DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

Notation: λ denotes Lebesgue measure on $\mathbb{R}$ (or a subset of it).

Let λ denote Lebesgue measure on $\mathbb{R}$ (or a subset of it). Let $\mathcal{H}$ be (an infinite dimensional) Hilbert space.

1. Prove or disprove: If $(X, \mathcal{M}, \mu)$ is a measure space, $A_1, \ldots$ are measurable and $\sum \mu(A_i)^2 < \infty$ then $\mu(\cap_{m=1}^{\infty} \cup_{i=m}^{\infty} A_i) = 0$.

2. Prove or give a counterexample: B_1, B_2 are Banach spaces and $A : B_1 \to B_2$ is continuous, linear, injective and $A(B_1)$ is dense then A is a bijection.

3. Show that if $v_i \in \mathcal{H}$ and v_i converges weakly to u then $\liminf_{i\to\infty} \|v_i\| \geq \|u\|$ and that equality occurs only where there exists a subsequence v_{n_j} so that v_{n_j} converges to u in norm.

4. Prove that the convolution of two $L^1(\lambda)$ functions is in $L^1(\lambda)$.

5. This question considers different notions of convergence for $\ell^1(\mathbb{N})$:
   (a) Does pointwise convergence imply $\|\cdot\|_1$ convergence? Vice versa?
   (b) Does pointwise convergence imply convergence in measure? Vice Versa?

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let f be a function holomorphic for z in the annulus $r < z < R$. Show that if for some ρ with $r < \rho < R$,
$$\int_{|z|=\rho} z^n f(z)dz = 0$$
for all nonnegative n then f extends to a holomorphic function on the disk $|z| < R$.

7. Let f be an entire function such that for each $z \in \mathbb{C}$, $\Re f(z) \cdot \Im f(z) \neq 1$. Prove that f is a constant.

8. Using methods of complex analysis, evaluate
$$\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$$

9. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that:
   (a) f is continuous;
   (b) f is a holomorphic at all points z not on the real axis;
   (c) if z is on the real axis then $f(z)$ is real.
   Show that $f(\bar{z}) = \overline{f(z)}$ and that f is holomorphic on all of $\mathbb{C}$.

10. Determine the number of zeroes of the polynomial
$$z^5 + z^3/3 + z^2/4 + z/3$$
in the region $1/2 < |z| < 1$.