Probability Qualifying Exam

August 2024

Instructions (Read before you begin)

- You may attempt all 6 problems in this exam. However, you can turn in solutions for **at most** 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let g be an integrable random variable. Prove that for each $\epsilon > 0$ we can pick $\delta > 0$ so that $\mu(A) < \delta$ implies $\int_A |g|\, d\mu < \epsilon$. Let $f \geq 0$.
   (Hint: Show first that for any $f \geq 0$, $\int f \wedge n\, d\mu \nearrow \int f\, d\mu$ as $n \to \infty$.)

2. At time $n = 0$, an urn contains $B_0 = 1$ blue, and $R_0 = 1$ red ball. At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and returned to the urn, together with a new ball of the same color. Let B_n and R_n be the number of blue, respectively, red balls in the urn after the n-th turn of this procedure. (Note that $B_n + R_n = n + 2$.) Denote by $\mathcal{F}_n = \sigma(B_j, 0 \leq j \leq n) = \sigma(R_j, 0 \leq j \leq n)$, $n \geq 0$, the natural filtration of the process. Let

$$M_n = \frac{B_n}{B_n + R_n} = \frac{B_n}{n+2}$$

   be the proportion of blue balls in the urn just after time n.

   (a) Show that $(M_n)_{n \geq 0}$ is a martingale in the filtration $(\mathcal{F}_n)_{n \geq 0}$.

   (b) Show that $P(B_n = k) = 1/(n+1)$ for all integers $k \in [1, n+1]$.
   (Hint: Write down the probability of choosing k blue and $n - k$ red balls in whatever fixed order [hence, ending up with $k+1$ blue and $n-k+1$ red balls].)

   (c) Show that $M_\infty = \lim_{n \to \infty} M_n$ exists almost surely and compute its distribution.
   (Hint: use part (b).)

3. Let $(a_n)_{n\geq 1}$ be a <u>nondecreasing</u> sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_n^2}{\sum_{k=1}^n a_k^2} = 0.$$

   Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and variance 1. Prove that

$$\frac{\sum_{k=1}^n a_k X_k}{\sqrt{\sum_{k=1}^n a_k^2}}$$

   converges weakly to the standard normal distribution.

4. Let A_n be a sequence of independent events with $P(A_n) < 1$ for all n. Show that $P(\cup A_n) = 1$ implies $\sum_n P(A_n) = \infty$.

5. Let $X_1, X_2, \ldots$ be i.i.d. with characteristic function ϕ.
   (a) Show that if $\phi'(0) = ia$ and $S_n = X_1 + \ldots + X_n$, then $S_n/n \to a$ in probability. (Hint: first show that if $c_n \to c \in \mathbb{C}$, then $(1 + \frac{c_n}{n})^n \to e^c$.)
   (b) Show that if $S_n/n \to a$ in probability, then $\phi(t/n)^n \to e^{iat}$ as $n \to \infty$ through the integers.

6. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $\mathbb{E}[Y_m] = 1$ and $P(Y_m = 1) < 1$.
   (a) Show that $X_n = \prod_{m\leq n} Y_m$ defines a martingale.
   (b) Show that $X_n \to 0$ a.s. (Hint: note that $|X_{n+1} - X_n| = X_n |Y_{n+1} - 1|$.)
   (c) Use the strong law of large numbers to conclude $\frac{1}{n}\log(X_n) \to c < 0$.