Ph.D. Qualifying Examination in Statistics
August, 2008

You need at least 50 points to pass.

1. Let $U_1, U_2, \ldots, U_n$ be independent random variables uniform on $[0, 1]$ and let $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ be the corresponding order statistics.
a. Compute the asymptotic distribution of $nU_{2,n}$. (5 points)
b. Show that $nU_{1,n}$ and $n(1 - U_{n,n})$ are asymptotically independent. (5 points)

2. Let $y_i = \alpha x_i + \epsilon_i, 1 \leq i \leq n$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent identically distributed normal $N(0, \sigma^2)$ random variables.
a. Determine the maximum likelihood estimators for α and σ^2. (5 points)
b. Compute the distribution of the maximum likelihood estimator for α. (5 points)
c. Compute the distribution of the maximum likelihood estimator for σ^2. (5 points)

3. Let X and Y be two independent random variables with distribution functions F and G and density functions f and g.
a. Determine the distribution function of XY. (5 points)
b. Does XY always have a density function ? (5 points)
c. Assuming that XY has a density function, compute it. (5 points)

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density function

$$f(t;\theta) = \begin{cases} 0, & \text{if } t < \theta \\ e^{-(t-\theta)}, & \text{if } t > \theta. \end{cases}$$

a. Find a moment estimator for θ. (5 points)
b. Find the maximum likelihood estimator for θ. (5 points)
c. Determine the asymptotic efficiency between the two estimators. (5 points)

5. The number of customers entering a store on a given day is a Poisson random variable with parameter θ. The money spent by a customer is uniformly distributed on $[0, \eta]$. The number of customers and the money spent in the store are independent. We observe $Z_1, Z_2, \ldots, Z_n$, the total spending (the money collected by the store) on n days. We can assume that $Z_1, Z_2, \ldots, Z_n$ are independent.
a. Provide estimators for θ and η using the method of moments. (10 points)
b. Provide estimators for θ and η using the likelihood method. (10 points)

6. Let Φ and ϕ denote the standard normal distribution and density functions.
(a) Prove that for all $x > 0$

$$1 - \Phi(x) \leq \frac{1}{x}\phi(x).$$

(5 points)
Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ denote the order statistics from a sample of n independent identically distributed standard normal random variables.
(b) Show that $X_{n,n}$ converges in probability to infinity. (5 points)
(c) Show that for all $\epsilon > 0$

$$\lim_{n\to\infty} P\{X_{n,n} \leq \sqrt{(2+\epsilon)\log n}\} = 1.$$

(5 points)

7. Let $X_1, \ldots, X_n$ be independent identically distributed Poisson random variables with parameter θ_1. Let $Y_1, \ldots, Y_m$ be independent identically distributed Poisson random variables with parameter θ_2. We assume that the two samples are independent. We wish to test $H_0 : \theta_1 = \theta_2$ against the alternative that H_0 is not true.
a. Derive the likelihood ratio test. (5 points)
b. Provide a large sample approximation for the rejection region. (2 points)

8. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with distribution function F. We assume that F is strictly increasing and continous on its support. Let $x_{1/2}$ denote the median.
a. Find a $1-\alpha$ confidence interval for $x_{1/2}$. (5 points)
b. Provide a large sample approximation for the confidence interval. (5 points)

9. Let $X_1, \ldots, X_N$ be independent random variables. The distibution of X_i is binomial(n_i, p_i). We wish to test $H_0 : p_1 = p_2 = \ldots = p_N$ agains the alternative that H_0 is not true.
a. Derive the likelihood ratio test. (5 points)
b. Provide a large sample approximation for the rejection region. (2 points)