Preliminary Exam, Numerical Analysis, August 2015

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1. (**Numerical Integration**).
Find a formula of the form

$$\int_0^{2\pi} f(x)dx = A_1 f(0) + A_2 f(\pi)$$

that is exact for any function having the form

$$f(x) = c + d\cos(x).$$

Problem 2. (**Matrix Properties**).
Show that if a matrix $B \in \mathrm{R}^{m\times m}$ is both upper triangular and orthogonal, then it is diagonal.

Problem 3. (**Singular Value Decomposition (SVD)**).
a) Consider $A \in \mathrm{C}^{m\times n}$. Define what we mean by the singular value decomposition of A.
b) Consider $A \in \mathrm{C}^{m\times m}$ and show that

$$|\det(A)| = \prod_{i=1}^{m} \sigma_i,$$

where $\{\sigma_i\}$ are the singular values of A.

Problem 4. (**Overdetermined Linear System**).
Solve the following overdetermined linear system in the sense of the least squares:
$z + y = 1, \quad z - y = 2, \quad 4z + 2y = 4.8.$

Problem 5. (**Interpolation**).
a) State the theorem about the existence and uniqueness of interpolating polynomial. Give a proof.
b) Let $f(x) = 5x^2 + x + 1$. Find the polynomial of degree 3 that interpolates the values of f at $x = -1, 0, 1, 2$.

Problem 6. (**Linear Multistep Methods**).
Define linear multistep method (give formula).
a) State necessary and sufficient condition for linear multistep method to be consistent.
b) State root condition for linear multistep method.
c) What can you say about consistency and stability properties of the method:

$$y_{n+1} - 3y_n + 2y_{n-1} = -hf(y_{n-1})?$$

Problem 7. (**Example of A-stable Linear Multistep Method**).
Give an example of A-stable linear multistep method. Justify your answer.

Problem 8. (**Upwind Scheme**).
Consider the advection equation

$$u_t - u_x = 0, \quad x_L < x < x_R, \quad 0 < t \le T,$$

where $u(x, 0) = g(x)$, and $u(x_R, t) = u_R(t)$ for $t > 0$
a) Write the Upwind Scheme for this problem.
b) What is the stencil of the scheme? What is the CFL condition for this method?
c) Investigate the stability of the method using Von Neumann Stability Analysis.