4;\mathbb{Z})$ are not isomorphic. Are $\mathbb{C}P^3$ and $S^2 \times S^4$ homotopy equivalent? You are allowed to use the standard cell complex structure of spheres and complex projective spaces.

16. Compute $\pi_2(X)$ where X is the wedge sum of S^1 and S^2.