UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Complex Analysis
August 17, 2023

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$.

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1. What is the group of all biholomorphisms of $\mathbb{D}$ that fix 0? Justify your answer.

2. Compute the following integral using methods of complex analysis.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx$$

3. Let $P(z)$ be a nonzero polynomial. Show that the equation

$$e^z = P(z)$$

has infinitely many solutions $z \in \mathbb{C}$.

4. Recall that the hyperbolic metric on $\mathbb{H}$ is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Compute the hyperbolic distance between i and $1 + i$.

5. Let f be an entire function and suppose that for every positive integer $n > 1$ and every $z \in \mathbb{C}$ with $|z| < n$ we have

$$|f'(z)| < n^2 \log n$$

Does this imply that f is a polynomial?

6. (i) (3 points) State Runge's theorem.

   (ii) (7 points) Does there exist a sequence of entire functions f_n that converges to 0 pointwise, but not uniformly on compact sets?