UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
August 2022

Instructions: This examination includes **five** problems but you are to work **three** of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 25 points. A pass is 57 or more points (out of 75). A high-pass is 66 or more points (out of 75).

1. Let (u_n) and (v_n) be two orthonormal sequences in a Hilbert space H. Assume that

$$\sum_{n=1}^{\infty} \|u_n - v_n\|^2 < 1.$$

Show that if (u_n) is a total orthonormal sequence, then (v_n) is also a total orthonormal sequence.

Hint: Assume for contradiction that (v_n) is not total.

2. Let X be a real Banach space. Assume $f \in X^*$ has a closed nullspace $\mathcal{N}(f)$. The goal of this problem is to show that f must be a bounded linear functional.

   (a) Let $x_0 \in X$ be such that $f(x_0) = 1$. Explain why there is an $\epsilon > 0$ for which the ball

$$B(x_0, \epsilon) \equiv \{x \in X \mid |x - x_0| < \epsilon\}$$

   satisfies $B(x_0, \epsilon) \subset X - \mathcal{N}(f)$.

   (b) Prove that $f(x) > 0$ for all $x \in B(x_0, \epsilon)$, where ϵ is as in part (a).
   To prove this, you may assume for contradiction that there is some $y \in B(x_0, \epsilon)$ with $f(y) \leq 0$.

   (c) Any $x \in B(x_0, \epsilon)$ can be written as $x = x_0 + \epsilon u$ for some u with $\|u\| < 1$.
   Use the result of part (b) to show that $|f(u)| < 1/\epsilon$.

   (d) To conclude, give an upper bound for $\|f\|$.

3. Let H be a separable Hilbert space with (e_n) being a total orthonormal sequence of H. Let $T : H \to H$ be a bounded linear operator satisfying

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty.$$

Show that T is compact by approximating T by a sequence (T_k) of finite rank operators that converges to T in an appropriate sense.

4. Let X be a non-trivial normed vector space. Use the Hahn-Banach theorem to show that for any $x \in X$ we have that

$$\|x\| = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|}.$$

5. Consider the left shift operator $L : \ell^2 \to \ell^2$, defined for a sequence $(x_n) \in \ell^2$ by

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

   (a) Prove that $\|L\| = 1$.
   (b) Explain why if $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, then $\lambda \in \rho(L)$, the resolvent set of L.
   (c) Prove that that if $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then $\lambda \in \sigma_p(L)$, the point spectrum of L.
   (d) Deduce what is $\sigma(L)$, the spectrum of L.