UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Complex Analysis
August, 2024.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

$B(0,1)$ denotes the open unit disk in $\mathbb{C}$. For $z \in \mathbb{C}$ let $Re(z)$ denote the real part of z.

1. State and prove Riemann's theorem on removable singularities.

2. Let $f : B(0,1) \to \mathbb{C}$ be holomorphic. Describe all such f that satisfy $f(\frac{1}{n}) = \frac{n^2-2n-1}{n^2}$ for every $n \in \{2, 3, ...\}$.

3. Let f be an entire function with $f(0) = 0$. Define

$$M_f(r) := \sup\{|f(z)| : |z| = r\}.$$

   Show that $M_f(r)$ is non-decreasing. What can you say about f if it is not strictly increasing?

4. Let $f : \mathbb{C} \setminus \{0, 1, 2\} \to \mathbb{C}$ be a holomorphic function. Show that if f omits at least 4 values then it is constant.

5. Show that if $f : B(0,1) \to \mathbb{C}$ is holomorphic and $Re(f'(z)) > 0$ for all $z \in B(0,1)$ then f is injective.

6. State and prove Rouché's theorem.