This exam is closed book, closed notes, and no calculators are allowed. There is a formula sheet appended to this exam that you may use to complete the problems. You have two hours to complete this exam.

There are 4 problems below. You must complete 3 of them. Each problem is worth 20 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 52 (out of 60) is a *high pass.*
- A score of 48 (out of 60) is a *pass.*

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1. Complete the following:
   (a) (10 pts) Use de Moivre's Theorem to show that the two expressions,
$$\cos(n\theta), \qquad \frac{\sin((n+1)\theta)}{\sin\theta},$$
   for arbitrary $n \in \mathbb{N}$, are both degree-n polynomials of $\cos\theta$.
   (b) (10 pts) Prove the triangle inequality: Given $z_1, z_2 \in \mathbb{C}$, then
$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

2. (20 pts) Prove the Cauchy Integral Formula: Let f be analytic in an open domain D, and let $z \in D$. Then for any non-negative integer n,
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$
where C is any simple contour in D enclosing z. You may take the $n = 0, 1$ versions of this formula as given.

3. Complete the following:
   (a) (10 pts) Prove Rouché's theorem: Assume f and g are analytic functions inside and on a simple closed contour C, and suppose $|f(z)| > |g(z)|$ for all $z \in C$. Then f and $f + g$ have the same number of zeros inside C.
   (b) (10 pts) Prove the following variant of the Argument Principle: Let C be simple closed curve with counterclockwise orientation, and suppose f is analytic inside and on C, except at M poles inside C (poles are counted with weight corresponding to their multiplicity). Let $a \in \mathbb{C}$, and assume $f(z) + a$ has no zeros on C and has N_a zeros inside C (again, counting according to multiplicity). Without citing the Argument Principle, prove that,
$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z) + a} dz = N_a - M.$$

4. (a) (10 pts) Show that
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - \frac{\pi^2}{4}} dx = -2.$$
   (b) (10 pts) Compute the complete infinite asymptotic expansion of
$$I(k) = \int_0^{\pi} e^{-kt^2} t^{-\frac{3}{4}} \sin t \, dt.$$

The Euler Gamma function for real inputs is defined as,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \qquad\qquad x > 0.$$

In what follows, C_R denotes a semicircular arc of radius R in the upper half-plane centered at the origin. The contour C_ϵ is a circular arc of radius ϵ centered around a point z_0 that sweeps out an angle of ϕ.

1. Suppose f is analytic on an open domain containing a simple closed loop C. Then for all integers $n \geq 0$ and all z enclosed by C,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w,$$

2. The coefficients for a Laurent series of the function f are given by,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w$$

3. If a continuous f is bounded over a contour C of finite length, i.e., $|f(z)| \leq M < \infty$ for all $z \in C$ and $\int_C |\mathrm{d}z| = L < \infty$, then

$$\left| \int_C f(z)\mathrm{d}z \right| \leq ML$$

4. Suppose $f(z) = P(z)/Q(z)$ is a rational function with $\deg Q \geq \deg P + 2$. Then,

$$\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{d}z = 0.$$

5. (Jordan's Lemma) Suppose that $f(z) \to 0$ uniformly for $z \in C_R$ as $R \to \infty$. Then for any $k > 0$,

$$\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z)\mathrm{d}z = 0.$$

6. Suppose that $(z - z_0)f(z) \to 0$ uniformly for $z \in C_\epsilon$ as $\epsilon \to 0$. Then,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\mathrm{d}z = 0.$$

7. Suppose that f has a simple pole at $z = z_0$. Then

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\mathrm{d}z = i\phi \mathrm{Res}(f; z_0).$$

8. With C_R any origin-centered circular arc (not necessarily in the upper half-plane), if $zf(z) \to 0$ uniformly on C_R as $R \to 0$, then,

$$\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{d}z = 0.$$

Laplace-type integrals

These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t) e^{-k\phi(t)} \mathrm{d}t$ for $a < b$.

(1) (Watson's Lemma) Set $a = 0$ and $\phi(t) = t$. Assume f is integrable with the series expansion,

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad t \to 0^+, \quad \alpha > -1, \quad \beta > 0.$$

In addition, if $b < \infty$ then assume $|f(t)| \leq M < \infty$ for $t \in [a, b]$, and if $b = \infty$ then assume $f(t) = \mathcal{O}(e^{ct})$ as $t \to \infty$ for some $c \in \mathbb{R}$. Then,

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{k^{\alpha + \beta n + 1}}.$$

(2) (Laplace's Method) Assume $b < \infty$, and that $\phi \in C^4([a, b])$ and $f \in C^2([a, b])$. Suppose that for some $c \in [a, b]$, we have $\phi'(c) = 0$ and $\phi''(c) > 0$. Also, assume that $\phi'(t) \neq 0$ for all $t \in [a, b] \backslash \{c\}$. Then,

$$I(k) \sim G(c) e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi''(c)}} + \mathcal{O}\left(\frac{e^{-k\phi(c)}}{k^{G(c)+1/2}}\right), \qquad G(c) := \begin{cases} 1, & c \in (a, b) \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Fourier-type integrals

These are formulas regarding asymptotic $(k \to \infty)$ behavior of $I(k) := \int_a^b f(t) e^{ik\phi(t)} \mathrm{d}t$ for $a < b$.

(1) Set $a = 0$, and $\phi(t) = \mu t$, where $\mu = \pm 1$, and $k > 0$. Suppose f vanishes infinitely smoothly at $t = b$, that $f \in C^\infty((0, b])$, and that for some $\gamma > -1$, $f(t) \sim t^\gamma + o(t^\gamma)$ as $t \to 0^+$. Then,

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma + 1) e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}).$$

(2) (Stationary phase) Suppose $c \in (a, b)$ is the only value of t where $\phi'(t)$ vanishes. Assume that f vanishes infinitely smoothly at both $t = a$ and $t = b$, and that both f and ϕ are C^∞ on the intervals $[a, c)$ and $(c, b]$. Suppose that there is some $\gamma > -1$ such that as $t \to c$,

$$\phi(t) - \phi(c) \sim \alpha(t - c)^2 + o((t - c)^2),$$
$$f(t) \sim \beta(t - c)^\gamma + o((t - c)^\gamma).$$

Then with $\mu = \operatorname{sgn}\alpha$,

$$\int_a^b f(t) e^{ik\phi(t)} \mathrm{d}t \sim e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma + 1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right).$$