DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

Notation: λ denotes Lebesgue measure on $\mathbb{R}$ (or a subset of it).

1. Recall that (a version of) Lusin's theorem states that if $f : [0,1] \to \mathbb{C}$ is measurable then for all $\epsilon > 0$ there exists $g : [0,1] \to \mathbb{C}$ continuous so that $\lambda(\{x \in [0,1] : g(x) \neq f(x)\}) < \epsilon$. Prove that Lusin's Theorem implies that if $f : \mathbb{R} \to \mathbb{C}$ is measurable then there exists a continuous function $g : \mathbb{R} \to \mathbb{C}$ so that $\lambda(\{x : f(x) \neq g(x)\}) \leq 1$.

2. Let μ be a non-atomic Borel probability measure on $\mathbb{R}$. Prove that there exists $A \subset \mathbb{R}$ with $\mu(A) = \frac{1}{2}$.

3. Prove that for every Hilbert space, the set of isometries is closed in the strong operator topology. Show that there exists Hilbert spaces where the set of isometries is not closed in the weak operator topology.

4. Show that if $f : [0,\infty) \to \mathbb{C}$ is continuous and compactly supported then $\lim\limits_{\rho \to 1} \int |f(x^\rho) - f(x)| d\lambda = 0$. Use this to show that if $f \in L^\infty([1,2],\lambda)$ we have that $\lim\limits_{\rho \to 1^+} \int |f(x^\rho) - f(x)| d\lambda = 0$.

5. Prove the Banach-Steinhaus theorem. That is, show that if $(B, \|\cdot\|_B)$ and $(\hat{B}, \|\cdot\|_{\hat{B}})$ are Banach spaces and $A_i : B \to \hat{B}$ are a sequence of continuous linear maps from B to $\hat{B}$ then either there exists M so that $\|A_i\|_{op} < M$ for all i or there exists $x \in B$ so that $\sup \|A_i x\|_{\hat{B}} = \infty$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$.

6. Let f be a holomorphic function on D such that $|f(1/n)| \leq \frac{1}{2^n}$ for all positive integers n. Show that f is the zero function.

7. Calculate
$$\int_\gamma (1 + z^2)e^{1/z} dz$$
where γ is the unit circle traversed once in the counter-clockwise direction.

8. Determine the poles and their orders of the function
$$\frac{1}{e^z - 1} - \frac{1}{z}$$

9. Let $f(z) = nz + z^n$, where n is a positive integer. Show that f is one-to-one on D.

10. Let f be a holomorphic function on D such $f^{(n)}(0) \neq 0$ for an infinite number of positive integers n. Show that there exists a point $z_0 \in D$ such that $f^{(n)}(z_0) \neq 0$ for all $n > 0$.