Probability Prelim Exam

August 2018

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let μ be a translation invariant σ-additive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu((0,1]) < \infty$. Prove that there exists a $c \in (0,\infty)$ such that $c\mu$ is Lebesgue measure.

2. Let $\{X_i : i \geqslant 1\}$ be a sequence of non-negative, i.i.d. random variables such $\mathrm{E}[X_1] = \infty$. Prove that
$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad \text{almost surely.}$$

3. Let μ and ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\lambda = \frac{1}{2}\mu + \frac{1}{2}\nu$. Let $\mathscr{F}$ be a σ-algebra contained in $\mathcal{B}(\mathbb{R})$. Let $\mu_{\mathcal{F}}$, $\nu_{\mathcal{F}}$, and $\lambda_{\mathcal{F}}$ be the restrictions of μ, ν, and λ to $\mathcal{F}$, respectively.

   a) Prove that $\mu_{\mathcal{F}} \ll \lambda_{\mathcal{F}}$ and $\nu_{\mathcal{F}} \ll \lambda_{\mathcal{F}}$.

   b) What is $f_\mu = \frac{d\mu_{\mathcal{F}}}{d\lambda_{\mathcal{F}}}$ in terms of $\frac{d\mu}{d\lambda}$? Of course, a similar relationship holds for $f_\nu = \frac{d\nu_{\mathcal{F}}}{d\lambda_{\mathcal{F}}}$.

   c) For a bounded $\mathcal{B}(\mathbb{R})$-measurable function g, compute $\mathrm{E}^\lambda[g \,|\, \mathcal{F}]$ in terms of $\mathrm{E}^\mu[g \,|\, \mathcal{F}]$, $\mathrm{E}^\nu[g \,|\, \mathcal{F}]$, f_μ, and f_ν?

4. Let $\{X_i : i \geqslant 1\}$ be i.i.d. integrable random variables. Prove that
$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} = \mathrm{E}[X_1].$$

5. Let $\{X_i : i \geqslant 1\}$ be i.i.d. random variables distributed uniformly on $[0,1]$. Prove that the distribution of
$$\frac{4\sum_{i=1}^n iX_i - n^2}{n^{3/2}}$$
converges weakly and identify the limiting distribution.

6. Let $\{X_n : n \geqslant 1\}$ be a stochastic process adapted to a filtration $\{\mathcal{F}_n : n \geqslant 1\}$ and satisfying $\mathrm{E}[|X_n|] < \infty$ for all $n \geqslant 1$. Prove that $\{X_n : n \geqslant 1\}$ is a martingale if and only if $\mathrm{E}[X_T] = \mathrm{E}[X_1]$ for all bounded stopping times T.

7. Let $\{X_n : n \geqslant 1\}$ be a martingale with respect to filtration $\{\mathcal{F}_n : n \geqslant 1\}$, and let T be a stopping time with respect to $\{\mathcal{F}_n : n \geqslant 1\}$. Prove the following two forms of the optional stopping theorem:

   a) If T is almost surely bounded then $\mathrm{E}[X_T] = \mathrm{E}[X_1]$,

   b) If $\mathrm{E}\big[\max_{1\leqslant n\leqslant T} |X_n|\big] < \infty$, then $\mathrm{E}[X_T] = \mathrm{E}[X_1]$.

8. Prove the identity
$$\frac{\sin\theta}{\theta} = \int_0^1 e^{i\theta(2x-1)}\,dx.$$
Then use this to show that
$$\frac{\sin\theta}{\theta} = \prod_{n=1}^{\infty}\cos(\theta/2^n).$$
Hint: Use that a Uniform$(0,1)$ random variable can be written as an infinite weighted sum of iid Bernoulli$(1/2)$ random variables.

9. Let $Z_1, Z_2, \ldots$ be iid $N(0,1)$ random variables. First prove *Mills' ratio*
$$\mathrm{P}(Z_i > \lambda) \leq \frac{1}{\lambda\sqrt{2\pi}}e^{-\lambda^2/2}$$
for any $\lambda > 0$. Use this to show that for any $\epsilon > 0$
$$\mathrm{P}\left(\limsup_{n\to\infty}\frac{\max_{k\leq n} Z_k}{\sqrt{(2-\epsilon)\log n}} > 1\right) = 1.$$

10. Show that for every $\rho \in (-1,1)$ the function
$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)$$
is a pdf on $\mathbb{R}^2$, and that if (X,Y) is a pair of random variables with pdf f then
$$\mathrm{P}(XY < 0) = \frac{1}{\pi}\arccos\rho.$$