Probability Prelim Exam

August 2013

Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam problems begin here:
Throughout, $(\Omega, \mathcal{F}, \mathrm{P})$ denotes the underlying probability space.

1. Let $X, X_1, X_2, \ldots$ be independent random variables, all distributed according to the standard normal distribution.

   (a) Prove that there exist finite constants $c_1, c_2, x_0 > 0$, such that
   $$\frac{c_1}{x}\mathrm{e}^{-x^2/2} \leqslant \mathrm{P}\{X > x\} \leqslant \frac{c_2}{x}\mathrm{e}^{-x^2/2} \qquad \text{for all } x \geqslant x_0.$$

   (b) Use the inequalities of part (a) in order to prove that if $X_1, X_2, \ldots$ are independent standard normal random variables then
   $$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} = -\liminf_{n\to\infty} \frac{X_n}{\sqrt{2\ln n}} = 1 \qquad \text{a.s.}$$

2. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathrm{P}\{X_1 = 1\} = 1/4$ and $\mathrm{P}\{X_1 = 0\} = 3/4$. Let N denote the smallest integer $k \geqslant 2$ such that $X_{k-1} = 0, = X_k = 1$; that is, N is the first time the pattern "01" occurs in the infinite sequence $X_1, X_2, \ldots$. Compute $\mathrm{E}(N)$.

3. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, and $1 \leqslant X_1 \leqslant 2$ a.s. Compute, for every $1 \leqslant i \leqslant j \leqslant n$, the quantity

$$\mathrm{E}\left(\frac{X_i + \cdots + X_j}{X_1 + \cdots + X_n}\right).$$

4. Suppose μ is a probability measure on the Borel subsets of $(0\,,1)$ such that $\int |\ln(x)|\,\mu(\mathrm{d}x) < \infty$. Prove that

$$I(n) := \left(\int x^{1/n}\,\mu(\mathrm{d}x)\right)^n \to \exp\left(\int \ln(x)\,\mu(\mathrm{d}x)\right) \qquad \text{as } n \to \infty.$$

[Hint: $I(n) = \mathrm{E}(\prod_{j=1}^n X_j^{1/n})$, where $X_1, \ldots, X_n$ are i.i.d.]

5. Let $\boldsymbol{X}_n := (X_{1,n}, \ldots, X_{d,n})$ be a random variable in $\mathbf{R}^d$ for every $n \geqslant 1$. Let $\boldsymbol{Y} := (Y_1, \ldots, Y_d)$ be another random variable in $\mathbf{R}^d$. Prove that $\boldsymbol{X}_n \Rightarrow \boldsymbol{Y}$ if and only if the 1-dimensional random variable $\sum_{i=1}^d \lambda_i X_{i,n}$ converges weakly to $\sum_{i=1}^d \lambda_i Y_i$ for every $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_d) \in \mathbf{R}^d$.

6. For every random vector $(X\,,Y)$ in $\mathbf{R}^2$ we can define a function $\mathscr{Q}$ of two variables as follows:

$$\mathscr{Q}(a\,,b) := \mathrm{P}\{X > a\,, Y > b\} - \mathrm{P}\{X > a\}\mathrm{P}\{Y > b\}.$$

   (a) Suppose $f, g : \mathbf{R} \to \mathbf{R}$ are bounded and have integrable, bounded, and continuous derivatives. Then prove that

$$\mathrm{E}[f(X)g(Y)] - \mathrm{E}[f(X)]\mathrm{E}[g(Y)] = \iint_{\mathbf{R}^2} \mathscr{Q}(a\,,b) f'(a) g'(b)\,\mathrm{d}a\,\mathrm{d}b.$$

   (b) Suppose $\mathscr{Q}(a\,,b) \geqslant 0$ for all $a, b \in \mathbf{R}$. Prove that if X and Y have two finite moments, then $\mathrm{Cov}(X\,,Y) \geqslant 0$ and there exists a finite constant α, depending only on f and g, such that

$$|\mathrm{E}[f(X)g(Y)] - \mathrm{E}[f(X)]\mathrm{E}[g(Y)]| \leqslant \alpha \mathrm{Cov}(X\,,Y).$$

   [In other words, if $\mathscr{Q} \geqslant 0$ and X and Y have a small covariance, then X and Y are almost independent.]

7. Suppose that, for every integer $n \geqslant 1$, $X(n)$ has a Gamma distribution with parameters $n+1$ and 1; that is, the probability density function of $X(n)$ is

$$f_n(x) := \frac{x^n \mathrm{e}^{-x}}{n!} \mathbf{1}_{(0,\infty)}(x).$$

   (a) Compute $\mathrm{E}(X(n))$ and $\mathrm{Var}(X(n))$ for every integer $n \geqslant 1$.

(b) Prove that, as $n \to \infty$,

$$\frac{X(n) - n}{\sqrt{n}} \Rightarrow \mathrm{N}(0\,,1).$$

8. Let $\{W(t)\}_{t \geqslant 0}$ be a standard Brownian motion, and define

$$U(t) := \mathrm{e}^{-t/2} W(\mathrm{e}^t) \qquad \text{for all } t \geqslant 0.$$

(a) Prove that $\{U(t)\}_{t \geqslant 0}$ is Gaussian process; compute its mean and covariance functions.

(b) Prove that

$$\limsup_{t\to\infty} \frac{U(t)}{\sqrt{2\ln t}} = -\liminf_{t\to\infty} \frac{U(t)}{\sqrt{2\ln t}} = 1 \qquad \text{a.s.}$$

9. Let X be an integer-valued random variable that is non-degenerate; this means that there exist at least two integers n, m such that $\mathrm{P}\{X = n\}$ and $\mathrm{P}\{X = m\}$ are both strictly positive. Prove that the set

$$\mathcal{V} := \left\{ t \in \mathbf{R} : \; |\mathrm{E}[\mathrm{e}^{itX}]| = 1 \right\}$$

has zero Lebesgue measure. [Hint: If $t \in \mathcal{V}$ then $Z := \mathrm{e}^{it(X-Y)}$ has mean one, where Y is an independent copy of X. Proceed by proving that $\mathrm{E}(|Z-1|^2) = 0$.]

10. Consider i.i.d. $X_1, X_2, \ldots$ that are strictly positive, and define

$$S_n := \sqrt{X_1 + \sqrt{X_2 + \sqrt{X_3 + \sqrt{\cdots + \sqrt{X_n}}}}}, \qquad \text{for all } n \geqslant 1.$$

In other words, if $f(a\,,b) := \sqrt{a + \sqrt{b}}$ for all $a, b > 0$, then $S_1 := f(X_1\,,0)$ and $S_{k+1} := f(S_k\,,X_{k+1})$ for all $k \geqslant 0$.

Prove that $\lim_{n\to\infty} S_n$ exists a.s. and is finite a.s.