Probability Qualifying Examination

January 8, 2009

There are 10 problems, of which you should turn in solutions for **exactly** 6 (your best 6, in your opinion). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2, and put $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$ for each $n \geq 1$. Let N be a nonnegative-integer-valued random variable independent of $X_1, X_2, \ldots$. Find the mean and variance of S_N by conditioning on N.

2. (a) Find a closed-form expression (i.e., one with no summation sign) for $\mathrm{E}[(X+1)^{-1}]$ when X is binomial(n, p) with $n \geq 1$ and $0 < p < 1$.

   (b) Find a closed-form expression for $\mathrm{E}[(X+1)^{-1}]$ when X is Poisson(λ) with $\lambda > 0$. You may use the result of part (a) here.

3. For $n = 1, 2, \ldots$, let X_n have density

$$f_n(x) := \frac{n}{\pi(1 + n^2 x^2)}, \qquad -\infty < x < \infty.$$

   (a) Does X_n converge to 0 in probability?

   (b) Does X_n converge to 0 in $L^1(\mathrm{P})$?

   (c) If $X_1, X_2, \ldots$ are independent, does X_n converge to 0 a.s.?

4. Let $X_1, X_2, X_3, Y_1, Y_2, Y_3$ be independent $N(0\,,1)$ random variables. Let P_1, P_2, and P_3 be the points in the plane with coordinates $(X_1\,,Y_1)$, $(X_2\,,Y_2)$, and $(X_3\,,Y_3)$. Let M be the midpoint of the segment P_1P_2 and let r be half its length. Show that the probability that the point P_3 lies within the circle of radius r centered at M is equal to 1/4.

5. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of integer-valued random variables, and let λ_0 be a root of the equation $\mathrm{E}[\lambda^{X_1}] = 1$. (If λ_0 is complex, it suffices to notice that expectations, conditional expectations, and martingales extend easily to complex-valued random variables.) Let $S_n := X_1 + \cdots + X_n$ and $M_n := \lambda_0^{S_n}$ for each $n \geq 0$, where $S_0 := 0$.

(a) Show that $\{M_n\}_{n\geq 0}$ is a martingale with respect to $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$.

(b) Consider the special case that $\mathrm{P}\{X_1 = 1\} = p$ and $\mathrm{P}\{X_1 = -1\} = 1 - p$, where $0 < p < 1$. Show that, in this case, $\lim_{n\to\infty} M_n$ exists a.s. and is finite.

(c) Find another example to show that $\lim_{n\to\infty} M_n$ need not exist a.s.

6. Suppose $X_n \Rightarrow X$ as $n \to \infty$, and $\sup_{n\geq 1} \mathrm{E}(X_n^2) < \infty$. Prove that $X \in L^1(\mathrm{P})$ and $\lim_{n\to\infty} \mathrm{E}X_n = \mathrm{E}X$.

7. Choose and fix a sequence of real numbers $a_1, a_2, \ldots$ such that: (i) $0 < a_k < 1$ for all $k \geq 1$; and (ii) $\sum_{k=1}^{\infty} a_k^2 = \infty$. Let $X_1, X_2, \ldots$ be independent random variables, where each X_k is distributed uniformly on $(-a_k\,, a_k)$. Prove that $n^{-1/2}(X_1 + \cdots + X_n)$ converges weakly to $N(0\,, \sigma^2)$ and compute σ.

8. Suppose f is a probability density function on $\mathbf{R}$, and define for all $x, y \in \mathbf{R}$,

$$g(x\,,y) := \begin{cases} 2f(x)f(y) & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that g is the probability density function of some random variable $(X\,,Y)$ which takes values in $\mathbf{R}^2$.

(b) Are X and Y independent? Prove or disprove.

9. Suppose $X_1, X_2, \ldots$ are independent, identically-distributed random variables such that

$$\mathrm{P}\{X_1 > x\} = \begin{cases} x^{-\alpha} & \text{if } x \geq 1, \\ 1 & \text{if } x < 1, \end{cases}$$

where $\alpha > 0$ is fixed. Prove that

$$n^{-1/\alpha} \max(X_1\,, \ldots, X_n) \Rightarrow X,$$

and compute $\mathrm{P}\{X > x\}$ for all $x \in \mathbf{R}$.

10. Let α, λ be fixed, strictly positive, and finite. The Gamma$(\alpha\,,\lambda)$ density function is

$$f(x) := \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} \mathrm{e}^{-\lambda x}, & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} \mathrm{e}^{-t}\, \mathrm{d}t$. This defines a probability density function.

(a) Compute the characteristic function of f.

(b) Suppose $X_1, \ldots, X_n$ are independent with common density Gamma$(1\,,\lambda)$. Prove that $S_n := X_1 + \cdots + X_n$ has a Gamma density; also compute its paramaters.

(c) Compute $\mathrm{P}\{S_n < k < S_{n+1}\}$ for all integers $k \geq 1$.