UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
August, 2022.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$ be the 2-sphere and let $M_a = f_a^{-1}(0)$ where $a \in \mathbb{R}$ and $f_a \colon \mathbb{R}^3 \to \mathbb{R}$ is the function $f_a(x,y,z) = x^2+y^2-z+a$. For what values of $a \in \mathbb{R}$ are S^2 and M_a transverse? Justify your answer.

2. Let $\mathcal{D}$ be the 2-dimensional distribution in $\mathbb{R}^3$ spanned by the vector fields $V = \frac{\partial}{\partial x}$ and $W = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Let M be a smooth 2-manifold and $\phi \colon M \to \mathbb{R}^3$ a smooth map such that $\phi_*(p)(T_p) \subset \mathcal{D}_{\phi(p)}$ for all $p \in M$. Show that the derivative $\phi_*(p) \colon T_pM \to T_{\phi(p)}\mathbb{R}^3$ is not injective for any $p \in M$.

3. Let X and Y be smooth vector fields on smooth manifolds M and N that are f-related by a smooth map $f \colon M \to N$. If ψ_t is the flow generated by X and ϕ_t is the flow generated by Y show that $f \circ \psi_t = \phi_t \circ f$.

4. Let G_0 and G_1 be Lie groups. Then the product $G_0 \times G_1$ is a Lie group and we let $p_i \colon G_0 \times G_1 \to G_i$ be the projection to each coordinate. Let X be a vector field on $G_0 \times G_1$ and Y_0 and Y_1 vector fields on G_0 and G_1 such that X and the Y_i are p_i-related. Show that X if left-invariant if and only if both Y_0 and Y_1 are left-invariant.

5. Let $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}^n$ a smooth immersion. Then for each $x \in U$ we have that T_xU is canonically isomorphic to $\mathbb{R}^n$ so we can define a smooth map $F \colon U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by $F(x,v) = (f(x), f_*(x)(v))$. Show that the derivative $F_*(x,v) \colon T_{(x,v)}U \times \mathbb{R}^n \to T_{F(x,v)}\mathbb{R}^n \times \mathbb{R}^n$ has positive determinant as a linear map from $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ to itself. Use this to show that for any smooth manifold M its tangent bundle TM is orientable.

6. (a) Construct an explicit n-form ω on $\mathbb{R}^n$ with compact support such that ω is closed but there does not exists an $(n-1)$-form η with compact support and $d\eta = \omega$. You can use the existence of bump functions in your construction but you need to prove that your construction works.

   (b) Let M be a smooth, compact orientable n-manifold without boundary and let (U, ϕ) be a chart for M. Show that there exists a closed n-form on M with support in U that is not exact.