

Preliminary Exam, Numerical Analysis, Winter 2020

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1(**Rank-One Perturbation of the Identity**).

If x and y are m -vectors, the matrix $A = I + xy^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \beta xy^*$ for some scalar β , and give an expression for β . For what x and y is A singular? If it is singular, what is $\text{null}(A)$?

Problem 2(**Properties via SVD**).

Prove that any matrix in $\mathbb{C}^{m \times n}$ is the limit of a sequence of matrices of full rank. Use the 2-norm for your proof.

Problem 3(**Numerical Integration**).

Determine a formula

$$\int_{-\pi}^{\pi} f(x) \cos(x) dx \approx A_0 f\left(-\frac{3}{4}\pi\right) + A_1 f\left(-\frac{1}{4}\pi\right) + A_2 f\left(\frac{1}{4}\pi\right) + A_3 f\left(\frac{3}{4}\pi\right)$$

that is exact when f is a polynomial of degree 3.

Problem 4(**Numerical Algorithm to Find Eigenvalues/Eigenvectors**).

State the Rayleigh Quotient Iteration Algorithm to find eigenvalues/eigenvectors for general real symmetric matrix $A \in \mathbb{R}^{m \times m}$.

Problem 5(**Interpolation**).

Let $x_0, \dots, x_n$ be distinct real points, and consider the following interpolation problem. Choose a function

$$F_n(x) = \sum_{j=0}^n a_j e^{jx}$$

such that

$$F_n(x_i) = y_i \quad i = 0, 1, \dots, n$$

with the $\{y_i\}$ given data. Show there is a unique choice of $a_0, \dots, a_n$.

Problem 6(**Unstable Multistep Method**).

Consider the numerical method

$$y_{k+1} = 3y_k - 2y_{k-1} + \frac{h}{2}(f(x_k, y_k) - 3f(x_{k-1}, y_{k-1})), \quad k \geq 1$$

Illustrate with an example of a simple initial value problem that the above scheme is unstable.

Problem 7(**Linear Multistep Methods**).

- a) Define linear multistep method (give formula). Give definition of the region of absolute stability.
- b) Show that the region of absolute stability for the trapezoidal method is the set of all complex $h\lambda$ with $\text{Real}(\lambda) < 0$.

Problem 8(**Heat Equation and Stability of the Scheme**).

Consider the implicit in time, Backward Euler method for the solution of the heat equation:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - 4 \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} = f_m^{n+1},$$

$$u_m^0 = g_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad n = 0, 1, \dots, [T/\Delta t] - 1,$$

and investigate the stability of the scheme using the von Neumann analysis.