Statistics Qualifying Exam

January, 2010

You need to correctly solve 8 of the following problems to guarantee a "pass".

1. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with distribution function

$$F(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ t^3, & \text{if } 0 \le t \le 1 \\ 1, & \text{if } t > 0. \end{cases}$$

Show that

$$Y_n = n^{1/3} X_{1,n}$$

converges in distribution, where $X_{1,n} = \min\{X_1, X_2, \ldots, X_n\}$.

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t, \theta) = \begin{cases} 0, & \text{if } t \notin [-\theta, \theta] \\ \frac{5}{2}\theta^{-5}t^4 & \text{if } -\theta \le t \le \theta. \end{cases}$$

Find a moment estimator for θ.

3. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with density function

$$h(t, \theta) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ \theta(t+1)^{-\theta-1} & \text{if } 0 \le t < \infty, \end{cases}$$

$\theta > 0$.
(a) Find the maximum likelihood estimator for θ.
(b) Is the estimator unbiased?
(c) Find the asymptotic variance of the maximum likelihood estimator for θ.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t,\theta) = \begin{cases} 0, & \text{if } t \notin [0,\theta] \\ 2\theta^{-2}t & \text{if } 0 \le t \le \theta. \end{cases}$$

Find the uniformly minimum variance unbiased estimator for θ. Explain your answer. You need to prove directly that the sufficient statistic is also complete in this case.

5. Let X and Y be two independent random variables with density functions

$$f(t) = \begin{cases} 0, & \text{if } t \notin [0,4] \\ 1/4, & \text{if } 0 \le t \le 4 \end{cases}$$

and

$$h(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ 2e^{-2t} & \text{if } 0 \le t < \infty. \end{cases}$$

Compute the density of $X - Y$.

6. Let X_1 and X_2 be independent random variables. The density function function of X_1 is

$$f(t) = \begin{cases} 0, & \text{if } t \notin [0,1] \\ \dfrac{e^t}{e-1} & \text{if } 0 \le t \le 1. \end{cases}$$

The distribution of X_2 is

$$P\{X_2 = 1\} = p \quad \text{and} \quad P\{X_2 = -1\} = q, \quad p + q = 1.$$

Compute the moment generating function of $X_1 X_2$.

7. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a UNIF$(0\,,\theta)$ distribution, where $\theta > 0$ is unknown. Find a 95% confidence interval for θ.

8. Let $X_1, \ldots, X_n$ denote an independent sample from an exponential distribution with [unknown] mean $\theta > 0$. What does the Neyman–Pearson lemma say about $H_0:\ \theta = 1$ versus $H_a:\ \theta = 2$? Explain carefully, and identify explicitly the rejection region.

9. Let m denote the median distance [in 1000 miles] required for a certain brand of automobile tires to wear out. Test to see whether or not $m \le 29$, based on the following random sample:

23 20 26 25 48 26 25 24 15 20

10. Derive, using only first principles, the least-squares estimators of the slope and the intercept of a linear regression problem. What can you say about the optimality properties of those estimators?

11. The following data are times (in hours) between failures of air conditioning equipment in a particular airplane:

    74 57 48 29 502 12 70 21 29 386 59 27 153 26 326.

    Assume that the data are observed values of an i.i.d. random sample from an exponential distribution, $X_i \sim \text{EXP}(\theta)$. Test $H_0 : \theta = 125$ versus $H_a : \theta \neq 125$. (A chi-square table is provided.)

12. A sample of 400 people was asked their degree of support of a balanced budget and their degree of support of public education, with the following results:

Education/Budget	Strong	Undecided	Weak
Strong	100	80	20
Undecided	60	80	20
Weak	20	50	5

    Test the hypothesis of independence at $\alpha = 0.05$. (A chi-square table is provided.)

TABLE 4

100 × γth Percentiles $\chi^2_\gamma(\nu)$ of the chi-square distribution with ν degrees of freedom

$$\gamma = \int_0^{\chi^2_\gamma(\nu)} h(y;\, \nu)\, dy$$

ν	γ = 0.005	0.010	0.025	0.050	0.100	0.250	0.500	0.750	0.900	0.950	0.975	0.990	0.995
1					0.02	0.10	0.45	1.32	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	0.58	1.39	2.77	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	1.21	2.37	4.11	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	1.92	3.36	5.39	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	2.67	4.35	6.63	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	3.45	5.35	7.84	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.22	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.39	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.55	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	7.58	10.34	13.70	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	8.44	11.34	14.85	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	9.30	12.34	15.98	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	10.17	13.34	17.12	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	11.04	14.34	18.25	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	11.91	15.34	19.37	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	12.79	16.34	20.49	24.77	27.59	30.19	33.41	35.73
18	6.26	7.01	8.23	9.39	10.86	13.68	17.34	21.60	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	14.56	18.34	22.72	27.20	30.14	32.85	36.19	35.58
20	7.43	8.26	9.59	10.85	12.44	15.45	19.34	23.83	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	16.34	20.34	24.93	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	17.24	21.34	26.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	18.14	22.34	27.14	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	19.04	23.34	28.24	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	19.94	24.34	29.34	34.38	37.65	40.65	44.31	46.93
30	13.79	14.95	16.79	18.49	20.60	24.48	29.34	34.80	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	33.66	39.34	45.62	51.80	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	42.94	49.33	56.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	52.29	59.33	66.98	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	61.70	69.33	77.58	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	71.14	79.33	88.13	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	80.62	89.33	98.64	107.56	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	90.13	99.33	109.14	118.50	124.34	129.56	135.81	140.17

For large ν, $\chi^2_\gamma(\nu) \doteq \nu[1 - (2/9\nu) + z_\gamma\sqrt{(2/9\nu)}]^3$.