Statistics Prelim Exam
University of Utah
Department of Mathematics

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Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let $X_1, X_2, \ldots, X_n$ be iid random variables with probability mass function

$$f(t;\theta) = \theta(1-\theta)^{t-1}, \quad t = 1, 2, 3, \ldots$$

Find the uniformly minimum variance unbiased estimator of θ.

2. Let $X_1, X_2, \ldots, X_n$ be iid random variables with density function

$$h(t;\eta) = e^{-(t-\eta)} I\{t \geq \eta\},$$

where I indicates the indicator function. We want to test $H_0 : \eta = \eta_0$ against $H_A : \eta \neq \eta_0$. Find a test using the likelihood method and compute the power function.

3. Let $X_1, X_2, \ldots, X_n$ be iid random variables with density

$$h(t;\theta) = \theta t^{\theta-1} I\{0 \leq t \leq 1\}, \quad \theta > 0.$$

Find the maximum likelihood estimator of θ and compute the asymptotic variance of the maximum likelihood estimator.

4. Let $X_1, X_2, \ldots, X_n$ be iid with density

$$h(t; \theta) = \frac{1}{2} I\{\theta - 1 \leq t \leq \theta + 1\}.$$

Find all maximum likelihood estimators of θ. Show that if $\hat{\theta}_n$ is a maximum likelihood estimator then $\mathrm{E}[\hat{\theta}_n] \to \theta$ as $n \to \infty$.

5. Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \ldots, Y_m$ be iid $N(\mu_2, \sigma_2^2)$. The X and the Y are also independent of each other. Assume that σ_1^2, σ_2^2 are known while μ_1, μ_2 are unknown. We wish to test $H_0 : \mu_1 = \mu_2$ against $H_A : \mu_1 \neq \mu_2$. Find a test using the likelihood method. Compute the distribution of the likelihood ratio test.

6. Let X_1 be $N(\mu_1, \sigma_1^2)$ and X_2 be $N(\mu_2, \sigma_2^2)$ and assume X_1 and X_2 are independent. Show that $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ are independent if and only if $\sigma_1 = \sigma_2$.

7. Let $X_1, X_2, \ldots, X_n$ be iid with density

$$f(x) = 2x/\theta^2, \quad 0 < x < \theta.$$

Let $Y = X_{(n)} = X_{n:n}$ be the largest order statistic of the X_i. Show that Y/θ is a pivotal quantity for θ and use it to construct a 99% confidence interval for θ.

8. Let $X_1, X_2, \ldots, X_n$ be iid with a Poisson(θ) distribution, so that their common probability mass function is

$$\mathbb{P}(X_i = k) = e^{-\theta} \frac{\theta^k}{k!}, \quad k = 0, 1, 2, \ldots, \ \theta > 0.$$

Find a sufficient statistic for θ.

9. Assume that the lifetimes of 100 lightbulbs are iid random variables with an Exponential(5) distribution, meaning that their common pdf is

$$f(x; \theta) = \frac{1}{5} e^{-x/5}, \ x > 0.$$

Use normal approximation to estimate the probability that in total the 100 lightbulbs last longer than 550 hours. You can express your answer as an integral.

10. Let $X_1, X_2, \ldots, X_n$ be iid with common pdf

$$f(x; \theta) = \frac{\theta}{(1 + x)^{\theta + 1}}, \quad 0 < x < \infty, \theta > 0.$$

What is a lower bound for the variance of an unbiased estimator of θ?