University of Utah, Department of Mathematics
August 2022, Algebra #1 (6310) Qualifying Exam

There are five problems on the exam. You may attempt as many problems as you wish; three correct solutions count as a high pass. 2 correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Find every prime ideal Q of $R = \mathbb{Z}[x]/(x^3 - 3x, 4 + x)$ and count the number of elements in each quotient ring R/Q.

2. Let $k = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements and let $R = k[x]$. Let M be the cokernel of the mapping from R^3 to R^3 given by the matrix
$$\begin{bmatrix} x^2 & x^2 & 0 \\ x^2 & x^2+x & x+1 \\ 0 & 0 & x+1 \end{bmatrix}$$
How many elements are in $\operatorname{Hom}_R(k[x]/(x), M)$?

3. Let $R = \mathbb{R}[x,y]$, let $I = (x,y)$ be the ideal generated by x and y and let $M = R/I$. Compute $\dim_{\mathbb{R}} \operatorname{Tor}_i^R(I,M)$ for all i.

4. Suppose that R is a commutative ring and $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \longrightarrow 0$ is a short exact sequence of R-modules and L is an R-module. Prove, via an explicit argument involving module homomorphisms (and without citing properties of the Hom functor), that there is an exact sequence of Abelian groups:
$$0 \longrightarrow \operatorname{Hom}_R(P,L) \xrightarrow{\overline{\beta}} \operatorname{Hom}_R(N,L) \xrightarrow{\overline{\alpha}} \operatorname{Hom}_R(M,L)$$
You must concretely explain how to induce the maps $\overline{\alpha}$ and $\overline{\beta}$ from α and β respectively.

5. Let k be the field with 2 elements and let $R = k[x]$. Identify, up to isomorphism, all R-modules M with 8 elements (that is $|M| = 8$) and such that $M \otimes_R \left(R/(x^2(x+1))\right)$ also has 8 elements.