DEPARTMENT OF MATHEMATICS

University of Utah
Real and Complex Analysis Preliminary Examination
January 2, 2013

Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. List on the front of your blue book the seven problems to be graded.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let $f \in L^p(\mathbb{R})$ (with respect to the Lebesgue measure), and $1 \leq p \leq \infty$, and let

$$F(x) = \int_0^x f(t)\, dt\,.$$

Prove that F defines a continuous function on $\mathbb{R}$.

2. Let (X, M, μ) be a measure space, and let $f \geq 0$ be an integrable function on X. Suppose that $f_n \geq 0$ is a measurable function, $n = 1, 2, \ldots$, and that

$$f_1 \leq f_2 \leq \cdots \leq f\,.$$

If $\int f_n\, d\mu \to \int f\, d\mu$, prove that $f_n \to f$ a.e.

3. Let (X, M, μ) be a measure space, $1 < p < \infty$, and $q = \frac{p}{p-1}$. Suppose that $f_n \in L^p(X)$, $n = 1, 2, \ldots$, and that for each $g \in L^q(X)$,

$$\sup_n \left| \int f_n g\, d\mu \right| < \infty\,.$$

Show that $\sup_n \|f_n\|_p < \infty$.

4. Let H be a Hilbert space, with U and V closed subspaces which are orthogonal. Prove that $U + V$ is a closed subspace of H.

5. Consider $\mathbb{R}$ with the Lebesgue measure, and recall the definitions of convolution and the Fourier Transform, for $f, g \in L^1(\mathbb{R})$:

$$f * g(x) = \int f(x-y)g(y)\, dy, \qquad \hat{f}(\xi) = \int f(x)e^{-2\pi i x\xi}\, dx\,.$$

Show that for $f, g \in L^1(\mathbb{R})$, $f * g \in L^1(\mathbb{R})$, and that $\widehat{f * g} = \hat{f}\hat{g}$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

1. Find the Taylor series of the function

$$f(z) = \frac{z}{2+z}$$

   at the point $a = 1$. What is the radius of convergence of that series?

2. Let f be a holomorphic function and a an isolated singularity of f. Prove that the following statements are equivalent:

   (a) there exists a disk $D(a, \epsilon) = \{z \in \mathbb{C} \mid |z - a| < \epsilon\}$ and a constant $C > 0$ such that

$$|f(z)| \geq \frac{C}{|z-a|^n}$$

   for all $z \in D(a, \epsilon) - \{a\}$;

   (b) f has a pole of order $m \geq n$ at a.

3. Find the Laurent series of the function

$$f(z) = \frac{1}{(z - z^2)^2}$$

   at $z = 0$.

   (a) What kind of isolated singularity f has at 0?

   (b) What is the residue of f at 0?

4. Evaluate the integral

$$\int_0^\infty \frac{x^2+1}{x^4+1} dx\,.$$

5. Let f be a holomorphic function in a neigborhood of the closed unit disk centered at 0. Assume that $|f(z)| < 1$ for any z on the unit circle centered at 0. How many fixed points f has inside this unit circle?