Preliminary Exam, Numerical Analysis, August 2017

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1(**Rank-One Perturbation of the Identity**).
If u and v are n-vectors, the matrix $B = I + uv^\star$ is known as a *rank-one perturbation of the identity*. Show that if B is nonsingular, then its inverse has the form $B^{-1} = I + \beta uv^\star$ for some scalar β, and give an expression for β.
For what u and v is B singular? If it is singular, what is $null(B)$?

Problem 2(**Properties via SVD**).
Prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. Use the 2-norm for your proof.

Problem 3(**Numerical Integration**).
a) Establish a numerical integration formula of the form

$$\int_a^b f(x)dx \approx Af(a) + Bf'(b)$$

that is accurate for polynomials of as high a degree as possible.
b) Determine the quadrature formula of the form

$$\int_{-2}^{2} f(x)dx \approx A_0 f(-1) + A_1 f(0) + A_2 f(1)$$

that is accurate for polynomials of degree <= 2.

Problem 4(**Interpolation**).
a) State the theorem about the existence and uniqueness of interpolating polynomial. Give a proof.
b) Let $f(x) = 7x^2 + 2x + 1$. Find the polynomial of degree 3 that interpolates the values of f at $x = -1, 0, 1, 2$.

Problem 5(**Unstable Multistep Method**).
Consider the numerical method

$$y_{p+1} = 3y_p - 2y_{p-1} + \frac{h}{2}(f(x_p, y_p) - 3f(x_{p-1}, y_{p-1})), \quad p \geq 1,$$

for the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Illustrate with an example of a simple initial value problem that the above numerical scheme is unstable.

Problem 6(**Linear Multistep Methods**).
a) Define linear multistep method (give formula). Give definition of the region of absolute stability.
b) Show that the region of absolute stability for the trapezoidal method is the set of all complex $h\lambda$ with Real(λ)< 0.

Problem 7(**Heat Equation and Stability of the Scheme**).
Consider the implicit in time, Backward Euler method for the solution of the heat equation:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} = f_m^{n+1},$$

$$u_m^0 = g_m, \quad m = 0, \pm 1, \pm 2, ..., \quad n = 0, 1, ..., [T/\Delta t] - 1.$$

and investigate the stability of the scheme using the von Neumann analysis.

Problem 8(**Upwind Scheme**).
Consider the advection equation

$$u_t - 5u_x = 0, \quad x_L < x < x_R, \quad 0 < t \leq T,$$

where $u(x, 0) = g(x)$, and $u(x_R, t) = u_R(t)$ for $t > 0$
a) Write the Upwind Scheme for this problem.
b) What is the stencil of the scheme? What is the CFL condition for this method?
c) Investigate the stability of the method using Von Neumann Stability Analysis.