DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

August 2010

Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. Give a nowhere vanishing vector field on $SL_n(\mathbb{R})$.
2. Suppose M is a smooth manifold with a smooth plane field. If the plane field is integrable, then prove that the bracket of any pair of smooth vector fields on M that are tangent to the plane field is another smooth vector field on M that is tangent to the plane field.
3. Give an example of a smooth plane field on a manifold that is not integrable.
4. Given a Lie group G with Lie algebra $\mathfrak{g}$ and some $v \in \mathfrak{g}$, let $\{\theta_t^v\}_{t\in\mathbb{R}}$ be the corresponding 1-paramater group of diffeomorphisms of G. Prove that $\theta_t^v(gh) = g\theta_t^v(h)$ for any $t \in \mathbb{R}$ and $g, h \in G$.
5. Prove that there are uncountably many 3-dimensional foliations on a 5-dimensional torus.
6. Let U be a connected Lie subgroup of $GL_n(\mathbb{R})$. Suppose that every element of U has all of its entries below the main diagonal equal to 0, and all of its entries on the main diagonal equal to 1. Why is U diffeomorphic to $\mathbb{R}^k$ for some k?
7. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, prove there is a Lie subgroup $H \leq G$ whose Lie algebra is $\mathfrak{h}$. (The entire proof with all details would be best. If not, then try to write the main ideas of the proof.)
8. Let X be a smooth vector field on S^2. Prove that $X(p) = 0$ for some $p \in S^2$.

B. Answer all of the following questions.

9. Let X be a topological space and $x_0 \in X$ a basepoint.
   (a) Define $\pi_1(X, x_0)$ (describe it as a set and define the group operation; you don't have to prove that it is well defined or that it is a group).
   (b) If X is path-connected and $x_1 \in X$ prove that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ (write down an explicit isomorphism and check that it works).
10. (a) Define a covering space $p : \tilde{X} \to X$.
   (b) Let $p : \tilde{X} \to X$ be a covering space, $\tilde{x}_0 \in \tilde{X}$, $x_0 \in X$, and $p(\tilde{x}_0) = x_0$. Show that $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective (carefully state the lifting property you are using).

11. Let X be the cell complex obtained from the circle S^1 by attaching two 2-cells e_2^2 and e_3^2 with attaching maps of degrees 2 and 3 respectively. Compute the fundamental group of X. Carefully state any theorems you are using.

12. (a) Define the concepts of chain morphisms and chain homotopies.
    (b) Show that every map $S^2 \to T^2$ from the 2-sphere to the 2-torus is null-homotopic.
    (c) Write down explicit cellular chain complexes $C(S^2)$ and $C(T^2)$ for S^2 and T^2 and a chain morphism $\Phi : C(S^2) \to C(T^2)$ which is not chain homotopic to a chain morphism representing a constant map.

13. Let X be the space obtained from two n-spheres by identifying them along their equatorial $(n-1)$-spheres. Using any method you like compute $H_i(X)$ for all i. State any theorems you are using (e.g. Mayer-Vietoris).

14. (a) State the universal coefficient theorem for cohomology.
    (b) Suppose X is a space such that $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2\mathbb{Z}$ and $H_i(X) = 0$ for $i > 1$. Compute $H^i(X; \mathbb{Z}/2\mathbb{Z})$ for all i.