University of Utah, Department of Mathematics
August 2024, Algebra I Qualifying Exam

There are five problems on the exam. You may attempt as many problems as you wish; three correct solutions count as a high pass. Show all your work, and provide reasonable justification for your answers.

1. Identify all the ideals in the ring $R = \mathbb{Z}[x]/(6, x^3)$.

2. Let $V := \mathbb{Z}^3$ and L the submodule of V spanned by the columns of the matrix
$$A := \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 6 \\ 2 & 4 & 4 \end{bmatrix}$$
Find invertible matrices Q and P such that the matrix QAP is diagonal.

3. Determine all the $\mathbb{Z}$-submodules of the $\mathbb{Z}$-module $\mathbb{Z}/(3) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(9)$ consisting of 3 elements.

4. Determine all the possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$.

5. Prove the following assertions for the ring $S := \mathbb{R}[x]/(x^2+1)^2$.

   (a) There exist exactly two ring homomorphisms $\pi \colon S \longrightarrow \mathbb{C}$ such that $\pi|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$, the identity map on $\mathbb{R}$.

   (b) Choose one of the homomorphisms in (a), and call it π. Prove that there exists a unique homomorphism of rings $\sigma \colon \mathbb{C} \longrightarrow S$ such that $\sigma|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$ and $\pi\sigma = \mathrm{id}_{\mathbb{C}}$.

   (c) Prove that σ extends to an isomorphism of rings $\mathbb{C}[y]/(y^2) \longrightarrow S$.