DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

May, 2018

Instructions: This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. To pass the exam, you will have to pass each part. To pass each part, you will have to demonstrate mastery of the material. It will be up to the faculty grading the exam to determine if sufficient understanding of the material is demonstrated. The exam should be viewed as a dialogue, questions asked and questions answered. The exam will not be graded using a numeric system. There is not a set standard of how many questions need to be answered correctly, so please answer as many questions as you can. Not all questions are equally difficult, and the grading will take this into account.

Section A.

1. Let M be the smooth submanifold of $\mathbb{R}^3$ consisting of points that satisfy $x^2 + y^2 - z^2 = 1$. Let N be the smooth submanifold of $\mathbb{R}^3$ consisting of points that satisfy $x + y + z = 0$. Prove that $M \cap N$ is a smooth manifold.

2. Let M be a smooth manifold containing a point p, and let $C_p^\infty(M)$ be the germs of smooth real-valued functions on M at the point p. Prove, using only the definition of a derivation, that $\delta([\lambda]) = 0$ for any constant $\lambda \in \mathbb{R}$ and any derivation $\delta : C_p^\infty(M) \to \mathbb{R}$.

3. Let the inclusion of SO(2) into SO(3) be induced by the inclusion of $\mathbb{R}^2$ into $\mathbb{R}^3$. Find all connected, 2-dimensional Lie subgroups of SO(3) that contain SO(2).

4. Let
$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$
Find $\exp(A)$.

5. Let M and N be smooth manifolds, and let $f : M \to N$ be smooth. Prove that the set of all $p \in M$ for which the differential $\mathrm{D}_p f : T_pM \to T_{f(p)}N$ is injective is an open set.

6. Let $f : M \to N$ and $g : N \to P$ be diffeomorphisms of smooth manifolds. Let $\sigma(f)$ equal 1 if f preserves orientation and -1 otherwise. Defining $\sigma(g)$ and $\sigma(g \circ f)$ similarly, prove that $\sigma(g \circ f) = \sigma(g)\sigma(f)$.

Section B.

7. Let X be a path-connected topological space.
   (a) Let x_0, x_1 be points in X, and let $\alpha\colon I \to X$ be any path from x_0 to x_1. Explain how α induces an isomorphism of groups $\alpha_*\colon \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x_1)$.

(b) Give an example where $x_1 = x_0$ (so α is a loop at x_0), but the map $\alpha_* \colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ is not the identity map.

(c) Give an example of a space X with $H_1(X; \mathbb{Z}) = 0$ but $\pi_1(X, x_0) \neq 0$. State carefully the results you are using.

8. Let $n \geq 1$ be a positive integer.
   (a) Give an elementary description of the universal cover $X \to \mathbb{RP}^n$ of $\mathbb{RP}^n$.
   (b) Describe a CW structure on $\mathbb{RP}^n$.
   (c) Compute the cohomology ring $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ and the map (from (a)) $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)$.

9. (a) State the van Kampen theorem for fundamental groups. Use it to compute, $\pi_1(K, *)$, where K is the Klein bottle, and $\pi_1(S_g, *)$, where S_g is the orientable surface of genus g. (Here $*$ is any base-point.)
   (b) Show that S_g is indeed orientable; carefully state the results you are using.

10. Let $C_\bullet$ be a chain complex of finitely-generated $\mathbb{Z}$-modules such that $C_i = 0$ for all $|i| \gg 0$ (i.e., the complex $C_\bullet$ is bounded). Define the Euler characteristic of C to be

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{rk}(C_i),$$

where $\mathrm{rk}(C_i)$ denotes the rank of the (finitely-generated) abelian group C_i.

   (a) Let X be a finite CW complex, i.e. one having finitely many cells in total. Define $\chi(X)$ to be $\chi(C_\bullet^{\mathrm{CW}}(X))$, where $C_\bullet^{\mathrm{CW}}(X)$ denotes the cellular chain complex of X. Show that

$$\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{rk}(H_i(X)),$$

   and that this quantity equals $\sum_{i \in \mathbb{Z}} (-1)^i \dim_F(H_i(X; F))$ for any field F.

   (b) Let X be a connected compact manifold of odd dimension; you may assume X is homeomorphic to a finite CW complex. Show that $\chi(X) = 0$.