DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. The following will count as a pass: 5 problems completely correct total including at least two problems completely correct from each section. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

Section A.

Notation: λ denotes Lebesgue measure on $\mathbb{R}$ (or a subset of it).

Let λ denote Lebesgue measure on $\mathbb{R}$ (or a subset of it). Let $\mathcal{H}$ be (an infinite dimensional) Hilbert space.

1. Let μ be the outer measure on $\mathbb{R}$ defined by
$$\mu(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \mid A \subseteq \cup_{i=1}^{\infty}(a_i, b_i],\, a_i \leq b_i\}$$
for every $A \subset \mathbb{R}$. Prove that $\mu((a, b]) = b - a$. You may use that the length of a segment covered by finitely many segments is less than the sum of lengths of segments in the covering.

2. Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = e^{-x}$. Explain why f is Lebesgue integrable and compute its integral.

3. Let $T : V \to W$ be a bounded map, where V and W are normed spaces. Let U be the kernel of T. Prove that $||S|| = ||T||$ where $S : V/U \to W$ such that $S(x + U) = T(x)$, for all $x \in V$.

4. Let V be a Banach space, and $T : V \to V$ a bounded linear map. Assume that for every $v \in V$ there exists a non-negative integer n such that $T^n(v) = 0$. Prove that there exists an integer n such that $T^n(v) = 0$ for all $v \in V$.

5. Let H be a Hilbert space and $e_1, e_2, \ldots$ and an orthonormal basis. Prove that 0 is a weak limit of the sequence e_n. Prove that the unit ball B, $||x|| \leq 1$, is closed in the weak topology. Prove that the unit sphere S, $||x|| = 1$, is dense in B in the weak topology.

Section B.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

6. Calculate

$$\int_\gamma (1 + z^2) \sin(1/z)\, dz$$

where γ is the unit circle traversed in the counter-clockwise direction.

7. Let f be a nonconstant entire function. Assume that the set

$$\{z \in \mathbb{C} : |f(z)| < 1\}$$

is unbounded. Show that f had an essential singularity at ∞.

8. Let $f : D \to D$ be holomorphic. Show that

$$|f(z) - f(0)| \leq |z(1 - \overline{f(0)} f(z))|$$

for all $z \in D$

9. Show that the equation

$$e^z = 2z + 1$$

has exactly one solution in D.
Hint: Note that $e^z - 1 = \int_0^1 e^{tz} z dt$. Conclude that $|e^z - 1| \leq e - 1$.

10. Find a biholomorphism φ of $D - \mathbf{R}^-$ onto D. Here $\mathbf{R}^- = \{x \in \mathbf{R} : x \leq 0\}$.
(It will suffice to give explicitly finitely many maps whose composition is equal to φ).