UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
August 2023

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Let X be a real Banach space and X^* denote its dual. Let $C \subset X$ be a convex set containing the origin. Define

$$C^* = \{f \in X^* \mid f(x) \leq 1 \text{ for all } x \in C\} \text{ and}$$
$$C^{**} = \{x \in X \mid f(x) \leq 1 \text{ for all } f \in C^*\}.$$

The goal of this problem is to prove that $C^{**} = \overline{C}$.

(a) Show that C^{**} is closed.

(b) Show that $\overline{C} \subset C^{**}$.

(c) Show that $\overline{C}$ is convex.

(d) Assume there is $x_0 \in X$ such that $x_0 \in C^{**}$ but $x_0 \notin \overline{C}$. Show there exists an $f \in X^*$ such that for all $x \in \overline{C}$ we have:

$$f(x) < 1 < f(x_0).$$

(e) Explain the contradiction in (d) and how it can be used to conclude with the desired result.

2. Let X be a Banach space over the complex numbers. Let $X_{\mathbb{R}}$ be the vector space over the reals obtained by restricting the multiplication by scalars used to define X to the real line. Let $f : X \to \mathbb{C}$ be a bounded linear functional on X and let $\phi : X \to \mathbb{R}$ be its real part, defined by $\phi(x) = \operatorname{Re} f(x)$ for all $x \in X$.

(a) Prove that ϕ is a linear functional on $X_{\mathbb{R}}$.

(b) Prove that f can be reconstructed from ϕ by:

$$f(x) = \phi(x) - i\phi(ix), \text{ for } x \in X.$$

(c) Show that f as a linear functional on X and ϕ as a linear functional on $X_{\mathbb{R}}$ have the same norm, i.e.

$$\|f\|_{X^*} = \|\phi\|_{X^*_{\mathbb{R}}},$$

where X^* is the dual of X and $X^*_{\mathbb{R}}$ is the dual of $X_{\mathbb{R}}$.

Hint: To prove one of the inequalities, it may be helpful to find $\lambda \in \mathbb{C}$ such that $|f(x)| = f(x/\lambda)$ for a fixed x with $f(x) \neq 0$.

3. Let (λ_j) be a sequence that is dense in $[0, 1]$ and consider the linear operator $T : \ell^2 \to \ell^2$ defined by its action on a sequence $(\xi_j) \in \ell^2$ by

$$T(\xi_j) = (\lambda_j \xi_j).$$

Find the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, continuum spectrum $\sigma_c(T)$, residual spectrum $\sigma_r(T)$ and resolvent $\rho(T)$.

4. Let H be a separable Hilbert space with (e_n) being a total orthonormal sequence of H. Let $T : H \to H$ be a bounded linear operator satisfying

$$\sum_{n=1}^{\infty} \|Te_n\| < \infty.$$

Show that T is compact by approximating T by a sequence of finite rank operators (T_k) that converges to T in an appropriate sense.

5. Let X be a Banach space. Let $y \in X$ be fixed and let $f : X \to \mathbb{R}$ be a bounded linear functional. Show that there is a constant $C > 0$ such that for all scalars α with $|\alpha| < C$ the equation

$$x + \alpha f(x)x = y$$

has a unique solution on the open ball $\{x \in X : ||x - y|| < 1\}$.