UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Complex Analysis
August 17, 2022.

Instructions: Problems will be scored out of 10 points and four (4) problems will be graded. Clearly identify which four (4) problems you want to be graded. Partial credit will be given for *significant* progress towards the solution.

Three (3) completely correct problems will be a High Pass and two (2) completely correct problems with at least 26 total points will be a Pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Notation: The open disk centered at z of radius r is denoted $D(z,r)$, $\mathbb{D} = D(0,1)$ is the unit disk.

1. Let Ω be a simply connected domain in $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic and has finitely many zeros in Ω. What is the supremum of $m \in \mathbb{N}$ so that $f(z) = g(z)^m$ for some g holomorphic in Ω? Identify this value explicitly in terms of the orders of the zeros of f and prove that it is correct.

2. Show that if f is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$ and $|f(z)| = 1$ on $|z| = 1$ then f is a rational function.

3. Let Ω a bounded domain in $\mathbb{C}$ and $f_n : \Omega \to \mathbb{C}$ a sequence of holomorphic functions such that
$$\int_\Omega |f_n - f_m|^2 \, dA(z) \to 0 \quad \text{as} \quad n, m \to \infty.$$
Here $dA(z)$ is the Lebesgue area integral. Show that there is f holomorphic on Ω such that $f_n \to f$ locally uniformly on Ω.

4. Let $a, b > 0$. Using the residue calculus evaluate the improper integral
$$\int_0^\infty \frac{x \sin(ax)}{x^2 + b^2} dx.$$

5. Prove the following special case of Runge's theorem: Suppose that $f : D(0,4) \setminus (\overline{D(1,.1)} \cup \overline{D(-1,.1)}) \to \mathbb{C}$ is holomorphic and $\varepsilon > 0$ is given. Then there is a rational function q such that $|f(z) - q(z)| \le \varepsilon$ for $z \in K = \overline{D(0,2) \setminus (D(1,.2) \cup D(-1,.2))}$.

6. Let f be an entire function with the property that for all $z \in \mathbb{C}$ there is n such that the n-th derivative $f^{(n)}(z) = 0$. Show that f is a polynomial.