DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS

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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Decide whether the following statement is true or false. If true, prove it, if false, provide a counterexample.

   Let $\{f_n\}$ be a sequence of measurable functions (with respect to Lebesgue measure) with $0 \leq f \leq 1$ such that $\int_0^1 f(x)\,dx \to 0$. Then $f \to 0$ almost everywhere.

2. Let (X, μ) be a measure space with $\mu(X) = 1$, and let$f \in L^\infty(X)$. Prove that $\|f\|_p \to \|f\|_\infty$ where $\|f\|_p$ denotes the L^p norm of f.

3. Let X and Y be Banach spaces, and suppose that $T : X \to Y$ is a compact surjective linear map. Show that Y is finite dimensional. (Recall that a linear map is compact if the image of any bounded set has compact closure.)

4. Let H be a Hilbert space and suppose that $\{v_n\}$ is a sequence in H with the following property. For each $u \in H$

   $$\sup_n | < v_n, u > | < \infty$$

   Prove that $\sup_n \|v_n\| < \infty$.

5. Let $f \in L^1(\mathbb{R})$ such that

   $$\int_{-\infty}^{\infty} |x\hat{f}(x)|\,dx < \infty$$

   Prove that there is a differentiable function g on $\mathbb{R}$ such that $f = g$ almost everywhere.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let f be a holomorphic function on $\mathbb{C}\backslash\{0\}$ such that $f(z)\sin z \to 1$ as $z \to 0$. Calculate
$$\int_\gamma f(z)dz$$
where γ is the unit circle traversed counter-clockwise.

7. Let f be a meromorphic function on $\mathbb{C}$ such that $|f(z)| < B$ if $|z| > R$. Show that f is a rational function (a quotient of polynomials).

8. Let $f : \Omega \longrightarrow \mathbb{C}$ be a complex-valued function on an open domain $\Omega \subset \mathbb{C}$. Assume that for all closed contours γ that consists of line segments parallel to the real or imaginary axes
$$\int_\gamma f(z)dz = 0.$$
Show that there exists a holomorphic function $F : \Omega \longrightarrow \mathbb{C}$ such that $F'(z) = f(z)$.

9. Let Q be a rectangle in $\mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ a holomorphic map. Assume that there exists a $z \in Q$ such that $f(z) = z$ and $f'(z) = 1$. Show that f is the identity map.

10. Suppose a is a real number greater than 1. Compute
$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}$$
using residues. Hint: $\sin\theta = \frac{z-z^{-1}}{2\imath}$ where $z = e^{\imath\theta}$.